

M4

Time: 3 Hours

Questions: 30

Max Marks: 100

INSTRUCTIONS

1. Use of mobile phones, smartphones, tablets, calculators, programmable wrist watches or any other electronic devices is **STRICTLY PROHIBITED**. Only ordinary pens and pencils are allowed inside the examination hall.
2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.
3. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.
4. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.

INSTRUCTIONS

1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.

WRONG METHODS	CORRECT METHOD
	

4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.

Q. 1	Q. 2
47 ☐0☐0 ☐1☐1 ☐2☐2 ☐3☐3 ☒4☐4 ☐5☐5 ☐6☐6 ☐7☒7 ☐8☐8 ☐9☐9	05 ☒0☐0 ☐1☐1 ☐2☐2 ☐3☐3 ☐4☐4 ☒5☐5 ☐6☐6 ☐7☐7 ☐8☐8 ☐9☐9

5. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.
6. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 to 30 carry 5 marks each.
7. All questions are compulsory.
8. There are no negative marks.
9. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.
10. After the exam, you may take away the Candidate's copy of the OMR sheet.
11. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.
12. You may take away the question paper after the examination.

Note:

1. Unless otherwise stated all numbers are written in decimal notation, and by digits of a number we mean the digits when written in the decimal notation.
2. When the value of an angle is written followed by $^\circ$, we mean it's measure in degrees. Eg. 90° is the angle formed by two perpendicular line segments.

Questions

1. The central square of a 9×9 chessboard is white. How many white squares are there on the board? (The squares of the chessboard are coloured alternately black and white.)
2. An integer M is divisible by 4 but not by 256. What is the number of distinct possible remainders when M is divided by 256?
3. If $x_1, x_2, \dots, x_{49}$ are non-zero integers such that $\sum_{i=1}^{49} x_i = 0$, then what is the minimum possible value of $\sum_{i=1}^{49} x_i^2$?
4. All six digits of three 2-digit numbers are different. If N is the largest possible sum of three such numbers, what is the sum of digits of N ?
5. In triangle ABC , we are given that $\angle CAB = 80^\circ$. Let the perpendicular bisector of BC meet the circumcircle of triangle ABC in N , where we assume that A and N lie on the same side of the chord BC . Then what is the measure of $\angle NBC$ in degrees?

Space for rough work

6. What is the number of integers in the set $\{0, \dots, 20\}$ which can be expressed as the sum of two square integers?
7. Find the number of 2-digit positive integers n such that $n = 26 + (a \times b)$, where a and b are the two digits of n .
8. In trapezium $ABCD$, it is given that AB is parallel to CD . Assume that $AB = 3CD$, $CD = DA$ and $\angle CDA = 120^\circ$. If the largest angle of $ABCD$ is x° and the smallest angle is y° , what is the value of x/y ?
9. Let N be the smallest positive integer whose digits add up to 2026. What is the leading digit of $N + 1$?

Space for rough work

10. Find the number of positive integers n satisfying all the following conditions:
- a) The digits of n lie in the set $\{1, 2, 4, 8\}$. (Digits may be repeated.)
 - b) The sum of the digits is 14.
 - c) If 1 occurs as a digit, then it can occur only immediately to the right of 8.
11. In an isosceles triangle ABC , with $\angle ACB = 90^\circ$. The point D is on the side BC such that $\angle ADC = 75^\circ$. If the area of triangle ADC is 81, what is the length of segment BD ?
12. Let a, b, c, d be positive integers such that $a^2 + b^2 - cd^2 = 2026$. Find the minimum possible value of $a + b + c + d$.
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Space for rough work

13. Find the number of non-constant polynomials $P(x)$, with real coefficients, such that $P(x^2) = P(P(x))$
14. A 7×7 board is divided into 49 unit squares. We place checkers on the board, at most one per square. Find the largest number of checkers that can be placed on the unit squares so that each row, as well as each column, contains an even number of checkers.
15. Let A be a 3-digit number with distinct nonzero digits and B be the number obtained by reversing the digits of A . Determine the largest possible prime factor of $|A - B|$.

Space for rough work

16. A sequence $a_1, a_2, a_3, \dots$ of real numbers satisfies

$$\frac{a_{n+3} - a_{n+2}}{a_n - a_{n+1}} = \frac{a_{n+3} + a_{n+2}}{a_n + a_{n+1}}$$

for all $n \geq 1$. Suppose $a_{55} = 6$, $a_{66} = 2$ and $a_{77} = 1$. Let N denote the sum $a_1^2 + a_2^2 + \dots + a_{2026}^2$. What is the sum of the digits of N ?

17. Let P be a regular polygon with 8 vertices. By a labelling of P we mean an assignment of integers $1, 2, \dots, 8$ to the vertices in some order. A labelling is considered **good** if the path consisting of line segments 1 to 2, 2 to 3, ..., and 7 to 8 does not self-intersect. If N is the number of good labellings, what is the remainder when N is divided by 100?

18. Find the number of ordered triples (x, y, z) of positive integers such that $1 \leq x, y, z \leq 8$ and

$$|x - y| + |y - z| + |z - x| = 8.$$

Space for rough work

19. Let M be the smallest positive integer with the following two properties:
- a) The leading digit of M is equal to 3.
 - b) If N is the number obtained by moving this leading 3 to the units place, and shifting all the other digits one place to the left, then $N = M/4$.

What is the sum of the digits of M ?

20. Four points A, B, C and D lie on a straight line, in this order. A point E , not on the line, satisfies $\angle AEB = \angle BEC = \angle CED = 45^\circ$. Let F and G be the midpoints of AC and BD , respectively. If $\angle FEG = x^\circ$, what is the value of x ?
21. A 1×5 rectangle is divided into five 1×1 squares by drawing four line segments parallel to the shorter side of the rectangle. Each of the resulting sixteen unit-length line segments is coloured red, blue or green. A 1×1 square is called **colourful** if all the three colours are used in colouring its sides. If N is the number of ways of colouring such that all the five 1×1 squares are colourful, find the remainder when N is divided by 100.

Space for rough work

22. Let $ABCD$ be a rectangle and let E be a point on BD such that AE is perpendicular to BD . If $AE = 12$ and $CE = \sqrt{193}$, compute the area of the rectangle $ABCD$.

23. Complex numbers x, y, z satisfy the following system of equations:

$$x^2 + y^2 + z = xy$$

$$x + y^2 + z^2 = yz$$

$$x^2 + y + z^2 = xz$$

Determine the sum of all distinct possible values of $|(x^2 - y)(y^2 - z)(z^2 - x)|$.

24. Let $E = \{p^4 + p^2 - 2 \mid p \text{ is a prime, } p > 3\}$. What is the largest positive integer that divides all the numbers in E ?

Space for rough work

25. Let N be the number of distinct 8-digit numbers obtained by arranging the six numbers 0, 1, 2, 3, 10, 23, where the first digit of the 8 digit number is not zero. Find the sum of the digits of N .
26. There are n points in the plane, no three of which are collinear. Every pair of points is joined by a segment which is coloured red or blue such that the following conditions hold:
- (a) If A, B, C are three points such that AB is red and BC is blue, then AC is red.
 - (b) For any point A , there are exactly three points B, C, D such that AB, AC, AD are red.
- Find the sum of all possible values of n .
27. The lengths of the sides of a convex quadrilateral are $\sqrt{a}$, $\sqrt{a+3}$, $\sqrt{a+2}$ and $\sqrt{2a+5}$, in this order. The length of each diagonal is $\sqrt{2a+5}$. If θ° is the difference between the largest angle and the second largest angle of the quadrilateral then determine the value of θ .

Space for rough work

28. Let $a_1, a_2, \dots$ and $b_1, b_2, \dots$ be strictly increasing sequences of positive integers such that

(a) $a_{n+1} = a_n + a_{n-1}$ for $n \geq 2$

(b) $b_n = 2b_{n-1}$ for all $n \geq 2$

(c) $a_{10} = b_{10} < 2026$.

Find the sum of all possible values of $a_1 + b_1$.

29. Let $n = \frac{4^{31} - 1}{3}$. Find the remainder when 2^{n-1} is divided by n .

30. In triangle ABC , it is given that $\angle CAB = 50^\circ$ and $\angle ABC = 70^\circ$. Points D and E are chosen on sides BC and AC respectively such that $\angle ABE = \angle DAB = 30^\circ$. If $\angle DEB = x^\circ$, what is the value of x ?

Space for rough work