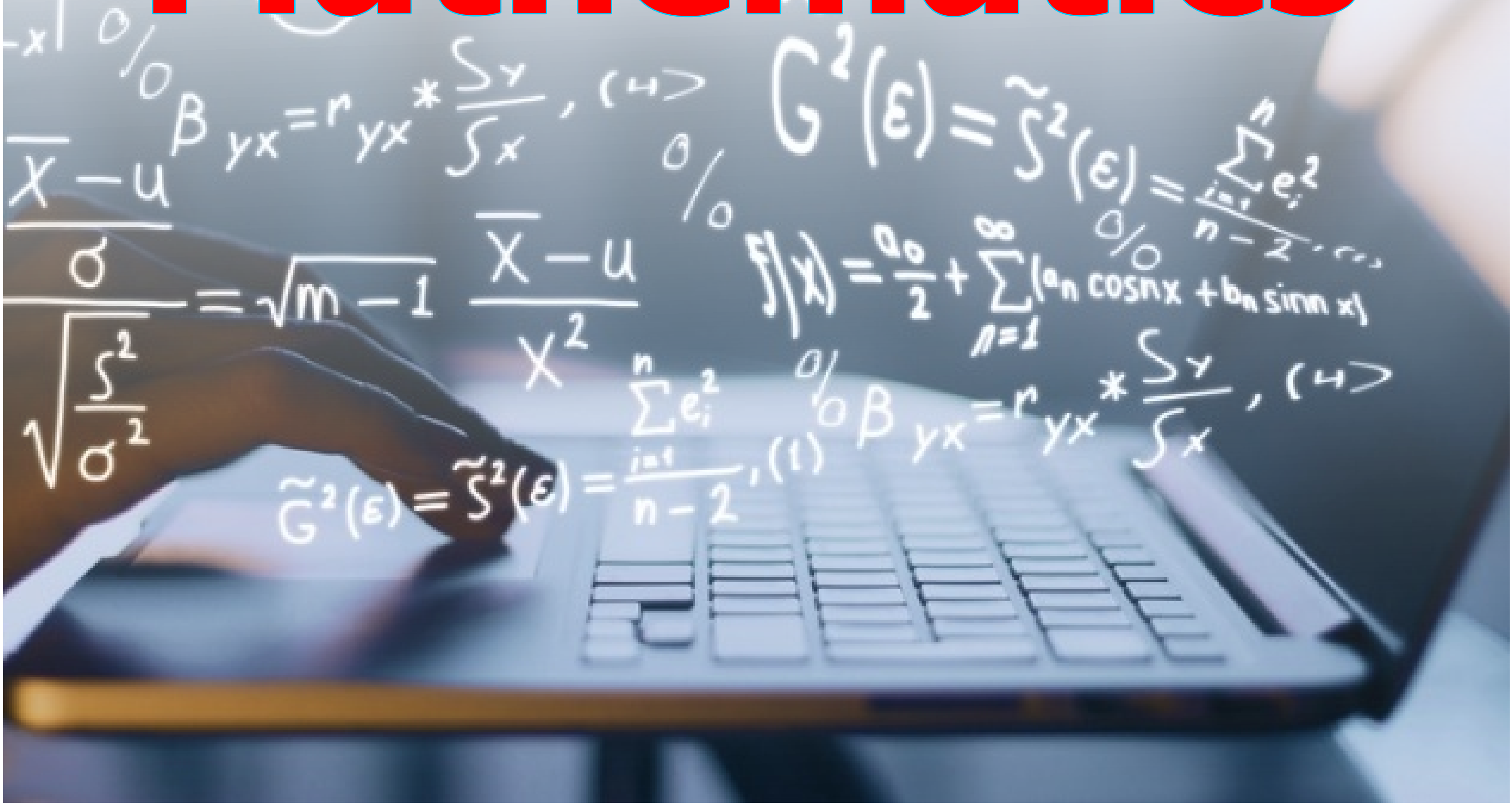
Engineering Mathematics



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ENGINEERING MATHEMATICS

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1 BASIC CALCULUS

1.1. Introduction

1.1.1 Limits, Continuity and Differentiability

(a) As x tends to a $(x \to a)$ $\Rightarrow$ x is moving towards a

A value l is said to be limit of a function $f(x)$ at $x \to a$ if $f(x) \to l$ as $x \to a$.

It is mathematically defined as

$$\lim_{x \to a} f(x) = l = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

That is, Limit exist at any point, if LHL = RHL

A function f(x) is said to be continuous at x = a, if

$$\lim_{x \to a} f(x) = l = f(a) = f(x)|_{x=a}$$

That is, for a function to be continuous at any point, RHL = LHL = Value of function at point $x = a$.

Note:
- For $\lim_{x \to a} f(x)$ to exist, the function need not be continuous at $x = a$.
- But for $f(x)$ to be continuous at x = a, $\lim_{x \to a} f(x)$ should exist.

- Continuity from Left : $\lim_{x \to a^-} f(x) = f(a)$
- Continuity from Right : If $\lim_{x \to a^+} f(x) = f(a)$

Continuity in an Open Interval

A function 'f' is said to be continuous in open interval (a, b), if it is continuous at each point of open interval.

Continuity in a Closed Interval

Let 'f' be a function defined on the closed interval (a, b) then 'f' is said to be continuous on the closed interval [a, b], if it is :

1. Continuous from the right at a and
2. Continuous from the left at b and
3. Continuous on the open interval (a, b).

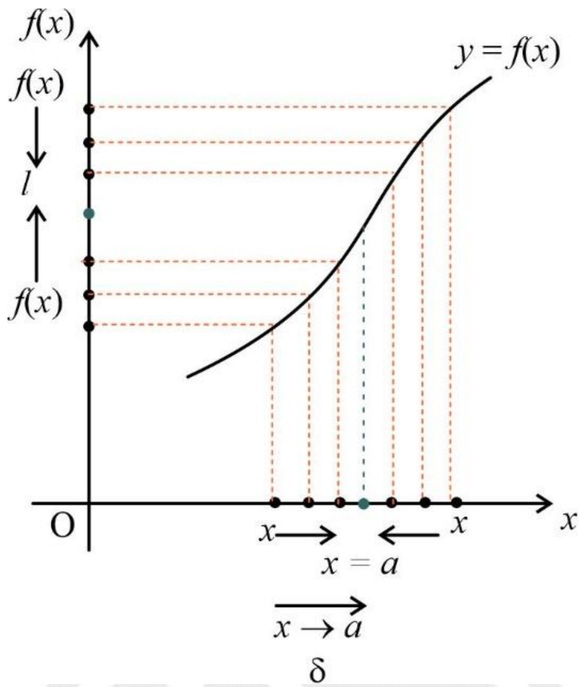

Fig. 1.1

(b) Concept of differentiability

A continuous function $f(x)$ is said to be differentiable at $x = a$, if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, that is, RHL and LHL exist at a point under consideration in $f'(x)$.

$$f'(x)|_{x=a} = f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$f'(a) = \tan\theta$, where θ is the angle made by the tangent to the curve at x=a with x – axis.

(c) Some Standard Derivatives

(i) $\frac{d}{dx}(x^n) = n.x^{n-1}$

(ii) $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\frac{d}{dx}(\cos x) = -\sin x$

(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$

(v) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(vi) $\frac{d}{dx}(\sec x) = \sec x . \tan x$

(vii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; -1 < x < 1$

(ix) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

(x) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xi) $\frac{d}{dx}(cot^{-1} x) = \frac{-1}{1+x^2}$

(xii) $\frac{d}{dx}(sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

(xiii) $\frac{d}{dx}(cosec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

(xiv) $\frac{d}{dx}(log_a x) = \frac{1}{x\, log_e a}$

(xv) $\frac{d}{dx}(log_e x) = \frac{1}{x}$

(xvi) $\frac{d}{dx}(a^x) = a^x . log_e a$

(xvii) $\frac{d}{dx}(e^x) = e^x$

(xviii) $\frac{d}{dx}(|x|) = \frac{|x|}{x}, \ (x \neq 0)$

(xix) $\frac{d}{dx}(x^x) = x^x(1 + log_e x)$

(xx) $\frac{d}{dx}(sinh\, x) = cos\, h\, x$

(d) Product rule of differentiation

(i) $\frac{d}{dx}\big(f(x).g(x)\big) = f(x).g'(x) + f'(x).g(x)$

(ii) $d(uvw) = uvw' + uv'w + u'vw$

(e) Quotient rule of differentiation

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).f'(x) - f(x).g'(x)}{\big(g(x)\big)^2}, (g(x) \neq 0)$$

(f) Logarithmic differentiation:

Taking log might help in differentiation of a function. For example if $y = v^u$ then we can take log both side and differentiable to get $\frac{dy}{dx}$

(g) Differentiation in parametric from :

If we write x and y in term of find variable 't' that is x = f(t), y = ω(t), then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

(h) Greatest Integer function / step function / integer part function

$f(x) = [x] = n, \forall\, n \leq x < n + 1 \text{where}, n \in Z$

$\lim_{x \to a}[x] = \nexists$ if a is an integer (∴ ∄ = do not exist)

L.H.L. = $\lim_{x \to a^-}[x] = a - 1$

R.H.L. = $\lim_{x \to a^+}[x] = a$

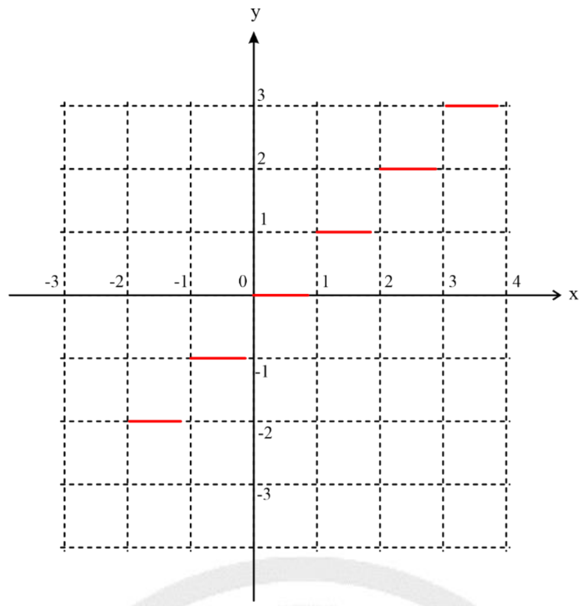

Fig.1.2. Greatest Integer

(i) Properties of Limits

(i) $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}\big(f(x).g(x)\big) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \left(\lim_{x\to a} g(x) \neq 0\right)$

(iv) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x) = \nexists$, then $\lim_{x\to a} f(x).g(x)$ **may exist**

Example: Let $f(x) = \sin x,\ g(x) = \frac{1}{x},\ \lim_{x\to 0} f(x) = 0,\ \lim_{x\to 0}\frac{1}{x} = \nexists$

But $\lim_{x\to 0} \sin x.\frac{1}{x} = 1$

(v) Indeterminate form III (0^0, 1^∞, ∞^0)

If $y = \lim_{x\to a}\left[f(x)\right]^{\phi(x)}$

Then, $\log y = \lim_{x\to a} \phi(x)\log\left[f(x)\right]$

Thus 0^0, 1^∞, ∞^0 will convert into $\infty\times 0$ from which can be solved easily.

(vi) If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \neq \left(\frac{0}{0}\right)$

If $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$ and so on

(vii) If $\lim_{x\to a}\big(f(x).g(x)\big) = 0 \times \infty \Rightarrow \lim_{x\to a}\frac{f(x)}{\left(\frac{1}{g(x)}\right)} = \frac{0}{0}$ (Apply L- Hospital Rule again)

(j) Some Standard Limits

(i) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

(iii) $\lim_{x\to 0} \frac{1-\cos ax}{x^2} = \frac{a^2}{2}$

(iv) $\lim_{x\to \infty} \frac{\sin x}{x} = 0$

(v) $\lim_{x\to \infty} \frac{\cos x}{x} = 0$

(vi) $\lim_{x\to 0}(1+ax)^{b/x} = e^{ab}$

(vii) $\lim_{x\to \infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$

(viii) $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x} = \sqrt{ab}$

(ix) $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+\ldots+n^x}{n}\right)^{1/x} = \sqrt[n]{n!}$

(x) $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a\,;\, \lim_{x\to 0}\frac{e^x-1}{x} = 1$

(xi) $\lim_{x\to 0} x.\sin\left(\frac{1}{x}\right) = 0$

1.2 Mean Value Theorems

1.2.1 Lagrange's Mean Value Theorem (LMVT):

If $f(x)$ is continuous in [a, b] and it is differentiable in (a, b) then ∃ at least one point 'c' such that c ∈ (a, b) and

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Here $f'(c)$ slope of tangent to $f(x)$ at $x = c$.

Tangent at x = c is parallel to the line connecting the points A and B

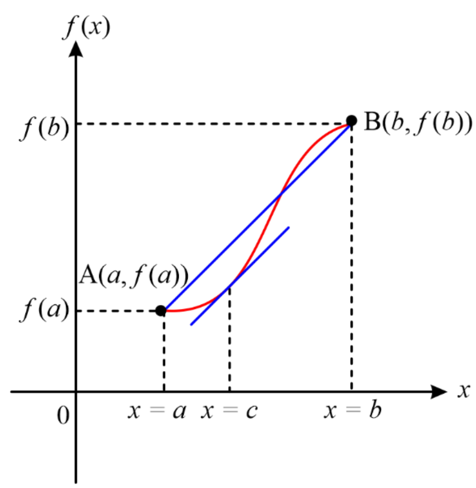

Fig.1.3. LMVT

1.2.2 Rolle's Mean Value Theorem

If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and $f(a) = f(b)$ then ∃ at least one-point c ∈ (a, b) such that $f'(c) = 0$.

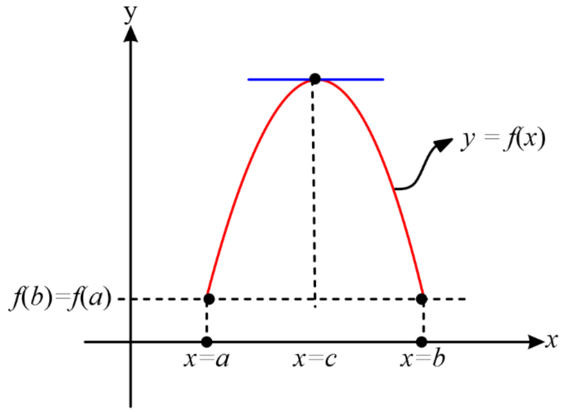

Fig. 1.4. Rolle's mean value

1.2.3 Cauchy's Mean Value Theorem

If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b) then ∃ at least one value of 'c' such that $c \in (a, b)$ and $\frac{g'(C)}{f'(C)} = \frac{g(b)-g(a)}{f(b)-f(a)}$

1.3 Increasing and Decreasing Functions

1.3.1 Increasing Functions

A function $f(x)$ is said to be increasing, if $f(x_1) < f(x_2)\ \forall\ x_1 < x_2$

Or

A function $f(x)$ is said to be increasing, if $f(x)$ increases as x increases.

For a function $f(x)$ to be increasing at the point x=a, $f'(a) > 0$.

Example:

e^x, $\log_e x$ → Monotonically increasing functions

$\sin x$ in $(0, \pi/2)$ → non-monotonic functions

1.3.2 Decreasing Functions

A function $f(x)$ is said to be a decreasing function, if $f(x_1) > f(x_2) \forall x_1 < x_2$

A function $f(x)$ is said to be decreasing function, if $f(x)$ decreases as x increases.

Example: e^{-x} →Monotonically decreasing function, $\sin x$ in $\left(\frac{\pi}{2}, \pi\right)$

1.4. Concept of Maxima and Minima

Let $f(x)$ be a differentiable function, then to find the maximum (or) minimum of $f(x)$.

(1) Find $f'(x)$ and equate to zero.

(2) Solve the resulting equation for x. Let its roots be $a_1, a_2, \ldots$ then $f(x)$ is stationary at $x = a_1, a_2, \ldots\ldots$ Thus $x = a_1, a_2, \ldots\ldots$ are the only points at which $f(x)$ can be maximum or a minimum.

(3) Find $f''(x)$ and substitute in it by terms $x = a_1, a_2, \ldots\ldots$ wherever $f''(x)$ is negative, we have a maximum and wherever $f''(x)$ is positive, we have a minimum.

(4) If $f''(a_1) = 0$, find $f'''(x)$ put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. If $f'''(a_1) = 0$, find $f^{iv}(x)$ and put $x = a_1$ in it. If $f^{iv}(a_1)$ is negative, we have maximum at $x = a_1$, if it is positive there is a minimum at $x = a_1$. If $f^{iv}(a_1)$ is zero, we must find $f^{v}(x)$, and so on. Repeat the above process for each root of the equation $f'(x) = 0$.

Example: $x = 0$ is a critical point of $f(x) = x^3$

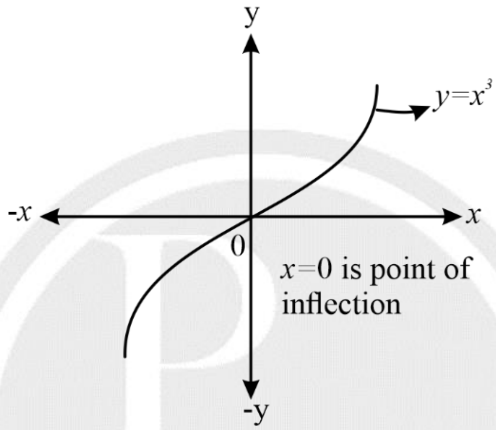

Fig. 1.5. Graph of x^3

$$f(x) = x^3$$

$$\Rightarrow \quad f'(x) = 3x^2 = 0 \quad \Rightarrow \quad x = 0$$

$$f''(x) = 6x \Rightarrow f''(0) = 6(0) = 0$$

- **Global maxima and minima :**
  We first find local maxima and minima and then calculate the value of 'f' at boundary points of interval given e.g. (a, b) we find f(a) and f(b) and compare it with the values of local maxima and minima. The absolute maxima and minima can be decided then.

1.5. Taylor Series

If $f(x)$ is continuously differentiable ($f'(x), f''(x), f'''(x), \ldots\ldots$ exists) then the Taylor series expansion of $f(x)$ about the point $x = a$ is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 . + \ldots \infty$$

If $a = 0$, then $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots\infty$ (Remember that Mc-Lauren Series is same as Taylor Series if a = 0)

The coefficient of $(x-a)^n$ in the Taylor series expansion of $f(x)$ is $\frac{f^n(a)}{n!}$.

The general expansion of Taylor series is given by $f(x+h) = f(x) + h.\frac{f'(x)}{1!} + h^2.\frac{f''(x)}{2!} + h^3.\frac{f'''(x)}{3!} + \ldots\ldots\infty$

- Finding the expansion of e^x about $x = 0$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1; f''(0) = f'''(0) = f''''(0) = \ldots.. = 1$

$f(x) = e^x = 1 + (x-0)\frac{1}{1!} + (x-0)^2.\frac{1}{2!} + (x-0)^3.\frac{1}{3!} + \ldots..$

$\Rightarrow \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$

1.6 Integral Calculus

If $F(x)$ is anti-derivative of $f(x)$. That is, continuous and differentiable in (a, b), then we write $\int_{x=a}^{x=b} f(x)\,dx = F(b) - F(a)$. Here $f(x)$ is integrand

If $f(x) > 0\ \forall a \le x \le b, then\ \int_a^b f(x)\,dx$ represents the shaded area in the given figure.

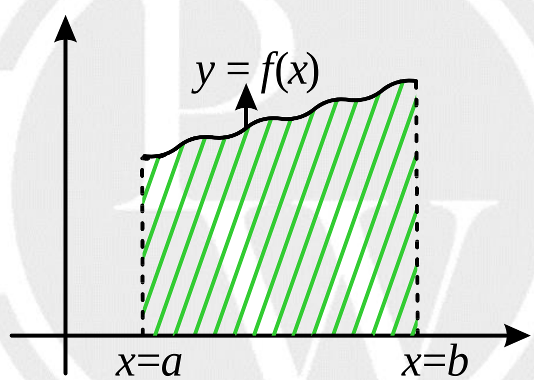

Fig.1. 6. Integration of continuous function

1.6.1 Mean Value Theorem of Integration

If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then '∃' atleast one-point $c \in (a, b)$ such that

$f(c) = \frac{\int_a^b f(x)dx}{(b-a)}$

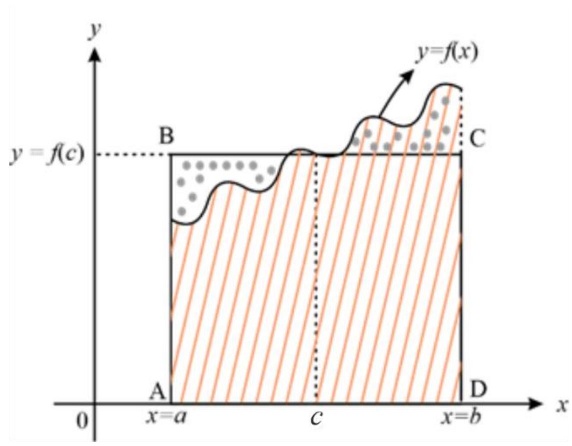

Fig. 1.7. Mean value of integration

1.7. Newton-Leibnitz Rule

If $f(x)$ is continuously differentiable and $\phi(x)$, $\Psi(x)$ are two functions for which the 1st derivative exists, then

$$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(x)dx\right) = f(\psi(x)).\psi'(x) - f(\phi(x)).\phi'(x)$$

Example: $\frac{d}{dx}\left(\int_x^{x^2} \sin x\, dx\right) = \sin(x^2).2x - \sin x.1 = 2x\sin(x^2) - \sin x$

1.8. Some Standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \ (n \neq -1)$

2. $\int \frac{1}{x} dx = \log_e |x| + C$

3. $\int \sin x\, dx = -\cos x + C$

4. $\int \cos x\, dx = \sin x + C$

5. $\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$

6. $\int \tan x\, dx = -\int -\frac{\sin x}{\cos x} dx = -\log_e|\cos x| + C$

$\Rightarrow \int \tan x\, dx = \log_e |\sec x| + C$

7. $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \log_e|\sin x| + C = -\log_e|\text{cosec } x| + C$

8. $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \log_e|\sec x + \tan x| + C$

9. $\int \text{cosec } x dx = \log_e|\text{cosec } x - \cot x| + C$

10. $\int a^x dx = \frac{a^x}{\log_e a} + C$

11. $\int \frac{1}{x.\log_e a} dx = \log_a x + C$

12. $\int x^x(1 + \log_e x)\, dx = x^x + C$

13. $\int f(x).f'(x)dx = \frac{1}{2}(f(x))^2 + C$

14. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2.\sqrt{f(x)} + C$

15. If $f(x)$, $g(x)$ are two functions. that are differentiable, then

$\int f(x)\, g(x)dx = f(x).\int g(x)\, dx - \int[f'(x)\, g(x)]dx + C$

Before integrating the product, the functions $f(x)$ and $g(x)$ are to be arranged according to the ILATE Principle.

Here, ILATE stands for INVERSE LOGARITHMIC ALGEBRAIC TRIGONOMETRIC EXPONENTIAL.

1.9 Properties of Definite Integrals

1. If $f(x)$ is differentiable in interval (a, b), then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. If $\exists$ a point $c \in (a, b)$ such that $f(x)$ is not differentiable, then

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

3. If $f(x)$ is continuously differentiable function,

$\int_{-a}^a f(x)\,dx = 2 \times \int_0^a f(x)\,dx$; if $f(-x) = f(x)$, ("$f(x)$ is even function")

$= 0$; if $f(-x) = -f(x)$, ("$f(x)$ is odd function")

4. $\int_0^{2a} f(x)\,dx = 2 \times \int_0^a f(x)\,dx$, if $f(2a - x) = f(x)$

5. $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

6. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \left(\frac{b-a}{2}\right)$

Example:

(i) $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} = \frac{\pi}{4}$

(ii) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}dx = \int_0^{\pi/2} \frac{1}{1+\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)}dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \frac{\pi}{4}$

(iii) $\int_2^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}} = \left(\frac{3-2}{2}\right) = \frac{1}{2}$

(iv) $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx = \frac{\pi}{4}$

7. $\int_0^{\pi/2} \sin^m x\,dx = \int_0^{\pi/2} \cos^m x\,dx = \frac{(m-1)\times(m-3)\times(m-5)}{m\times(m-2)\times(m-4)} \times \ldots \left(\frac{1}{2}\right)$ (or) $\frac{2}{3} \times K$

Where $K = \pi/2$ if m is even

$= 1$ if m is odd.

8. $\int_0^{\pi} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{\pi}{ab}$

9. $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{\pi}{2ab}$

1.10 Length of a Curve

(a) The length of the arc of the curve $y = f(x)$ between the points where x = a and x = b is $s = \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}dx$

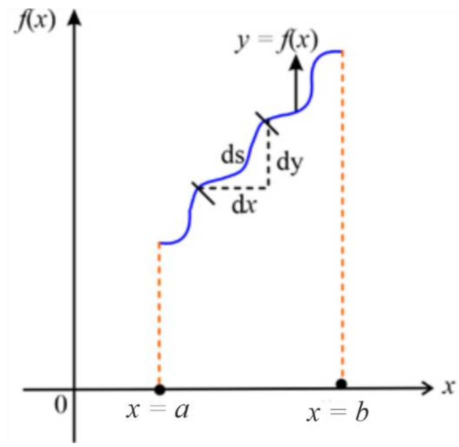

Fig.1.8. Length of the curve

(b) The length of the arc of the curve $x = f(y)$ between the points where y = a and y = b, is $s = \int_a^b \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]} dy$

(c) The length of the arc of the curve $x = f(t), y = f(t)$ between the points where t = a and t = b, is $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

(d) The length of the arc of the curve $r = f(\theta)$, between the points where $\theta = \alpha$ and $\theta = \beta$, is $s = \int_\alpha^\beta \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} d\theta$

1.11 Surface Area of Solid generated by revolving a curve about a fixed axis.

Elemental Surface Area

$dA = 2\pi y \times ds = 2\pi y ds$

$\Rightarrow$ Total surface area = A = $\int_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

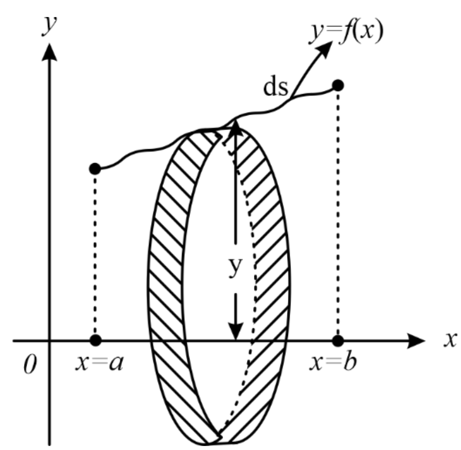

Fig.1.9. Surface area

1.12 Volume of the solid

A. The volume of the solid obtained by revolving the curve y = $f(x)$ between the lines $x = a$ and $x = b$ is given by

$\Rightarrow \quad V \approx \int_{x=a}^{x=b} \pi y^2 dx$

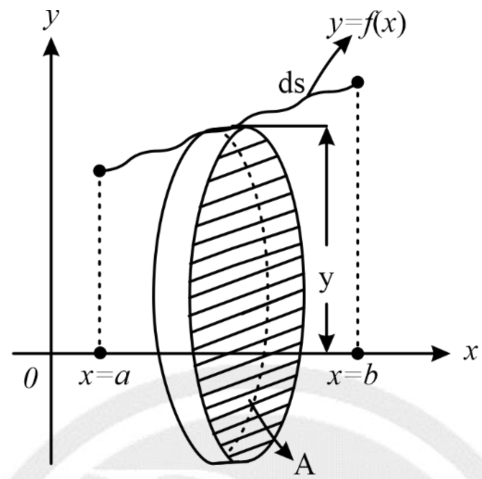

Fig. 1.10. Volume of the solid

B. **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the volume of the solid generated by the revolution, about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissa y = a, y = b is $\int_a^b \pi x^2 dy$.

1.13 Gamma Function

The integral $\int_0^\infty e^{-x}.x^{n-1}\,dx, (n > 0)$ is called Gamma function of n. It is denoted by $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$.

Note : $$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left[\frac{m+n+2}{2}\right]}$$

Where Γ(x) is called the gamma function.

1.13.1 Properties of Gamma Function

(i) $\Gamma n = (n-1)!$

(ii) $\Gamma(n+1) = (n)!$

(iii) $\Gamma(n+1) = n\Gamma n$

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

1.14 Beta Function

The function β (m, n) = $\int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ (m, n > 0) is called β function of m and n.

1.14.1 Properties of β function

(i) $\beta(m,n) = \frac{\Gamma m.\Gamma n}{\Gamma(m+n)}$

(ii) $\beta(m,n) = \beta(n,m)$

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$\beta(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

(iv) $sin^p\theta.cos^q\,\theta\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right), (p,q > -1)$

1.15 Area between the curves

If the function f(x) > g(x) for all values of x between x=a and x=b then

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx \quad \Rightarrow \quad A = \int_a^b \big(f(x) - g(x)\big)\,dx$$

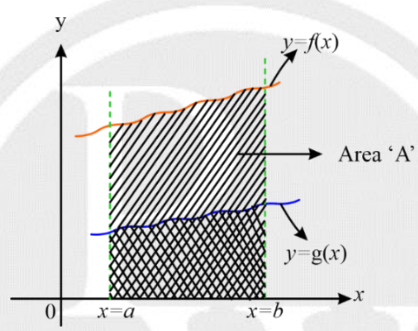

Fig. 1.11. Area under curve

Note : Area bounded by curve $r = f(\theta)$ between $\theta = \alpha$ and β is $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$

1.16. Multi Variable Calculus

(a) Continuity of a function

A function *f*(x, y) is said to be continuous at (*a*, *b*), if $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$

(b) Differentiation of a two-variable function

If *f*(*x*, *y*) is a continuous function, then the derivative of *f*(*x*, *y*) with respect to *x* treating *y* as constant is given by

$$p = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

The derivative of *f*(*x*, *y*) with respect to *y* treating *x* as constant is given by

$$q = \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

(c) Homogenous Function

A function f (x, y) is said to be homogenous function of degree 'n' if $f(kx,ky) = k^n.f(x,y)$.

Example: $f(x,y) = x^3 - 3x^2y + 3xy^2 + y^3$

$\Rightarrow$ $f(kx,ky) = (kx)^3 - 3(kx)^2(ky) + 3(kx).(ky)^2 + (ky)^3$

$= k^3(x^3 - 3x^2y + 3xy^2 + y^3)$

$= k^3.f(x,y) \Rightarrow f(x,y)$ is a homogenous function of degree '3'.

(d) Euler's Theorem

If $f(x, y)$ is a homogeneous function of degree 'n' then

(i) $x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nf$

(ii) $x^2.\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

If $f(x, y) = g(x, y) + h(x, y) + \phi(x, y)$ where $g(x, y)$, $h(x, y)$ and $\phi(x, y)$ are homogenous functions of degrees m, n and p respectively, then

$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = m.g(x,y) + n.h(x,y) + p.\phi(x,y)$

$x^2.\frac{\partial^2 f}{\partial x^2} + 2xy.\frac{\partial^2 f}{\partial x \partial y} + y^2.\frac{\partial^2 f}{\partial y^2} = m(m-1).g(x,y) + n(n-1).h(x,y) + p(p-1).\phi(x,y)$

(e) Total derivative:

(i) If u = f(x, y) and if x = ω(t), y = v(t) then $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

(ii) If u be a function of x and y, where y is a function of x, then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$

(iii) If u = f(x, y) and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1}$ and $\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}$

(iv) If x and y are connected by an equation of the form f(x, y) = 0, then $\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$

(f) Concept of Maxima and Minima in Two Variables

If $f(x, y)$ is a two-variable differentiable function, then to find the maxima (or) minima.

Step-1: Calculate $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$ and equate $p = 0$, $q = 0$

Let (x_0, y_0) be a stationary point.

Step-2: Calculate r, s, t where $r = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)}$; $s = \left.\frac{\partial^2 f}{\partial x.\partial y}\right|_{(x_0,y_0)}$; $t = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)}$

Case (i): If $rt - s^2 > 0$ and $r > 0$, then the function $f(x, y)$ has minimum at (x_0, y_0) and the minimum value is $f(x_0, y_0)$.

Case (ii): If $rt - s^2 > 0$ and $r < 0$, then the function $f(x, y)$ has maximum at (x_0, y_0) and the maximum value is

$f(x_0, y_0)$.

Case (iii): If $rt - s^2 < 0$; then we cannot comment on the existence of maxima and minima.

Such stationary points where $rt - s^2 = 0$ are called **saddle points**.

(g) Concept of Constraint Maxima and Minima (Method of Lagrange's unidentified multipliers).

If $f(x, y, z)$ is a continuous and differentiable function, such that the variables x, y and z are related/constrained by the equation $\phi(x, y, z) = C$ then to calculate the extreme value of $f(x, y, z)$ using Lagrange's Method of unidentified multipliers.

Step-1: Form the function $F(x, y, z) = f(x, y, z) + \lambda\{\phi(x, y, z) - C\}$, where λ is a multiplier.

Step-2: Calculate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ and equate them to zero

Step-3: Equate the values of λ from the above 3 equations and obtain the relation between the variables x, y and z.

Step-4: Substitute the relation between x, y and z in $\phi(x, y, z) = C$ and get the values of x, y, z. Let they be (x_0, y_0, z_0).

Step-5: Calculate $f(x_0, y_0, z_0)$

The value $f(x_0, y_0, z_0)$ is the extreme value of $f(x, y, z)$.

(h) Multiple Integrals

If $f(x, y)$ is continuous and differentiable at every point within a region 'R' bounded by some curves is given by
$I = \iint_R f(x, y)\, dxdy$

If there is a double integral,
$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x, y) dydx$ [Let $\psi(x) > \phi(x)$]

Then I = area of the region bounded by the curves, $y = \phi(x)$; $y = \Psi(x)$, $x = a$ and $x = b$ if $f(x, y) = 1$

Value of x – co-ordinate of centroid of the region bounded by $y = \phi(x)$; $y = \psi(x)$; $x = a$, $x = b$ if $f(x, y) = x$

(i) Change of Orders of Integration

$$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x, y) dydx \rightarrow I = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x, y) dxdy$$

In change of order of Integration, the order of the Integrating variables changes and the limits as well.

Note : When limits are constants, the order of integration does not matter,

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y)\, dxdy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x, y)\, dydx$$

1.17 Jacobian of the Transformation

(i) The Jacobian of the transformation, $x = f_1(u, v), y = f_2(u, v)$ is defined as,

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

(ii) The Jacobian of the transformation,

$x = f_1(u,v,w), y = f_2(u,v,w), z = f_3(u,v,w)$ is defined as

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

1.18 Change of Variables Formula

(i) $\iint_R f(x,y)dx\,dy = \iint_R f(f_1(u,v), f_2(u,v)|J|du\,dv$

(ii) $\iiint_R f(x,y,z)dxdydz = \iiint_R f(f_1(u,v,w), f_2(u,v,w), f_3(u,v,w))|J|du\,dv\,dw$

1.19 Change of Variables

(i) Cartesian to polar co-ordinates :

$x = r\cos\theta$

$y = r\sin\theta$

J = r

$dx\,dy = rdrd\theta$

(ii) Cartesian to cylindrical polar co-ordinate :

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$J = r$

$dxdydz = rdr\,d\theta\,dz$

(iii) Cartesian to spherical polar co-ordinates :

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$J = \rho^2\sin\phi$

$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

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2 ORDINARY DIFFERENTIAL EQUATION

2.1 Differential Equation

The equation involving differential coefficients is called a Differential Equation (DE).

1. $x^2.\frac{dy}{dx} + y^2 = 0$

2. $\frac{\partial^2 T}{\partial x^2} = k \cdot \frac{\partial T}{\partial x}$

3. $x^2.\frac{\partial^2 u}{\partial x^2} + y^2.\frac{\partial^2 u}{\partial y^2} = 0$

2.1.1 Ordinary Differential Equations (ODE)

The DEs involving only one independent variable is called ordinary differential equation.

Example:

(1) $x^2\frac{dy}{dx} + y^2 = 0;$

(2) $e^{-x} \cdot \frac{dy}{dx} + y^2 = e^x$

2.1.2 Partial Differential Equations

The DEs involving two (or) more independent variables are called Partial Differential Equations (PDEs).

Example:

$$\frac{\partial^2 u}{\partial x^2} = C^2 \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{K} \cdot \frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2.1.3 Order of a Differential Equation

The order of the highest derivative that occurs in a DE is called order of a DE.

Example:

(1) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - y = 0$ → Order = 2

(2) $\frac{dy}{dx} + 2 \cdot \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} - 3x^2 = e^x$ → Order = 3

(3) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \cdot \frac{\partial^2 u}{\partial t^2}$ → Order = 2

(4) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u}{\partial t}$ → Order = 2

2.1.4 Degree of a Differential Equation

The Degree of the highest order derivative that occurs in a DE, when the DE is free from fractional (or) radical powers.

Example:

(1) The Degree of the DE $\left(\frac{d^2y}{dx^2}\right)^1 + 2\left(\frac{dy}{dx}\right)^3 - 3y = 0$ is 1.

(2) The Degree of the DE $\left(\frac{d^2y}{dx^2}\right)^1 + \sqrt{\left(\frac{dy}{dx}\right)^3 + 4y} = 0$ is 2

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(-\sqrt{\left(\frac{dy}{dx}\right)^3 + 4y}\right)^2$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3 + 4y$$

(3) It is not possible every time that we can find the degree of a given differential equation. The degree of any differential equation can be found, when it is in the form of a polynomial; otherwise, the degree cannot be defined.

Example degree of the DE is not defined

$$\frac{d^2y}{dx^2} + \cos\frac{d^2y}{dx^2} = 5x$$

2.2. Formation of Differential Equations

If a solution $y = f(x)$ contains n arbitrary constants in it, then differentiate y for n times and calculate $y', y'', y''', \ldots . y^{(n)}$ So, from the $(n + 1)$ equations available, try to eliminate the arbitrary constants in $y = f(x)$

- The different equation formed for the solution, $y = C_1 e^{K_1 x} + C_2 e^{K_2 x}$ where C_1, C_2 are arbitrary constants is

  $\frac{d^2y}{dx^2} - (K_1 + K_2)\frac{dy}{dx} + (K_1 . K_2)y$

- If the solution is $y = C_1 e^{K_1 x} + C_2 e^{K_2 x} + C_3 e^{K_3 x}$ where C_1, C_2, C_3 are arbitrary constants, then the DE is $y''' - (K_1 + K_2 + K_3)y'' + (K_1K_2 + K_2K_3 + K_3K_1)y' - (K_1K_2K_3)y = 0$

2.2.1 First Order DE

The general form of a 1st order DE is given by $\frac{dy}{dx} = f(x, y)$

If $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x, y)$

- $N(x,y)dy + M(x,y)dx = 0$
- $Mdx + Ndy = 0$ where M, N are functions of x and y.

2.2.2 Linear ODE:

An ODE is said to be linear, if it do not contains the higher power terms of dependent variable $\left(y^2, y^3, y^4, \ldots, \left(\frac{dy}{dx}\right)^2, \left(\frac{dy}{dx}\right)^3, \ldots\right)$ and also the terms containing the product of dependent variable and its differential coefficient $\left(y.\frac{dy}{dx}, y^2\frac{dy}{dx}, y\left(\frac{dy}{dx}\right)^2, \ldots\right)$

Example:

(1) $x^2.\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 6y = 0$

(2) $\frac{dy}{dx} - 5y = \sin x$

- $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + \sin y = 0 \;\rightarrow$ Non-linear DE
  - Here, $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} \ldots$
- $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + \sin x = 4y \rightarrow$ Linear DE

2.3 Solving of Differential Equations

2.3.1 Solving of 1st Order DE

(i) Variable-separable form

If the 1st order DE is given by $\frac{dy}{dx} = \phi(x).\psi(y)$

$\Rightarrow \int \frac{1}{\Psi(y)} dy = \int \phi(x) dx$

On integrating we have solution of the given DE

(ii) Homogenous 1st Order

If the 1st order DE is of the form $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$

Such that both $M(x, y)$ and $N(x, y)$ are homogenous functions of same degree, then we say that the DE is homogeneous.

Example:

(1) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

(2) $\frac{dy}{dx} = \frac{ax+by}{a'x+b'y}$

(a and b are not zero at the same time)

If the DE $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$ is a homogeneous DE, then the equation can be converted to Variable Separable form if we substitute $y = Vx$; $\frac{dy}{dx} = V + x\frac{dV}{dx}$

2.3.2 Exact Differential Equations

The DE $Mdx + Ndy = 0$ where M, N are functions of x and y is said to be an Exact Differential Equation, if there exist a function f(x,y) such that $Mdx + Ndy = d\big(f(x,y)\big)$

Mathematical condition to check the Exactness of a differential equation is

$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

(i) Solution of an Exact DE

If $M(x, y)$ dx + $N(x, y)$ dy = 0 is an Exact differential Equation, then the solution of the DE is given by $\int_{y=\text{const}} M(x,y)dx + \int(\text{terms not containing } x \text{ in } N)\, dy = C$

(ii) Integrating Factor

The function which, when multiplied to a non-exact DE converts the DE to exact DE.

Example:

(1) $\frac{1}{y^2}$ is an integrating factor of $ydx - xdy = 0$

(2) $\frac{1}{y}$ is an integrating factor of $x^2dy - xydx = 0$

2.3.3 Methods of Writing the Integrating Factors (I.F.)

(i) If $M(x, y)dx + N(x, y)dy = 0$ is a homogeneous DE, then I.F. $= \frac{1}{Mx+Ny}$, $(Mx + Ny \neq 0)$

(ii) If $Mdx + Ndy = 0$ is of the form $yf(xy)dx + xg(xy)dy = 0$ then I.F. $= \frac{1}{Mx-Ny}$, $(Mx - Ny \neq 0)$

(iii) For a DE, $Mdx + Ndy = 0$, If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, then $e^{\int f(x)dx}$ is the integrating factor.

(iv) For the DE, $Mdx + Ndy = 0$, if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ then $e^{\int g(y)dy}$ is the integrating factor.

2.3.4 Leibnitz Linear Equation

The DE of the form $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x alone, is called Leibnitz Linear Equation

Integrating factor of the equation is $e^{\int Pdx}$

Solution of the Differential Equation is : $y.e^{\int Pdx} = \int Q.(e^{\int Pdx})\, dx$+ C, where C is arbitrary constant.

2.3.5. Non-linear Equations Convertible to Leibnitz Linear Form

Bernoullis Equation

$\frac{dy}{dx} + Py = Q.y^n, (n > 1, n \neq 1)$

(P, Q are functions of x alone)

$\Rightarrow \quad \frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^n}Py = \frac{Qy^n}{y^n}$

$\Rightarrow \quad y^{-n} \cdot \frac{dy}{dx} + y^{1-n}P = Q$

Let $\quad y^{1-n} = z$

$\Rightarrow \quad \frac{dz}{dx} + (1-n)Pz = (1-n)Q \quad$ [Leibnitz Linear Equation]

2.3.6 Applications of 1st order DE

Newton's Law of Cooling

The rate of change of temperature of a body placed in an ambience of temperature T_∞ is directional proportional to the temperature difference between the body and the ambience.

$\frac{dT}{dt} \propto -(T - T_\infty)$ where $T_\infty \rightarrow$ Ambient Temperature ($T > T_\infty$)

$\frac{dT}{dt} \propto (T_\infty - T)$

$\frac{dT}{dt} = -K(T - T_\infty)$

Radioactive Growth / Decay

The rate of growth/decay on any radioactive substance at any instant is directly proportional to concentration of the substance that is available at that instant.

- $\frac{dN}{dt} \propto N \rightarrow$ For growth

  $\Rightarrow \quad \frac{dN}{dt} = KN$

  $\Rightarrow \quad \int \frac{1}{N} dN = \int K dt$

- $\frac{dN}{dt} \propto -N \rightarrow$ For decay

  $\Rightarrow \quad \frac{dN}{dt} = -KN$

  $\Rightarrow \quad \log_e N = Kt + C$

  $\Rightarrow \quad N = e^{Kt}C$

2.4 Higher Order Differential Equations

The general form of Higher order Differential Equations is given by

$$K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + K_3 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + K_n y = X \qquad \ldots(1)$$

- If $K_1, K_2, K_3, K_4, \ldots K_n, X$ are functions of x alone then (1) is called Linear Higher Order Linear DE with variable coefficients.
- If $K_1, K_2, K_3, K_4, \ldots K_n$ are constants and X is a function of 'x' alone, then (1) is called Higher Order Linear DE with constant coefficients.

2.5 Higher Order Linear Differential Equations with Constant Coefficients

The DE $K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + K_3 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + K_n y = X$ …. (1) is said to be a higher order linear DE with constant coefficients if $K_1, K_2, K_3, K_4, \ldots K_n$ are constants and 'X' is a function of x alone.

If $X = 0$, then (1) is called Homogeneous DE

If $X \neq 0$, then (1) is called Non-Homogeneous DE.

2.5.1 Solution of Higher Order Linear Differential Equation

Y $= y_c + y_p$

$y_c \rightarrow$ Complimentary function; $y_p \rightarrow$ Particular Integral

(Solution of homogeneous part; ($X = 0$); (Solution of Non-Homogeneous Part; ($X \neq 0$))

If $K_1\frac{d^ny}{dx^n} + K_2\frac{d^{n-1}y}{dx^{n-1}} + \ldots + K_{n-1}\frac{dy}{dx} + K_ny = X$ …. (1) is a linear DE with constant coefficients.

2.5.2 Rules for Writing the Complete Solution of $(f(D))y = X$:

(i) Form the auxiliary equation of $(f(D))y = X$ i.e. $f(M) = 0$

(ii) Depending on the roots of the auxiliary equation $(f(M) = 0)$, we write the complimentary function.

(iii) Calculate the Particular Integral, $y_P = \frac{1}{(f(D))}X$.

(iv) Write the total solution of the equation $y = y_C + y_P$.

2.5.3 Rules for Writing the Complementary Function

(i) If the roots of $f(M) = 0$ are $M_1, M_2, M_3, \ldots$ ($M_1, M_2, M_3, \ldots \in$ Rational)

Then $y_C = C_1e^{M_1x} + C_2e^{M_2x} + C_3e^{M_3x} + \ldots$ where C_1, C_2, C_3,…. Are arbitrary constants)

(ii) If the roots of $f(M) = 0$ are $M_1, M_1, M_3 \ldots\ldots (M_1, M_3 \ldots\ldots \in$ Rational)

Then $y_C = (C_1x + C_2)e^{M_1x} + C_3.e^{M_3x} + \ldots$ Where C_1, C_2, C_3, …. are arbitrary constants).

(iii) If the roots of $f(M) = 0$ are $M_1, M_1, M_1, M_4 \ldots$ (Where $\in M_1, M_4 \ldots\ldots \in$ Rational)

Then $y_C = (C_1x^2 + C_2x + C_3)e^{M_1x} + C_4e^{M_4x} + \ldots$ (where C_1, C_2, C_3 …. Are arbitrary constants)

(iv) If the roots of $f(M) = 0$ are $\alpha + i\beta, \alpha - i\beta, M_3, M_4 \ldots$ then, $y_C = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3e^{M_3x} + C_4e^{M_4x} + \ldots$

(v) If the roots of $f(M) = 0$ are $\alpha + i\beta, \alpha - i\beta, \alpha + i\beta, \alpha - i\beta, M_5, M_6, \ldots$ then

$y_c = e^{\alpha x}((C_1x + C_2)\cos\beta x + (C_3x + C_4)\sin\beta x) + C_5e^{M_5x} + C_6e^{M_6x} + \ldots$

(vi) If the roots of $f(M) = 0$ are $\alpha + \sqrt{\beta}$, $\alpha - \sqrt{\beta}$, M_3, M_4, …. then $y_C = e^{\alpha x}\{C_1 \sinh\sqrt{\beta}x + C_2\cosh\sqrt{\beta}x\} + C_3e^{M_3x} + C_4e^{M_4x} + \ldots$

(vii) If the roots of $f(M) = 0$ are $\alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, \alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, M_5, M_6, \ldots$ then $y_C = e^{\alpha x}\{(C_1x + C_2)\sinh\sqrt{\beta}x + (C_3x + C_4)\cosh\sqrt{\beta}x\} + C_5e^{M_5x} + C_6.e^{M_6x} + \ldots$

2.5.4 Rules for writing the Particular Integral

(i) If $X = e^{ax}$,

$y_P = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{aX}$, (if $f(a) \neq 0$)

If $f(a) = 0$, then $y_P = x\frac{1}{f'(a)}e^{ax}$ (if $f'(a) \neq 0$)

If $f'(a) = 0$, then $y_p = x^2.\frac{1}{f''(a)}e^{ax}$ (if $f''(a) \neq 0$) and so on.

Solve $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + 6y = e^{2x}$

Sol. Aux. Eqn $\rightarrow M^2 - 5M + 6 = 0 \Rightarrow M = 2,3$

$y_C = C_1e^{2x} + C_2e^{3x}$

$y_P = \frac{1}{D^2-5D+6}e^{2x}$ *since* $f(2) = 0$

$\Rightarrow \quad y_P = x\frac{1}{(2D-5)}e^{2x} = x.\frac{1}{(2(2)-5)}e^{2x}$

$\frac{x}{-1}e^{2x} = -x.e^{2x}$

(ii) If $X = sin(ax + b)\,(or)\,\cos(ax + b)$

$y_P = \frac{1}{f(D)}sin(ax + b)$

Replace D^2 by $-a^2$ in $f(D)$

If the denominator is the form $cD + d$ then rationalize the denominator and replace D^2 by $-a^2$

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = sin(2x + 3)$

$y_p = \frac{1}{D^2-5D+6} \cdot sin(2x + 3)$

$a = 2 \Rightarrow -a^2 = -4$

$\Rightarrow \quad y_p = \frac{1}{-4-5D+6}sin(2x + 3)$

$\Rightarrow \quad y_p = \frac{1}{2-5D} \times \frac{2+5D}{2+5D}.sin(2x + 3)$

$\Rightarrow \quad y_p = \frac{2+5D}{4-25D^2}sin(2x + 3) = \frac{2+5D}{4-25(-4)}\sin(2x + 3) = \frac{1}{104}(2.\sin(2x + 3) + 10.\cos(2x + 3))$

(iii) If $X = x^m$

$y_P = \frac{1}{f(D)}x^m$

$\Rightarrow \quad y_P = [f(D)]^{-1}x^m$

Calculate y_P for the DE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

$y_P = \frac{1}{D^2-5D+6}x^2$

$= \frac{1}{6\left(1+\left(\frac{D^2-5D}{6}\right)\right)}x^2$

$= \frac{1}{6}\left(1 + \left(\frac{D^2-5D}{6}\right)\right)^{-1} x^2$

$= \frac{1}{6} \cdot \left\{1 - \left(\frac{D^2-5D}{6}\right) + \left(\frac{D^2-5D}{6}\right)^2 - \left(\frac{D^2-5D}{6}\right)^3 + \ldots\ldots\right\} x^2$

$= \frac{1}{6}\left\{x^2 - \frac{1}{6}\left(2 - 5(2x)\right) + \frac{1}{36}\{25(2)\}\right\}$

$= \frac{1}{6}x^2 - \frac{1}{18} + \frac{5x}{18} + \frac{25}{108}$

$= \frac{1}{6}x^2 + \frac{5x}{18} + \frac{19}{108}$

(iv) If $X = e^{ax}V$, then

$y_P = \frac{1}{f(D)} \cdot e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} V$

2.6 Euler Cauchy Equation (Higher order linear DE with Variable Coefficients)

The DE of the form $x^n \frac{d^n y}{dx^n} + K_1 \cdot x^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}} + K_2 \cdot x^{n-2} \cdot \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + K_{n-1} \cdot x \cdot \frac{dy}{dx} + K_n y = X$ Where $K_1, K_2, K_3, \ldots\ldots K_n$ are constants is called Euler-Cauchy Equation

2.6.1 Procedure to solve Euler Cauchy Equations

Let $x^n \frac{d^n y}{dx^n} + K_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + K_2\, x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + K_{n-1}\, x \frac{dy}{dx} + K_n y = X$ …(1)

$\left(x^n \frac{d^n}{dx^n} + K_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + K_2\, x^{n-2} \frac{d^{n-2}}{dx^{n-2}} + \ldots\ldots + K_{n-1}\, x \frac{d}{dx} + K_n\right) y = X$

Let $x = e^z \;\Rightarrow\; z = \log_e x$

$x \frac{d}{dx} = \frac{d}{dz} = D$

$x^2 \cdot \frac{d^2}{dx^2} = \frac{d}{dz}\left(\frac{d}{dz} - 1\right) = D(D-1)$

$x^3 \cdot \frac{d^3}{dx^3} = D(D-1)(D-2)$ and so, on

(1) $\Rightarrow \{D(D-1)(D-2)\ldots\ldots\ldots(D-(n-1)) + K_1 D(D-1)(D-2)\ldots\ldots(D-(n-2)) + \ldots\ldots K_{n-1}D + K_n\}y = X$

Where $D = \frac{d}{dz}$

$\Rightarrow$ $(f(D))y = z \rightarrow$ Higher order linear DE with constant

$f(D) \rightarrow$ Polynomial in terms of D with constant coefficients.

(2) If the differential equation is of form

$$(ax+b)^n \left(\frac{d^n y}{dx^n}\right) + k_1 (ax+b)^{n-1} \left(\frac{d^{n-1}y}{dx^{n-1}}\right) + \ldots + k_{n-1}(ax+b) \left(\frac{dy}{dx}\right) + k_n y = f(x)$$

i.e., $\left[(ax+b)^n D^n + k_1 (ax+b)^{n-1} D^{n-1} + \ldots. + k_{n-1}(ax+b)D + k_n\right] y = f(x)$

where, f(x) is a function of 'x'.

It can be reduced to linear differential equations with constant coefficients, by putting $(ax+b)=e^z$

Or z = log (ax + b)

Then, $(ax + b)\, Dy = aD_1 y$

$$(ax+b)^2 D^2 y = a^2 D_1 (D_1 - 1) y$$

$$(ax+b)^3 D^3 y = a^3 D_1 (D_1 - 1)(D_1 - 2) y$$

Where, $D = \dfrac{d}{dx}$ and $D_1 = \dfrac{d}{dz}$

❑❑❑

3 VECTOR CALCULUS

3.1 Basics of Vectors

1. **Equality of Vectors:** If two vectors have equal length and same direction then they are equal.

2. **Vector between two points P, Q :** Initial point P : (x_1, y_1, z_1) and terminal point $Q:(x_2, y_2, z_2)$ the three numbers,

   (a) $a_1 = x_2 - x_1, a_2 = y_2 - y_1, a_3 = z_2 - z_1$: are called the components of the vector a with respect to that coordinate system, and we write simply $a = [a_1, a_2, a_3]$.

   (b) $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

   (c) **Position vector:** The vector with origin as start point and A (x, y, z) as end point.

3. **Vector addition and multiplication:**

   (a) **Addition of vector:**

   $\vec{a} + \vec{b}$ of two vector $a = [a_1, a_2, a_3]$ and $b = [b_1, b_2, b_3]$

   $$\Rightarrow \quad \vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

   (b) **Vector addition is commutative** i.e., $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

   (c) **Vector addition is associative** i.e., $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

4. **Unit vector :** Unit vector along $\vec{A} = \frac{\vec{A}}{|A|}$, this vector is in direction of $\vec{A}$ but has unit magnitude

Note : **Length and Direction of Vectors**

Any vector $\vec{a}$ may be written as a product of its length and direction as follows :

$$\vec{a} = |\vec{a}|\left(\frac{\vec{a}}{|\vec{a}|}\right)$$

Here $|\vec{a}|$ is the length of vector and $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector in direction of '$\vec{a}$'.

3.2 Vector Product / Cross Product

If $\vec{a}$ and $\vec{b}$ are two vectors, then the cross product of the two vectors is denoted by $\vec{a} \times \vec{b}$ and it is given by $\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot sin\,\theta \cdot \hat{n}$

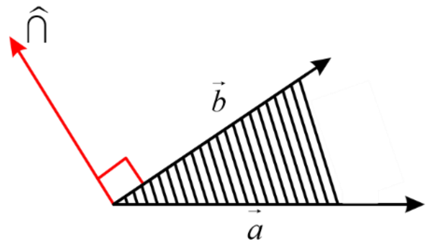


Fig. 3.1. Cross product

$\hat{n} \rightarrow$ unit vector passes through the intersection point of intersection of a and b and lies perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

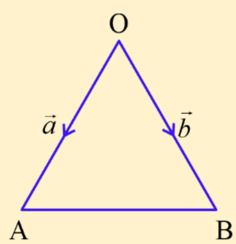
Properties :

(a) $\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$ not commutative rather anti commutative

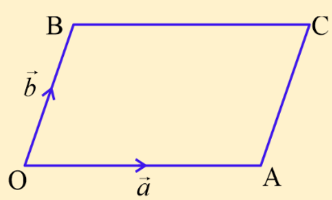
(b) $\vec{a} \times \left(\vec{b} \times \vec{c}\right) \neq \left(\vec{a} \times \vec{b}\right) \times \vec{c}$

Note :

- Area of triangle $OAB = \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right| = \frac{1}{2}\left|\vec{a} \times \vec{b}\right|$



- Area of rectangle or parallelogram $= \left|\vec{a} \times \vec{b}\right|$



- If $\vec{a}, \vec{b}, \vec{c}$ are the vectors then, volume of parallelepiped $= \left|\left[\vec{a}, \vec{b}, \vec{c}\right]\right| = \vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\vec{b}\vec{c}]$

3.3 Dot / Scalar Product:

(i) If $\vec{a}$ and $\vec{b}$ are two vectors, then the dot /scalar product of the two vectors is denoted by $\vec{a}.\vec{b}$ and it is given by $\vec{a}.\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot cos\,\theta$ where $\theta$is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Note:

$$\left|(\vec{a}\cdot\vec{b})\right|^2 + \left|\vec{a}\times\vec{b}\right|^2 = |\vec{a}|^2\cdot\left|\vec{b}\right|^2 . cos^2\theta + |\vec{a}|^2\cdot\left|\vec{b}\right|^2\cdot sin^2\theta = |\vec{a}|^2\cdot\left|\vec{b}\right|^2$$

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$ then $\vec{a}\cdot(\vec{b}\times\vec{c}) = [\vec{a}\vec{b}\vec{c}] \Rightarrow [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(ii) $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

(iii) If $\vec{a}.\vec{b} = 0$, then the two vectors are orthogonal

(iv) Properties of dot product:

(a) $\left(\vec{a}+\vec{b}\right).\vec{c} = \vec{a}.\vec{c} + \vec{b}.\vec{c}$ (Distributive)

(b) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (Symmetry)

(c) $\vec{a}.\vec{a} \geq 0$ (Positive-definiteness)

(d) $\vec{a}.\vec{a} = 0$ if and only if a = 0

(e) $\left|\vec{a}.\vec{b}\right| \leq |\vec{a}|\left|\vec{b}\right|$ (Schwarz inequality)

(f) $\left|\vec{a}+\vec{b}\right| \leq |\vec{a}| + \left|\vec{b}\right|$ (Triangle inequality)

(g) $\left|\vec{a}+\vec{b}\right|^2 + \left|\vec{a}-\vec{b}\right|^2 = 2\left(|\vec{a}|^2 + \left|\vec{b}\right|^2\right)$ (Parallelogram equality)

(v) Projection of vector $\vec{A}$ on $\vec{B}$ is $\left\{\dfrac{\vec{A}.\vec{B}}{|B|}.\vec{B}\right\}$

3.4 Scalar/ Vector triple product

Scalar Triple Product

- $\vec{a}.\left(\vec{b}\times\vec{c}\right) = \left[\vec{a},\vec{b},\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
- $\vec{a}.\left(\vec{b}\times\vec{c}\right) = \vec{b}.(\vec{c}\times\vec{a}) = \vec{c}\left(\vec{a}\times\vec{b}\right)$
- $\left[\vec{a},\vec{b},\vec{c}\right] = -\left[\vec{b},\vec{a},\vec{c}\right] = -\left[\vec{c},\vec{b},\vec{a}\right] = -\left[\vec{a},\vec{c},\vec{b}\right]$
- $\vec{a}.(\vec{b}\times\vec{c})$ = volume of parallelepiped created by $\vec{a},\vec{b},\vec{c}$.
- If $\vec{a}.(\vec{b}\times\vec{c}) = 0$ then the three vectors are coplanar
- If $\vec{a}.(\vec{b}\times\vec{c}) = 0$ then the three vectors are linearly dependent.

Vector triple product:

$\vec{a}\times\vec{b}\times\vec{c} = \vec{b}.(\vec{a}.\vec{c}) - \vec{c}\left(\vec{a}.\vec{b}\right)$

3.5 Differentiation of Vector Point functions

If $\vec{R}(t)$ is a vector point function, then the derivative of $\vec{R}(t)$ is given by

$$\frac{d\vec{R}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t}$$

If $\vec{R}(t) = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath}$ then $\frac{d\vec{R}(t)}{dt} = f'(t)\hat{\imath} + g'(t)\hat{\jmath}$

Example:

If $\vec{R}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} \Rightarrow \frac{d\vec{R}(t)}{dt} = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$

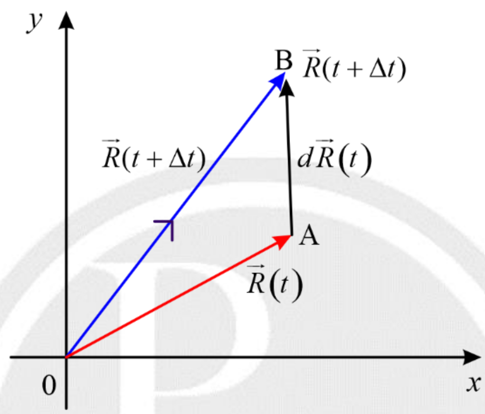


Fig. 3.2

3.5.1. Differentiation of Product of two vectors

$$\frac{d}{dt}\left(\vec{a}(t)\cdot\vec{b}(t)\right) = \vec{a}(t)\cdot\frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt}.\vec{b}(t)$$

$$\frac{d}{dt}\left(\vec{a}(t)\times\vec{b}(t)\right) = \vec{a}(t)\times\frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt}\times\vec{b}(t)$$

If $\vec{F}(t)$ is a vector point function with constant magnitude, then $\vec{F}(t)\cdot\frac{d}{dt}\vec{F}(t) = 0$.

If $\vec{F}(t)$ is a vector point function with constant direction, then $\vec{F}(t)\times\frac{d}{dt}\vec{F}(t) = \vec{0}$.

3.6 Dell operator

The Vector operator $\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$ is called the differential operator in vector and it is denoted as Del (or) ∇

$\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$.

3.6.1 Gradient of a Scalar Point function:

If $\phi(x,y,z)$ is a Scalar Point function, then the gradient (change) of $\phi(x,y,z)$ is denoted by grad $\phi(or)\nabla\phi$ and it is given by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right)\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k}.$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k}.$$

Note: If $\vec{F}(x, y, z)$ is an irrotational vector field ($\nabla \times \vec{F} = \vec{0}$), then definitely there exists a scalar point function $\emptyset(x, y, z)$ such that $\vec{F}(x, y, z) = grad\ \emptyset$.

If $\phi(x, y, z) = c$ is a level surface then $\nabla\phi|_{P(x_0, y_0, z_0)}$ gives the gradient of $\phi(x, y, z)$ at Point ' P '.

$$|\nabla\phi|_P| = \sqrt{\left(\frac{\partial\phi}{\partial x}\Big|_P\right)^2 + \left(\frac{\partial\phi}{\partial y}\Big|_P\right)^2 + \left(\frac{\partial\phi}{\partial z}\Big|_P\right)^2}.$$

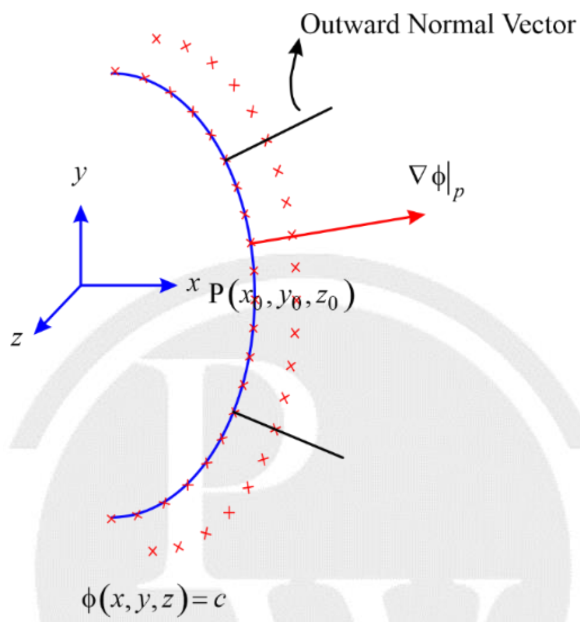


Fig. 3.3

→ $\nabla\phi|_p$ gives the change of $\phi(x, y, z)$ in the direction Normal to the surface $\phi(x, y, z) = c$ at P(x, y, z).

Note:

- Angle between two surfaces:
  Let $f(x,y,z)=C_1$ and $g(x,y,z)=C_2$ and be two surfaces and 'θ' be the angle between them, then $\cos\theta = \frac{\nabla f . \nabla g}{|\nabla f||\nabla g|}$.
- The directional derivative of $f(x,y,z)$ is maximum in the direction of ∇f.
- Maximum value of directional derivative $f(x,y,z) = |\nabla f|$.
- Gradient of a scalar field is normal to the surface given by scalar field.
- If any vector $\vec{A}$= Gradient of scalar field $f(x,y,z)$, then $f(x,y,z)$ is called potential of $\vec{A}$. Vector $\vec{A}$ is called conservative field.

3.7 Directional Derivative:

If $\phi(x, y, z) = c$ is a level surface, then the derivative of $\phi(x, y, z)$ at Point ' P ' in the direction of $\vec{a}$ is called Directional Derivative of $\phi(x, y, z)$ in the direction of $\vec{a}$.
It is given by

Direction Derivative = $\nabla\phi|_p \cdot \hat{a}$

$$= \nabla\phi\Big|_p \cdot \frac{\vec{a}}{|\vec{a}|} = \left|\nabla\phi|_p\right| \cdot |\vec{a}| \cdot \frac{\cos\theta}{|\vec{a}|} = \left|\nabla\phi|_p \cdot \cos\theta\right|$$

The directional Derivative of $\phi(x, y, z)$ at P in the direction of $\vec{a}$ is

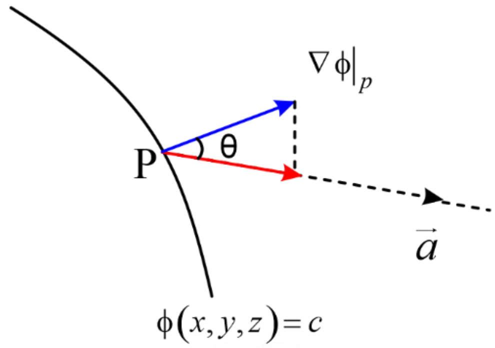


Fig. 3.4

Directional derivative
$DD = |\nabla\phi|_p| \cdot \cos\theta$ where ' θ ' is the angle between $\nabla\phi|_p$ and $\vec{a}$.

For Directional derivative to be maximum $\cos\theta = 1 \Rightarrow \theta = 0°$
$\Rightarrow$ The change of $\phi(x, y, z)$ at Point ' P ' is Maximum in the direction of Normal to $\phi(x, y, z)$

Maximum Change of $\phi(x, y, z)$ at $'p' = |\nabla\phi|_p|$

3.8. Del operated-on Vector Point functions

If Del is a differential operator and $\vec{F}(x, y, z)$ is a vector Point function then the Del operator is operated on $\vec{F}(x, y, z)$ in two Ways.

(i) $\nabla \cdot \vec{F} \rightarrow$ Divergence

(ii) $\nabla \times \vec{F} \rightarrow$ Curl

(i) Divergence of a Vector Point function:

If $\vec{F}(x, y, z)$ is a Vector Point function, then the divergence of $\vec{F}(x, y, z)$ is denoted by div $\vec{F}$(or)$\nabla \cdot \vec{F}$ and for any $\vec{F}(x, y, z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ the divergence is given by

$$\text{div} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

If div $\cdot \vec{F} = 0$, then $\vec{F}(x, y, z)$ is called a Solenoidal (or) Incompressible flow Vector.

(ii) Curl of a Vector Point Function:

If $\vec{F}(x, y, z)$ is a Vector Point function, then the curl of $\vec{F}(x, y, z)$ is denoted by $Curl\,\vec{F}(or)\nabla \times \vec{F}$ and for any $\vec{F}(x, y, z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$, the curl of $\vec{F}(x, y, z)$is given by

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

If curl $\vec{F} = \vec{0}$; then $\vec{F}$ is called Irrotational Flow Vector.

Note : Geometric interpretation of divergence:

Divergence (div) is the amount of flux per unit volume entering or leaving.

(i) $\nabla.\vec{F} > 0$ at point P, then P is source.

When the divergence is positive it gives the rate at which fluid is flowing away from the point per unit volume.

(ii) $\nabla.\vec{F} < 0$ at point P, then P is sink.

When the divergence is negative it gives the rate at which fluid is flowing towards the point per unit volume.

(iii) $\nabla.\vec{F} = 0$ at point P, then P is neither source or sink.

(iii) Properties of div, Curl & Grad:

If $\phi(x, y, z)$ and $\vec{F}(x, y, z)$ are a scalar point function and a vector point function respectively, then

(a) curl (grad ϕ)=$\vec{0}$

(b) div(curl $\vec{F}$) = 0

(c) div(grad ϕ) $= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$

Note : Curl measures the rotatory effects of the vector field.

(i) If curl F > 0 rotation is anticlockwise direction.

(ii) If curl F < 0 rotation is clockwise direction.

(iii) If there is no rotation of fluid anywhere then, $\nabla \times \vec{F} = \vec{0}$. Such a vector field is said to be irrotational or conservative.

(iv) Angular velocity $\omega = \frac{1}{2}\text{curl}\,\vec{v}$

(iv) Angle between two Intersecting Surfaces:

If $\phi_1(x, y, z) = c_1$ & $\phi_2(x, y, z) = c_2$ are two surfaces intersecting at 'P', then the angle of Intersection 'θ' is given by

$cos\,\theta = \frac{\nabla\phi_1|_p \cdot \nabla\phi_2|_p}{|\nabla\phi_1|_p| \cdot |\nabla\phi_2|_p|}$

3.9 Vector Integration:

3.9.1 Line Integrals

If $\vec{F}(x, y, z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point along the path C, then the Integral of $\vec{F}(x, y, z)$ from Point 'A' to point 'B' along a path is given by $\int_{A,C}^{B} \vec{F} \cdot d\vec{r}$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\int_{A,C}^{B} \vec{F} \cdot d\vec{r} = \int_{A,C}^{B} \left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) \cdot \left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right)$$

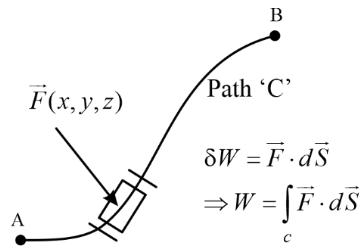


Fig. 3.5. Line Integral

If $\vec{F}(x, y, z)$ is an Irrotational Vector Point Function, (i.e., Curl $\vec{F} = \vec{0}$), then $\int_A^B \vec{F} \cdot d\vec{r}$ is independent of the path between points A and B.

If $\vec{F}(x, y, z)$ is an Irrotational Vector Point Function, then

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \nabla\phi \cdot d\vec{r} \text{ where} \vec{F} = \nabla\phi$$

$$= \phi|_B - \phi|_A$$

3.9.2 Surface Integral

If $\vec{F}(x, y, z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point on a surface 'S', then the surface integral of $\vec{F}(x, y, z)$ on the surface 'S' is given by $\int_S \vec{F} \cdot d\vec{s}$

Where $d\vec{s} = ds.\hat{n}$ and $\hat{n}$ is the outward unit normal vector to the surface at ds and

$ds = \frac{dx.dy}{|\hat{n}.\hat{k}|} = \frac{dy.dz}{|\hat{n}.\hat{i}|} = \frac{dx.dz}{|\hat{n}.\hat{j}|}$:

3.9.3 Volume Integral

If $\vec{F}(x, y, z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point over a volume V, then the volume integral of $\vec{F}(x, y, z)$ on the volume 'V' is given by $\int_V \vec{F} \cdot dv$.

3.9.4 Greens Theorem: (Connects closed line Integral to surface Integral)

If $\vec{F}(x, y) = F_x\vec{i} + F_y\vec{j}$ and if the first order derivatives of F_x& $F_y$are continuous at every point with in a region 'R' bounded by a closed path 'C', then

$$\oint_C \vec{F} d\vec{r} = \oint_C F_x dx + F_y dy = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy$$

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

3.9.5 Gauss – Divergence Theorem: (Connects closed surface integral to a Volume Integral)

If 'S' is a closed surface enclosing a volume 'V' and $\vec{F}$ is continuous and differentiable at every point on the closed surface 'S', then the closed surface integral $\oint_S \vec{F}.d\vec{s} = \iiint_V div\, \vec{F}.dV$

3.9.6 Stokes Theorem: (Connect Closed line integral to surface Integral)

If $\vec{F}$ is continuous and differentiable at every point with in a region 'R' (on a surface S) bounded by a closed path 'C', then $\oint_C \vec{F} d\vec{r} = \iint_R curl\, \vec{F} . d\vec{S}$

3.10 Properties of Position Vector

Position vector of a point (x, y, z) in space $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Length : $r = \sqrt{x^2 + y^2 + z^2}$

Results :

(a) (i) $\frac{\partial r}{\partial x} = \frac{x}{r}$ (ii) $\frac{\partial r}{\partial y} = \frac{y}{r}$ (iii) $\frac{\partial r}{\partial z} = \frac{z}{r}$

(b) $\text{grad}(r) = \nabla r = \frac{\vec{r}}{r}$

(c) $\text{div}(\vec{r}) = \nabla.\vec{r} = 3$

(d) $\text{curl}(\vec{r}) = \nabla \times \vec{r} = \vec{0}$

(e) $\nabla f(r) = f'(r)\nabla r$

(f) $|\vec{F}| = r^n$ if and only if $\vec{F} = r^{n-1}\vec{r}$

(g) $\nabla.\left(r^n\vec{r}\right) = 0$ if $n = -3$

❑❑❑

4 LINEAR ALGEBRA

4.1. Matrix

An array of elements in horizontal lines (Rows) and Vertical Lines (Columns) is called a Matrix.

Example: $A = \begin{bmatrix} i & n & d & i & a \\ j & a & p & a & n \end{bmatrix}$

4.1.1 Size of Matrix

If a matrix has 'm' rows and 'n' columns, then we say that the size of the matrix is m × n (read as m by n)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots\ldots\ldots a_{2n} \\ \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot\cdot \\ a_{m1} & a_{m2} & a_{m3}\ldots\ldots\ldots a_{mn} \end{bmatrix};\ A = [a_{ij}]_{m\times n} \text{ such that } 1 \le i \le m, 1 \le j \le n \text{ and } a_{ij} = f(i,j)$$

4.1.2 Addition of Matrices:

(i) Two matrices $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{p\times q}$ can be added only if m = p & n = q.

(ii) Matrix Addition is commutative (A + B = B + A)

(iii) Matrix Addition is Associative. A + (B + C) = (A + B) + C

(iv) Existence of additive identity : If O be m × n matrix each of whose elements are zero. Then, A + O = A = O + A for every m × n matrix A.

(v) Existence of additive inverse : Let $A=[a_{ij}]_{m\times n}$ then the negative of matrix A is defined as matrix $[-a_{ij}]_{m\times n}$ and is denoted by –A.

⇒ Matrix –A is additive inverse of A. Because (–A) + A = O = A + (–A). Here O is null matrix of order m × n.

(vi) Cancellation laws holds good in case of addition of matrices, which is X = –A.

⇒ A + X = B + X ⇒ A = B

⇒ X + A = X + B ⇒ A = B

(vii) The equation A + X = 0 has a unique solution in the set of all m × n matrices.

4.1.3 Multiplication of Matrices:

The multiplication of two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ $(\Rightarrow AB_{m\times q})$ is feasible only if n = P.

$A_{m\times n} \cdot B_{p\times q} = C$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times2} \quad A_{3\times3} \times B_{3\times2}$$

$$\Rightarrow \begin{bmatrix} a_{11}.b_{11} + a_{12}\cdot b_{21} + a_{13}.b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}.b_{11} + a_{32}\cdot b_{21} + a_{33}\cdot b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}_{3\times2}$$

4.1.4 Properties of Multiplication of Matrices:

(i) Matrix Multiplication Need not be commutative.

(ii) Matrix Multiplication is Associative (A(BC)) = ((AB)C)

(iii) Matrix Multiplication is distributive A(B + C) = (AB + AC)

(iv) The product of two Matrices $A_{m\times n}, B_{n\times q}$ (i.e. $AB_{m\times q}$) can be a zero matrix even if $A \neq O \& B \neq O$.

Example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- For the multiplication of two matrices $A_{m\times n}$ & $B_{n\times q}$
  (i) The No. of Multiplications required = m n q
  (ii) The number of Additions required = m (n –1) q

4.2 Types of Matrices:

(1) Upper triangular Matrix: A matrix $A = [a_{ij}]; 1 \leq i, j \leq n$ is said to be an upper triangular matrix if

$$a_{ij} = 0 \ \forall \ i > j$$

(2) Lower Triangular Matrix: A matrix $A = [a_{ij}]_{n\times n}; 1 \leq i, j \leq n$ is said to be a lower Triangular Matrix

if $a_{ij} = 0 \ \forall \ i < j$

(3) Diagonal Matrix: A matrix $A = [a_{ij}], \forall \ 1 \leq i, j \leq n$ is said to be a diagonal matrix if $a_{ij} = O \ \forall \ i \neq j$

Example: $A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$. The diagonal Matrix is also denoted as $A = diag\ [d_1, d_2, d_3]$

(4) Scalar Matrix: A Matrix 'A' = $[a_{ij}]$; $1 \leq i, j \leq n$ is said to be a scalar Matrix if $a_{ij} = \begin{cases} K; i = j \\ 0; 1 \neq j \end{cases}$

If K = 1, then A → Identity Matrix,

If K = 0, then A → Null Matrix.

(5) Idempotent Matrix:

A Matrix '$A_{n\times n}$' is said to be an idempotent matrix if $A^2 = A$.

Example: $A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

$\Rightarrow A \cdot A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A$

(6) Nilpotent Matrix: A matrix A is said to be nilpotent of class x or index if $A^x = 0$ and $A^{x-1} \neq 0$ i.e. x is the smallest index which makes $A^x = 0$.

Example: The matrix A = $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since A ≠ 0 and $A^2 \neq 0$, but $A^3 = 0$.

(7) Orthogonal Matrix: A matrix A is said to be orthogonal if $A.A^T = I$

Example: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(8) Involutory Matrix: A matrix A is said to be involutory if $A^2 = I$

Example: $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

4.3 Transpose of a Matrix:

For a given matrix = $[a_{ij}]; 1 \le i \le m, 1 \le j \le n$, we can say that 'B' where $B = [b_{ij}]$, $i \le i \le n \quad i \le j \le m$ is the transpose of the Matrix 'A' if $a_{ij} = b_{ji}$

4.3.1 Properties of Transpose of a Matrix:

(i) $(A^T)^T = A$

(ii) $(AB)^T = B^T \cdot A^T$

(iii) $(KA)^T = KA^T$ where 'K' is a scalar.

4.4 Conjugate of a matrix

The matrix obtained by replacing each element of matrix by its complex conjugate.

4.4.1 Properties of conjugate matrix

(a) $(\overline{\overline{A}}) = A$ (b) $(\overline{A+B}) = \overline{A} + \overline{B}$

(c) $(\overline{KA}) = \overline{K}\,\overline{A}$ (d) $(\overline{AB}) = \overline{A}\,\overline{B}$

(e) $\overline{A} = A$ if A is real matrix

$\overline{A} = -A$ if A is purely imaginary matrix

4.5 Transposed Conjugate of a Matrix

The transpose of conjugate of a matrix is called transposed conjugate. It is represented by A^θ.

(a) $(A^\theta)^\theta = A$

(b) $(A+B)^\theta = A^\theta + B^\theta$

(c) $(KA)^\theta = \bar{K}A^\theta$ (K : Complex number)

(d) $(AB)^\theta = B^\theta A^\theta$

4.6 Trace of a Matrix

Trace is simply sum of all diagonal elements of a matrix.

4.6.1 Properties of Trace of a matrix

Let A and B be two square matrices of order η and λ is scalar then

1. $Tr\,(\lambda A) = \lambda Tr\,(A)$
2. $Tr(A+B) = Tr(A) + Tr(B)$
3. $Tr(AB) = Tr(BA)$ [If both AB and BA are defined]

4.7. Type of Real Matrix

(a) Symmetric matrix : $(A)^T = A$

(b) Skew symmetric matrix : $(A^T) = -A$

(c) Orthogonal matrix : $(A^T = A^{-1}, AA^T = I)$

Note : (a) If A and B are symmetric, then (A + B) and (A – B) are also symmetric.

(b) For any matrix AA^T is always symmetric.

(c) For any matrix, $\left(\frac{A+A^T}{2}\right)$ is symmetric and $\left(\frac{A-A^T}{2}\right)$ is skew symmetric.

(d) For orthogonal matrices, $|A| = \pm 1$

(e) We can write any matrix A as a sum of symmetric and skew symmetric matrix $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$

4.8. Type of complex matrix

(a) Hermitian matrix : $(A^\theta = A)$

(b) Skew-Hermitian matrix: $A^\theta = -A$

(c) Unitary matrix : $(A^\theta = A^{-1}, AA^\theta = I)$

Note : (a) $\frac{A+A^\theta}{2}$ is Hermitian and $\frac{A-A^\theta}{2}$ is skew Hermitian matrix.

(b) We can write any matrix as a sum of Hermitian and skew Hermitian matrix $A = \frac{A+A^\theta}{2} + \frac{A-A^\theta}{2}$

4.9. Determinant

The summation of the product of elements of a row(or) column of a matrix with their corresponding Co-factors.

$$A \cdot adj(A) = |A| \cdot I$$

Determinant can be calculated only if matrix is a square matrix.

Suppose, we need to calculate a 3 × 3 determinant,

$$\Delta = \sum_{j=1}^{3} a_{1j} cof\,(a_{1j}) = \sum_{j=1}^{3} a_{2j} cof\,(a_{2j}) = \sum_{j=1}^{3} a_{3j} cof\,(a_{3j})$$

We can calculate determinant along any row or column of the matrix.

4.9.1 Properties of Determinants

(i) If 'A' is a Square Matrix of size ' $n \times n$ ' and 'k' is a Scalar then

$|K \cdot A_{n\times n}| = K^n \cdot |A_{n\times n}|$

(ii) $|adj(A)| = |A|^{(n-1)}$

(iii) $|adj(adj(A))| = (|A|)^{(n-1)^2}$

(iv) $|AB| = |A| \cdot |B|$

(v) $|(AB)^T| = |B^T| \cdot |A^T|$

(vi) If two rows (or) two columns of a determinant are interchanged, then the determinant changes its sign.

(vii) The determinant of an upper triangular Matrix/a lower triangular Matrix/a diagonal Matrix is the product of the principal diagonal elements of the Matrix.

(viii) The determinant of Every Skew-Symmetric Matrix of odd order $(A_{n\times n})('n' \, is \; odd)$ is zero.

(ix) The determinant of an orthogonal Matrix $A_{n\times n}$ is ± 1

(x) The determinant of an Idempotent Matrix is either 0 (or) 1.

(xi) The determinant of an Involuntary Matrix is ± 1

(xii) The determinant of a Nilpotent Matrix is always zero.

(xiii) If the product of two Non-zero Matrices $A_{n\times n} \neq 0; B_{n\times n} \neq 0$ is a zero Matrix $((AB)_{n\times n} = 0)$, then both $|A| = 0$ & $|B| = 0$.

(xiv) If two rows (or) two columns of a Matrix are either equal or Proportional, then the determinant of the Matrix is equal to zero.

(xv) The number of terms in the general expansion of an 'n × n' determinant is $\underline{n!}$

(xvi) Value of the determinant is invariant under row and column interchange i.e., $|A^T| = |A|$

(xvii) If any row or column is completely zero, then |A| = 0.

(xviii) If any single row or column of the matrix is multiplied by k then the determinant the of new matrix = K|A|

(xix) In a determinant the sum of the product of the element of any row or column with its cofactor gives a determinant of the matrix.

(xx) In determinant the sum of the product of the element of any row or column with a cofactor of another row or column will give zero.

(xxi) |AB| = |A| × |B|

(xxii) Elementary operations don't effect the determinant that is $A \xrightarrow{R_i = R_i + KR_j} B$ then |A| = |B|

$A \xrightarrow{C_i = C_i + KC_j} B$ then |A| = |B|

4.10. Minors, Cofactor and Adjoint of a Matrix

Minor of an element is equal to the determinant of the remaining elements of the matrix, after excluding the row and column containing the particular element. The cofactor of an element can be calculated from the minor of the element. The cofactor of an element is equal to the product of the minor of the element, and -1 to the power of position values of row and column of the element.

$$\text{Cofactor of an Element } = (-1)^{i+j} \times \text{ Minor of an Element}$$

Here i and j are the positional values of the row and column of the element.

Example :

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of element a_{21}: $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Co-factor of an element, $a_{ij} = (-1)^{i+j} M_{ij}$

- To design co-factor matrix, we replace each element by its co-factor.
- Adjoint of a matrix = transpose of cofactor matrix
- $A^{-1} = \dfrac{Adj(A)}{|A|}$

4.11 Inverse of a matrix

Inverse of a matrix only exists for square matrices.

$$\left(A^{-1}\right) = \frac{Adj(A)}{|A|} \text{ and } |A| \neq 0$$

Properties:

(a) $AA^{-1} = A^{-1}A = I$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(d) $(A^T)^{-1} = (A^{-1})^T$

(e) The inverse of 2 × 2 matrix should be remembered,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(i) Interchange the diagonal elements and put negative sign on the rest.

(ii) Divide by determinant.

4.12. Rank of a Matrix

- The rank of the matrix refers to the number of linearly independent rows or columns in the matrix. ρ(A) is used to denote the rank of matrix A.
- A matrix is said to be of rank zero when all of its elements become zero.
- The rank of the matrix is the dimension of the vector space obtained by its columns.
- The rank of a matrix cannot exceed more than the number of its rows or columns. The rank of the null matrix is zero.
- The nullity of a matrix is defined as the number of vectors present in the null space of a given matrix. In other words, it can be defined as the dimension of the null space of matrix A called the nullity of A. Rank + Nullity is the number of all columns in matrix A.

  A real Number 'r' is said to be the rank of a matrix '$A_{m\times n}$' if

  (1) There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.

  (2) If the matrix A contains any square sub-matrix of order (r + 1) and above, then the determinant of such a matrix should be zero.

  It is mathematically denoted by $\rho(A) = r$

4.12.1 Properties of Rank of a Matrix:

(i) $\rho(A_{m\times n}) \leq (m, n)$

(ii) $\rho(AB) \leq min\{\rho(A), \rho(B)\}$

(iii) Rank of transpose of matrix is equal to rank of matrix

(iv) Elementary operations do-not affect the rank the matrix

(v) $\rho(A+B) \leq \quad \{\rho(A)+\rho(B)\}$

4.12.2 Row Echelon Form

A Matrix $A_{m\times n}$ is said to be in row-echelon form if

(i) Number of zeroes before the 1st Non-zero element in any row is less than the number of such zeroes in its succeeding row.

(ii) Zero rows (if any) should lie at the bottom of the Matrix.

$\rho(A_{m\times n})$ = Number of non-zero rows in the Row-Echelon form of A.

4.13 System of Equations:

The given system of equations

$$a_{11}x_1 + a_{12}x_{12} + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written in Matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \boxed{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Ax = B

Coefficient Matrix — Variable Matrix — Constants Matrix

The system Ax = B is said to be a homogeneous system if B = 0.

The system of Ax = B is said to be a non-homogeneous system if $B \neq 0$.

4.13.1 Consistency of a non-homogeneous system of Equations:

For the above system of non – homogeneous equations, Ax = B; Augmented Matrix = [A/B] = $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

(i) If $\rho(A) = \rho(A/B) =$ Number of unknowns, then the system Ax = B has a unique solution.

(ii) If $\rho(A) = \rho(A/B) <$ Number of unknowns, then the system has infinitely many solutions.

(iii) If $\rho(A) \neq \rho(A/B)$, then the system has no solution.

Number of linearly independent solutions for a system of 'n' equations given by Ax = B is $n - \rho(A)$

4.13.2 Consistency of Homogeneous System of Equations:

$a_{11}x + a_{12}y + a_{13}z = 0$

$a_{21}x + a_{22}y + a_{23}z = 0$

$a_{31}x + a_{32}y + a_{33}z = 0$

$$\text{Ax} = 0 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; [A/B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix}_{3\times 4}$$

If $\rho(A) = \rho(A/B) = n$ $(i.e\ |A| \neq 0)$; the system has a unique solution.

(Trivial solution; x = 0, y = 0, z = 0)

If $\rho(A) = \rho(A/B) < n(|A| = 0)$; the system has infinitely many solutions (Non-trivial solution exists for the system).

4.14 Linear Combination of Vectors:

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' rows vectors, then the combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n$ is called a linear combination of $x_1, x_2, \ldots., x_n (k_1, k_2, k_3, \ldots.. k_n$ are scalars)

(1) The linear combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n$ is said to be linearly dependent if $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n = 0$ when $k_1, k_2, k_3, \ldots.., k_n$ (NOT All zeroes).

If $x_1 = [a_1 \quad b_1 \quad c_1]; x_2[a_2 \quad b_2 \quad c_2]; x_3 = [a_3 \quad b_3 \quad c_3]$, then the vectors x_1, x_2, x_3 are said to be linearly dependent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

(2) The combination $k_1x_1 + k_2x_2 + \ldots\ldots + k_nx_n$ is said to be linearly independent if $k_1x_1 + k_2x_2 + \ldots\ldots\ldots + k_nx_n = 0$ when $k_1 = k_2 = k_3 = \ldots\ldots k_n = 0$

4.14.1 Eigen Values and Eigen Vectors

For any square Matrix $A_{n\times n}$, the equation $|A - \lambda I| = 0$ where 'λ' is a scalar is called the characteristic equation.

The roots of the characteristic equation of a Matrix are called Eigen Values.

4.14.2 Properties of Eigen Values

(i) If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are 'n' Eigen Values of $A_{n\times n}$, then

(a) Sum of Eigen Values of 'A' = $\lambda_1 + \lambda_2 + \lambda_3 + \ldots. + \lambda_n = \sum_{i=1}^{n} \lambda_i = trace(A)$ = Sum of Principal diagonal elements

(b) Product of all the Eigen Values of 'A' = $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots\ldots \lambda_n = \prod_{i=1}^{n} \lambda_i = |A|$

(c) Eigen Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots\ldots \lambda_n^m$

(d) Eigen Values of adj(A) are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots\ldots, \frac{|A|}{\lambda_n}$

(e) Eigen Values of A & A^T are the same.

(f) Eigen Values of $k_1A + k_2I$ (Where k_1 and k_2 are scalar) are

$$k_1\lambda_1 + k_2, k_1\lambda_2 + k_2, k_1\lambda_3 + k_2, k_1\lambda_4 + k_2, \ldots\ldots\ldots k_1\lambda_n + k_2$$

(ii) '0' is always an Eigen Value of an odd-order Skew-Symmetric Matrix.

(iii) Eigen Values of a Real Symmetric Matrix are always real.

(iv) Eigen Values of the Skew-Symmetric Matrix are either zero (or) purely Imaginary.

(v) The Eigen values of an Orthogonal Matrix are of unit modulus.

(vi) If the sum of all the elements in a row (or Column) is constant (= k) for all the rows (or columns) in the matrix respectively, then 'k' is an Eigen Value of the Matrix.

Example: If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and if $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3 = k$,

then 'k' is an Eigen Value of 'A'.

(vii) The Eigen Values of an upper triangular Matrix, a lower triangular Matrix, a diagonal Matrix are the Principal diagonal elements of the Matrix.

4.15. Eigen Vector

A non-zero column vector $X_{n\times1}$ is said to be an Eigen Vector of $A_{n\times n}$ corresponding to the Eigen Value 'λ', if $AX = \lambda X (X \neq 0)$.

4.15.1 Properties of Eigen Vectors:

(i) Eigen Vectors of A & A^T are not the same.

(ii) Eigen Vectors of A & A^M are same.

(iii) The Eigen Vectors of a Real Symmetric Matrix are always orthogonal.

(iv) The number of linearly independent Eigen Vectors of '$A_{n\times n}$' is equal to the number of distinct Eigen Values of '$A_{n\times n}$'.

4.15.2 Cayley Hamilton Theorem:

Every Matrix satisfies its characteristic equation.

This means that, if $c_0\lambda^n + c_1\lambda^{-1} + ... + c_{n-1}\lambda + c_n = 0$ is the characteristic equation of a square matrix *A* of order *n*, then

$$c_0A^n + c_1A^{n-1} + ... + c_{n-1}\ A + c_nI = 0 \qquad ...(i)$$

Note: When λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by c_nI to get the result of the Cayley-Hamilton theorem, where *I* is the unit matrix of order *n*.

Also, 0 in the R.H.S. of (i) is a null matrix of order *n*.

4.16. Subspace (Basis of Dimensions)

4.16.1 Vector

An ordered n-tuple of numbers is called an n-vector.

4.16.2 Linearly Independent and Dependent Vector

Let X_1 and X_2 be the non-zero vectors:

- $\{x_1, x_2, ..., x_k\}$ are linearly independent if $r_1x_1 + r_2x_2 + ... + r_k x_k = 0$ only for $r_1 = r_2 = ... + r_k = 0$.
- The vectors $x_1, r_2, ..., x_k$ = are linearly dependent if they are not linearly independent; that is, if there exist scalars $r_1, r_2, ... , r_k$ which are not all zero such that

  $r_1x_1 + r_2x_2 + ... + r_k x_k = 0$

Note: Let $X_1, X_2.......X_n$ be 'n' vector of matrix *A*.

- If rank (*A*) = number of vectors then vector $X_1, X_2.....X_n$ are linearly independent.
- If rank (*A*) ≠ number of vectors then vector $X_1, X_2 ..... X_n$ are linearly dependent.

4.16.3 Vector Space R^n:

If n positive integer, then an ordered n-tuple is a sequence of n real numbers ($\alpha_1, \alpha_2, \ldots \alpha_n$). the set of all ordered *n*-tuples is called n-space and is denoted by $\mathbb{R}^n$.

4.16.4 Subspaces of an N-vector space V_n

A non-empty set S, of vectors of $V_n(F)$,is called a subspace of $V_n(F)$, if

- ξ_1, ξ_2 are any two members of S, then $\xi_1 + \xi_2$ is also a member of S; and
- ξ is a member of S, and k is a scalar then $k\xi$ is also a member of S.

  Briefly, we may say that a set S of vectors $V_n(F)$ is a subspace of $V_n(F)$ it closed w.r.t. the compositions of "addition" and "multiplication with scalars".

  Every subspace of V_n contains the zero vector; being the product of any vector with the scalar zero.

4.16.5 Construction of Subspaces

- **A subspace Spanned by a Set of Vectors:** A subspace that arises as a set of all linear combinations of any given set of vectors is said to be spanned by the given set of vectors.
- **Basis of a subspace:** A set of vectors is said to be a basis of a subspace, if
  - ➢ The subspace is spanned by the set, and
  - ➢ The set is linearly independent.

Note: If we have N vectors and they are independent then they span N-dimension space. But if they are dependent then they span only a subspace of N-dimension space.

4.16.6 Orthogonality of Vectors

- Two vectors are orthogonal if each is non-zero and $X_1^T X_2 = 0$
- If n vectors $X_1, X_2 \ldots X_n$ each of n dimensions is orthogonal then they are surely linearly independent and form the basis for n-dimension space.
- The set of the vector is orthonormal if they are orthogonal and have unit magnitude.

4.17. Similar Matrices

- Two matrix A and B are similar if there exist a non singular matrix P such that $B = P^{-1}AP$
- Similar matrix has same eigen valves
- If A is similar to B then B is also similar to A
- If A is similar to B and B is similar to C then A is similar to C.

4.18. Diagonalization of a matrix:

Finding a matrix D which is a diagonal matrix and which is similar to A is called diagonalization i.e., we wish to find a non-singular matrix M such that $A = M^{-1}DM$ where D is a diagonal matrix.

Condition for a Matrix to be Diagonalizable

1. A necessary and sufficient condition for a matrix $A_{n \times n}$ to be diagonalizable is that the matrix must have n linearly independent eigen vectors.

2. A sufficient (but not necessary) condition for a matrix $A_{n \times n}$ to be diagonalizable is that the matrix must have n linearly independent eigen values.

   This is because if a matrix has n linearly independent eigen values then it surely has n linearly independent eigen vectors (although the converse of this is not true).

❑❑❑

5 PROBABILITY AND STATISTICS

5.1. Random Experiment:

The experiment in which the outcome is uncertain is called a Random Experiment (RE).

Example: Flipping a coin, rolling a pair of dice, Picking a ball from a bag.

5.1.1 Sample Space

The set contains all the possible outcomes of a random experiment. It is denoted by 'S'.

If RE is flipping a coin, S = {Head, Tail}

If RE is rolling a dice, S = {1,2,3,4,5,6}

5.2. Event

Any subset of sample space 'S' is called an Event.

Example: If RE is flipping a coin, then the occurring of a Head is an Event.

If RE is rolling a dice, then getting an odd number is an Event.

5.2.1 Probability of an Event:

If 'A' is any event with in the sample space 'S' of a Random experiment, then the probability of event 'A' is given by

$$P(A) = \frac{\text{No. of outcomes favouring event 'A' to happen}}{\text{Total number of elements in 'S'}} = \frac{n(A)}{n(S)}$$

Probability of getting an Even Number when a dice is rolled.

P(Even Number) = $\frac{3}{6} = 0.5$

S = {1,2,3,4,5,6},

A = {2,4,6}

Note: Probability can also be expressed as odds if favour and odds against an event:

- **Odds is favour of an event:**

  Odds in **favour** of an event = Number of successes : Number of failures = m: (n – m).

- **Odds against an event:**

  Odds against an event = Number of failures : Number of successes = (n – m) : m.

5.2.2 Axioms Probability:

(i) If 'A' is any event with in the sample space 'S' of a RE, then $0 \le P(A) \le 1$

$$\frac{0}{n(S)} \le \boxed{\frac{n(A)}{n(S)}} \le \frac{n(S)}{n(S)}$$

$$\boxed{0 \le P(A) \le 1}$$

(ii) P(S) = 1

When a RE is conducted the experiment yields a possible outcome.

5.2.3 Types of Events:

(i) Mutually Exclusive Events:

If A, B are two events within a sample space 'S', then A & B are said to be mutually exclusive if $A \cap B = \phi$.

Example: If 'A' is the event of getting a prime number when a dice is rolled and 'B' is the event of getting a composite number when a dice is rolled then

$S = \{1,2,3,4,5,6\}$, $A = \{2,3,5\}$, $B = \{4,6\} \Rightarrow A \cap B = \phi \Rightarrow P(A \cap B) = 0$

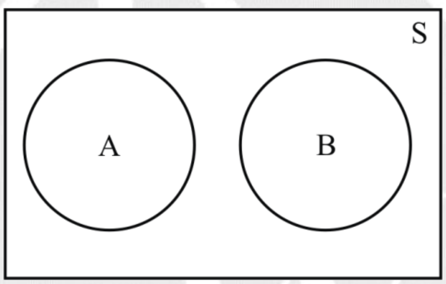


Fig. 5.1. Mutually exclusive event

(ii) Mutually Exhaustive Events:

If 'A', and 'B' are two events within a sample space 'S', then 'A' & 'B' are said to be mutually exhaustive if $A \cup B = S$

Example: If 'A' is the event of getting an odd number when a dice is rolled and 'B' is the event of getting an Even Number, then

$A \cup B = S$

$S = \{1,2,3,4,5,6\}$

$A = \{1,3,5\}$, $B = \{2,4,6\}$

$\Rightarrow \quad A \cup B = S$

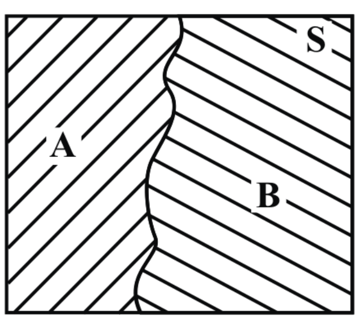


Fig. 5.2. Mutually exhaustive event

(iii) Independent Events:

Two events 'A' & 'B' within the sample space 'S' (or) within two different sample spaces 'S_1' & 'S_2' are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$.

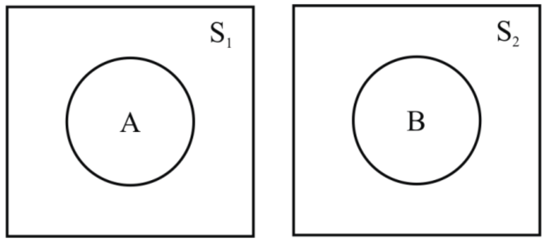


Fig. 5.3. Independent Event

(iv) Impossible Event (ϕ):

The event with zero probability is called an Impossible Event $P(\phi) = 0$.

5.3. Addition Theorem of Probability

If A, and B are two events with a sample space 'S' of a Random Experiment, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

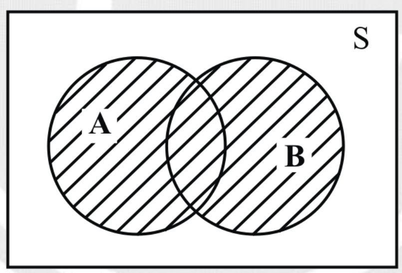


Fig. 5.4. Addition theorem

$\Rightarrow$ $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When A, and B are mutually exclusive events, $A \cap B = \phi$.

$\Rightarrow$ $$P(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B)$$

- If $E_1, E_2, E_3, \ldots\ldots E_n$ are mutually exclusive events ($E_i \cap E_j = \phi$), then $P(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$

$= P(E_1) + P(E_2) + P(E_3) + \ldots\ldots p(E_n)$

5.3.1. De Morgan's Law

- $(A \cup B)^C = A^C \cap B^C$
- $(A \cap B)^C = A^C \cup B^C$

5.3.2. Union and Intersection properties

For any two events A and B:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) $P(A^c \cap B^c) = 1 - P(A \cup B)$

For any three events A, B and C:

(a) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(b) $P(A^c \cap B^c \cap C^c) = 1 - P(A \cup B \cup C)$

5.3.3 Conditional Probability:

The probability of the happening of event 'A' when it is known that event 'B' has already occurred is given by P(A/B)

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}$$

5.3.4 Joint Probability:

- $f(x, y)$ is the joint probability of two RV'S x, y.
- If the two RV are Independent then
  $f(x, y) = f(x) \cdot f(y)$
- $P(a \le x \le b, c \le y \le d) = \int_a^b \int_c^d f(x, y) dy dx$
- $f(x) = \int_{-\infty}^{\infty} f(x, y) dy$
- $f(y) = \int_{-\infty}^{\infty} f(x, y) dx$

5.3.5 Multiplication Theorem of Probability:

If A, and B are two events within a sample space 'S', then P(A/B) · P(B) = P(B/A) · P(A)

P(A/B) = $\frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A/B) \cdot P(B) \rightarrow (1)$

P(B/A) = $\frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B/A) \cdot P(A) \rightarrow (2)$

From (1) & (2)

$P(A/B) \cdot P(B) = P(B/A) \cdot P(A)$

5.3.6 Total Theorem of Probability:

If E_1, E_2, E_3,……E_n are 'n' mutually exclusive ($E_i \cap E_j = \phi; \forall i \neq j$) and collectively exhaustive event ($E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n = S$) and 'A' is any event with in the sample space 'S', then

$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \ldots\ldots + P(E_n) \cdot P(A/E_n)$

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i) \bullet P(A/E_i)}$$

5.3.7. Baye's Theorem:

If $E_1, E_2, E_3, \ldots\ldots E_n$ are mutually exclusive $(E_i \cap E_j = \phi \forall i \neq j)$ and collectively exhaustive event $(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n = S)$ and 'A' is any event with in the sample space 'S', then

$$P(E_i/A) = \frac{P(E_i)\cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\cdot P(A/E_i)}$$

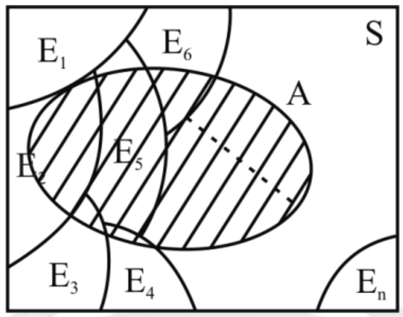


Fig. 5.5. Baye's theorem

5.3.8. Use of permutation and combination

What is combination?

A combination of 'n' objects taken 'r' at a time (r-combination of 'n' objects is an unordered selection of 'r' of the objects).

Number of ways of combining of 'r' object out of 'n' objects without repetition

$$^{n}C_r = \frac{n!}{(n-r)!r!}$$

What is permutation?

A combination of 'n' objects taken 'r' at a time (r-combination of 'n' objects is an ordered selection of 'r' of the objects).

Number of ways of selection of r object out of n objects without repetition

$$^{n}P_r = \frac{n!}{(n-r)!}$$

Result :

(i) $^{n}C_r = {}^{n}C_{n-r}$

(ii) $^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots + {}^{n}C_n = 2^n$

(iii) $^{n}C_0 + {}^{n}C_2 + {}^{n}C_4 + \ldots + = 2^{n-1}$

(iv) $^{n}C_1 + {}^{n}C_3 + {}^{n}C_5 + \ldots + = 2^{n-1}$

(v) $0.{}^{n}C_0 + 1.{}^{n}C_1 + 2.{}^{n}C_2 + \ldots + n.{}^{n}C_n = n.2^{n-1}$

Permutations with Repetition

The number of permutations of n objects, where p objects are of one kind, q objects are of another kind and the rest, if any, are of a different kind is $\frac{^nP_r}{p!q!}$.

Combination with Repetition

Number of combinations of 'n' distinct things taking 'r' at a time when each thing may be repeated any number of times is given by $^{n-1+r}C_r$.

5.4. STATISTICS

Statistics $\rightarrow$ Collection and Analysis of Data

Data → Ungrouped data / Row data
Data → Grouped Data

5.4.1. Analysis of Ungrouped Data:

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' observations, then

(1) The range of the data = R = max{$x_1, x_2, \ldots\ldots, x_n$} – min{$x_1, x_2, x_3, \ldots.., x_n$}

(2) Arithmetic mean **:** Mean of the data is equal to sum of observaions divided by the total number of observations.

$$\bar{x}(or)\mu = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \boxed{\frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} = \mu}$$

- The mean of 1st 'n' natural numbers = $\frac{\left(\frac{n(n+1)}{2}\right)}{n} = \frac{n+1}{2}$
- The mean of 1st 'n' odd numbers = $\frac{n^2}{n} = n$
- The mean of 1st 'n' even numbers = n +1

5.4.2 Median:

The middle most observation of the data $(x_1, x_2, x_3, \ldots\ldots x_n)$ When the data is arranged in either ascending or descending order.

If $x_1, x_2, x_3, x_4, \ldots\ldots\ldots x_n$ are 'n' observations that are arranged in ascending/descending order then

(i) Median of the Data = $\left(\frac{n+1}{2}\right)^{th}$ observation, if 'n' is odd.

(ii) Median of the Data = Mean of $\left(\frac{n}{2}\right)^{th}$ & $\left(\frac{n}{2}+1\right)^{th}$ observations, if 'n' is even.

5.4.3 Mode:

The observation with highest frequency is called mode.

Any Data with two Modes is called $\rightarrow$ Bimodel Data

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' data points, $\quad \bar{x} = \mu = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n}$

Mean Deviation of the observation $(x_i) = d_i = x_i - \bar{x}$

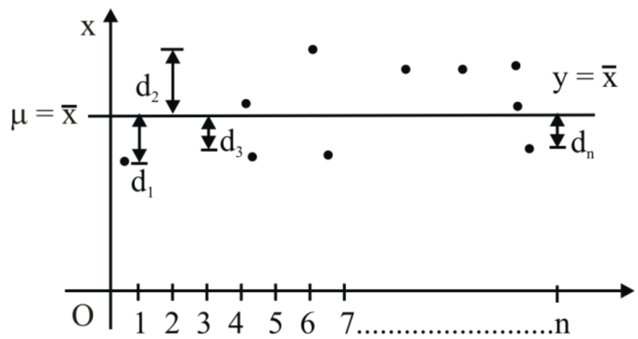


Fig. 5.6. Discrete data

Sum of derivations of all the observations = $\Sigma d_i = (x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots\ldots 0 + (x_n - \bar{x})$

$$= \Sigma d_i = (x_1 + x_2 + \ldots\ldots + x_n) - n\bar{x}$$

$$\boxed{\Sigma d_i = 0}$$

The sum of mean deviations of all the observations is equal to zero.

5.4.4 Absolute Mean Deviation:

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' data points with Mean = $\bar{x}$, then the absolute mean deviation of x_i about $\bar{x}$ is given by $|d_i| = |x - \bar{x}|$

The sum of absolute mean derivations of given data is not zero.

$$(\Sigma |d_i| \neq 0) \Rightarrow (|x_1 - \bar{x}| + |x_2 - \bar{x}| + \ldots\ldots + |x_n - \bar{x}| \neq 0)$$

5.4.5 Standard Deviation:

If $x_1, x_2, x_3, \ldots\ldots, x_n$ ('n' is very large), then the standard deviation of the data is given by

Population Standard Deviation $\sigma = \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}$, n → size of population

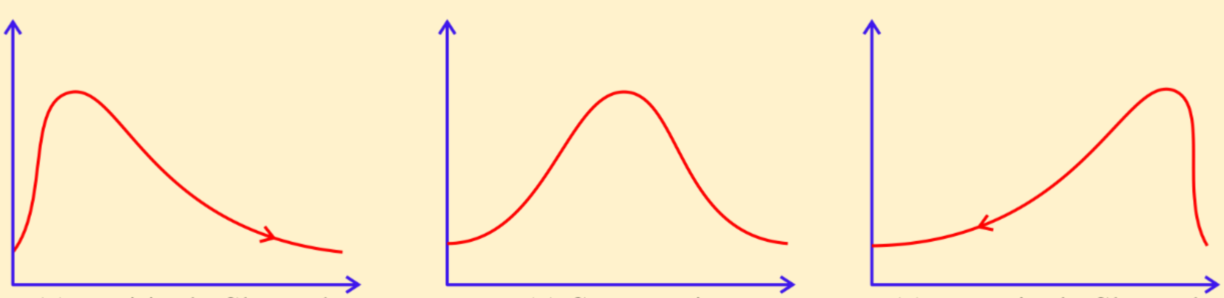
Sample Standard derivation: $\sigma = \sqrt{\frac{1}{(n-1)}\Sigma(x_i - \bar{x})^2}$, n→ size of sample

Generally (n > 29 → population) (n < 29 →sample)

Note: **Measures of skewness (The degree of asymmetry)**

A lack of symmetry is skewness.

- For symmetric distribution mean (M) = Median (M_d) = Mode (M_e)
- For negatively skewed distribution mean (M) < Median (M_d) < Mode (M_e)
- For positively skewed distribution Mean (M) > Median (M_d) > Mode M_e).



5.5. Random Variables

The variable that connects the outcome of a Random Experiment to a real number.

Example: 'x' is the value of the number that a dice shows when it is rolled.

Discrete RV → The RV whose value is obtained by counting, defined by PMF

Random Variable

Continuous RV → The RV whose value is obtained by Measuring, defined by PDF

- If a data consists of 'f_1' data points with value $'x_1', 'f_2'$ data points with value $'x_2'......'f_n'$ data point with value $'x_n'$, then
  (i) Expectation of 'x' $= E(x) = \sum_{i=1}^{n} x_i P(x = x_i)$
  (ii) Variance of 'x' = $\sigma^2 = E(x^2) - (E(x))^2$ and σ is the standard deviation.

5.5.1 Probability Mass Function (PMF)

The PMF p(x) of a discrete random variable X taking values $x_1, x_2, ... x_n$ is defined such that,

(i) $p(x_i) \geq 0$

(ii) $\sum_{i=1}^{n} p(x_i) = 1$

(iii) $p(x_i) = p(X = x_i)$

5.5.2 Probability Density Function (PDF)

The pdf f(x) of a continuous random variable X is defined such that,

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a < X < b) = \int_{a}^{b} f(x)dx$

5.5.3 Expected Value

1. Expected value of a random variable X, E [X], is defined as, $E[X] \begin{cases} \sum xp(x); & \text{X is discrete rv} \\ \int_{-\infty}^{\infty} xf(x)dx; & \text{X is continuous rv} \end{cases}$

2. Expected value of X^2 is,

$$E[X^2] = \begin{cases} \sum x^2 p(x); & \text{X is discrete rv} \\ \int_{-\infty}^{\infty} x^2 f(x)dx; & \text{X is continuous rv} \end{cases}$$

Note: $E[X^n]$ is called nth moment.

5.5.4 Mean of Random Variable 'X'

Mean $= \mu = E[X]$

5.5.5 Variance of a Random Variable 'X'

Var $(X) = E[(X-\mu)^2]$

Or, Var $(X) = E[X^2] - \mu^2$

5.5.6 Properties of Expectation

(i) E[c] = c, c is a constant.

(ii) E[ax] = aE[X]

(iii) E(aX + b) = aE(X) + b

(iv) If X and Y are random variable E[X ± Y] = E(x) ± E(Y).

(v) If X and Y are random variables E(X. Y) = E(X). E(Y / X).

(vi) If X and Y independent random variables E(X. Y) = E(X). E (Y).

5.5.7 Properties of Variance

(i) Var[C] = 0, C is constant.

(ii) Var (aX) = a^2V(X) where X is random variable and 'a' constant.

Var(–X) = $(-1)^2$ Var(X) = Var(X) Variance is always positive.

(iii) Var(ax + b) = a^2 Var(X) + 0

(iv) If X and Y are independent random variables.

Var(X + Y) = Var(X) + Var(Y)

Var(X – Y) = Var(X) + Var(Y)

(v) Var(ax + by) = a^2 v(x) + b^2 v(y) + 2ab Cov (x, y)

(vi) Cov (x, y) = E(x, y) – E(x) E(y)

(vii) For independent random variables Cov(x, y) = 0

5.5.8 Continuous RV

The value of the Random Variable is obtained by Measuring.

5.6. Probability Distribution Function (PDF)

A continuous & differentiable function P(x) is said to be a probability distribution/density function of a continuous random variable 'x' if $P(a \le x \le b) = \int_a^b P(x)dx$

5.6.1 Mean (or) Expectation

If P(x) is a probability distribution/density function of a continuous Random Variable 'x' then the Mean of 'x' = E(x) $= \int_{-\infty}^{\infty} x \cdot P(x) dx$

5.6.2 Median

The value of 'x' for which the total probability is exactly divided into two equal halves is called Median.

5.6.3 Mode

The valueof 'x' at which P(x) is maximum is called mode.

5.6.4 Variance

$= \sigma^2 = E(x^2) - (E(x))^2$

$$\Rightarrow \sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot P(x) dx - \left(\int_{-\infty}^{\infty} x \cdot P(x) dx\right)^2$$

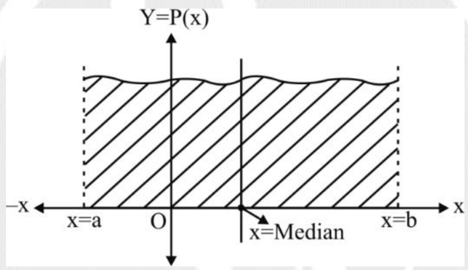


Fig. 5.7. Continuous random variables

5.7. Continous RV distributions

(1) Gaussian/Normal Distributon:

If 'x' is a continuous Random variable with mean 'μ' and standard deviation 'σ', then the probability distribution/density function of normally distributed variable 'x' is given by

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}$$

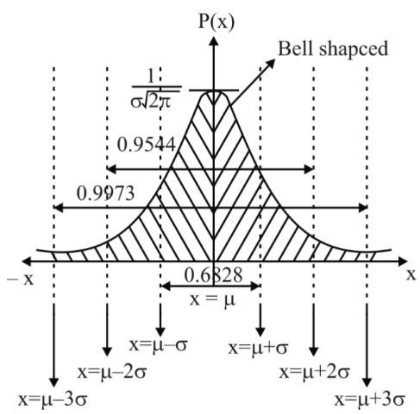


Fig. 5.8. Normal distribution

$$\text{Mean} = \text{Median} = \text{Mode} = \mu$$
$$P(\mu - \sigma \le x \le \mu + \sigma) = 0.6828$$
$$P(\mu - 2\sigma \le x \le \mu + 2\sigma) = 0.9544$$
$$P(\mu - 3\sigma \le x \le \mu + 3\sigma) = 0.9973$$
$$P(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} . e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

(2) Standard Normal Distribution:

Assuming $z = \frac{x-\mu}{\sigma}; \mu = 0; \sigma = 1, P(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}}$

$$P(-1 \le z \le 1) = 0.6828$$
$$P(-2 \le z \le 2) = 0.9544$$
$$P(-3 \le z \le 3) = 0.9973$$

Note:
1. The normal distribution curve is bell shaped curve
2. The points of infelection of the normal distribution curve are at $x = \mu + \sigma \ and \ x = \mu - \sigma$.
3. The cumulative function graph is of ‘S’ Shape.
4. For a given normal distribution, Mean = median = Mode

(3) Uniform Distribution:

If 'x' is a uniformly distrbuted random variable such that $a \le x \le b$ then the Pdf is given by

$$P(x) = \frac{1}{(b-a)}$$

Mean = $\int_a^b x \cdot P(x)dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{(b-a)} \int_a^b x \cdot dx$

$$\boxed{\left(\frac{b+a}{2}\right) = \text{Mean}}$$

$\Rightarrow$ $Variance = \sigma^2 = \frac{(b-a)^2}{12}$

Std. deviation = $\sigma = \frac{(b-a)}{\sqrt{12}}$

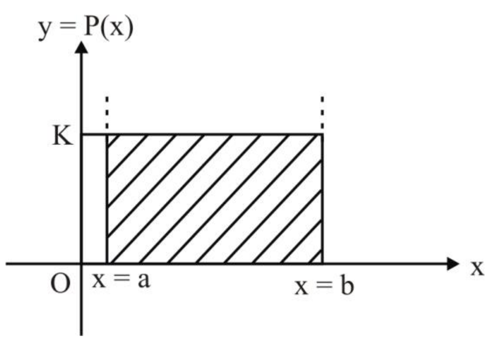


Fig. 5.9. Uniform Distribution

5.7.1 Properties of Mean and Variance:

$E(ax + by) = a \cdot E(x) + b \cdot E(y)$

$$V(ax + by) = a^2 \cdot V(x) + b^2 \cdot V(y) - 2abCOV(x, y)$$

where $COV(x, y) = E(xy) - E(x) \cdot E(y)$

If x,y are indpendent random variables, then $E(xy) = E(x) \cdot E(y) \Rightarrow COV(x, y) = 0$

(1) Exponential Dirtibution:

If 'x' is a continous random variable with mean as $\frac{1}{\lambda}$ then the exponential distribution of 'x' is given by the function

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & ; x \geq 0 \\ 0 & : otherwise \end{cases}$$

Mean = $\frac{1}{\lambda}$

$\sigma^2 = \frac{1}{\lambda^2}$

Mean = Standard Deviation = $\frac{1}{\lambda}$

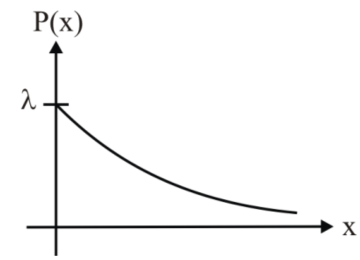


Fig. 5.10. Exponential distribution

5.8. Discrete Random Variable Distributions

If a Random experiment has **only two Possible outcomes**, (one is Success & other is failure) and the Probability of Success doesn't depend on time, then the probability of occuring of exactly 'r-successes' in 'n-trials' is given by

$$P(X = r) = {}^nC_r \cdot P^r \cdot q^{n-r}$$

Where, P → Probability of Success,

q → Probability of Failure

p + q = 1

Mean = np, Variance = npq = σ^2, standard deviation = $\sigma = \sqrt{npq}$

5.8.1 Poisson Distribution:

If a random experiment has only two possible outcomes, and the average number of successes in a given time 't' is λ, then the probability that exactly 'r' successes occur within the same time 't' given by

$P(x = r) \frac{e^{-\lambda}.\lambda^r}{r!}$

Mean = λ.

Mean = Variance = λ

$\Rightarrow \sigma = \sqrt{\lambda}$

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6 COMPLEX CALCULUS

A number of the form $z = x + iy$ where $x, y \in R$ is called a complex number.

x is called real part of z, $x = Re(z)$

y is called imaginary part of z, $y = Im(z)$

6.1. Modulus – Amplitude form of a Complex Number

Every Complex number $z = x + iy$ can be written as $z = r.e^{i\theta}$ where

$r = \sqrt{x^2 + y^2}$ is called the modulus of the complex number and

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called the amplitude (or) argument of the complex number.

$e^{i\theta} = \cos\theta + i.\sin\theta$ and

$e^{-i\theta} = \cos\theta - i.\sin\theta$

6.2. Arithmetic Operations with Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers then

(i) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

(ii) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1y_2 + x_2y_1)$

(iii) $\frac{z_1}{z_2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$

(iv) $|z_1 + z_2| \le |z_1| + |z_2|$

(v) $|z_1 - z_2| \ge \left||z_1| - |z_2|\right|$

(vi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

If r_1, θ_1 are the modulus and angle of a complex number z_1 and r_2, θ_2 are the modulus and angle of a complex number z_2 respectively, then

(i) The modulus of $z_1 \cdot z_2$ is $r_1 \cdot r_2$ and the angle of $z_1 \cdot z_2$ is $\theta_1 + \theta_2$

(ii) The modulus of $\frac{z_1}{z_2}$ is $\frac{r_1}{r_2}$ and the angle of $\frac{z_1}{z_2}$ is $\theta_1 - \theta_2$.

If $z = x + iy$ is a complex number, then the conjugate of the complex number is given by z^* (or) $\overline{z} = x - iy$.

$$Re(z) = \frac{z+z^*}{2} \quad \text{and} \quad Img(z) = \frac{z-z^*}{2i}$$

$$z \cdot z^* = |z|^2$$

- **nth root of unity :**

$Z = (1)^{1/n}$ We write $1 = e^{j2\pi k}$

Thus we get nth root of unity equal to $1, \alpha, \alpha^2, \alpha^3 .... \alpha^{n-1}$. Here $\alpha = e^{i2\pi/n}$

Properties :

1. n^{th} root of unity form a GP
2. Sum of all nth root of unity = 0
3. Product of all n^{th} root of unity is $(-1)^{n-1}$
4. n^{th} root of unity lie on circle of unit radius

$\log_e$ of a complex number

If z = x + iy, to find $\log_e(z)$ we write z in polar form so $z = re^{j\theta}, \log_e(z) = \log_e(re^{i\theta}) = \log_e r + i\theta.$

Since angle $\theta = \theta + 2n\pi$

So $\log(z) = \log_e r + i(\theta + 2n\pi)$

6.3. De-moivre's Theorem

For any complex number *x* and any integer *n*,

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

The cube roots of unity when plotted on an argand plane form an equilateral triangle.

6.4. Standard Complex Functions

If $z = x + iy$ is a complex number, then

(i) $\ln z = \frac{1}{2} \cdot \ln(x^2 + y^2) + i \cdot \tan^{-1}\left(\frac{y}{x}\right)$

(ii) $\exp(z) = e^x \cdot (\cos y + i \sin y)$

6.5. Periodic function

A complex function $f(z)$ is a periodic function if there exists a complex number 'k' such that $f(z) = f(z + k)$

Example: The function $f(z) = e^z$ is a periodic function with period $2\pi i$.

6.5.1 Analytic Functions

A function $f(z)$ is said to be analytic at a point $z = z_0$ if the fucntion $f(z)$ is differentiable at the pount $z = z_0$ and also at every point in the neighbourhood of z_0.

The mathematical conditions for a function $f(z) = u(x, y) + i.v(x, y)$ to be analytic at a point $z_0 = x_0 + i\, y_0$ is

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous and differentiable at (x_0, y_0)

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These set of equations are called Cauchy – Riemann (C-R) Equations .

Note: If the function $f(z) = u(x, y) + i.v(x, y)$ is analytic then

(i) Both $u(x, y)$ and $v(x, y)$ satisfy laplace equation.

i.e. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(ii) The family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal to each other.

Cauchy – Riemann Equations in polar form for the function $f(z) = u(x, y) + i.v(x, y)$ are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r \cdot \frac{\partial v}{\partial r}$$

6.6. Complex Integration

If $f(z) = u + iv$ is continuous and differentiable at every point along a path 'C' , then the evaluation of $f(z)$ along the pathe 'C' is given by

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (u\,dy + v\,dx)$$

Note: If the function $f(z)$ is analytic, then the integral $\int_{z_1}^{z_2} f(z)dz$ is independent of the path connecting the complex numbers z_1 and z_2.

6.6.1 Cauchy Integral Theorem:

If the function $f(z)$ is analytic at everypoint with in a closed path 'C' then $\oint_C f(z)dz = 0$.

Parametric integration of complex function:

Consider a contour C parameterized by z(t) = x(t) + iy(t) for a ≤ t ≤ b. We defined the integral of the complex function along C to be the complex number

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt .$$

Note : If F(z) is an analytic function then the integration $\int_a^b F(z)dz$ is independent of path followed to move from point a to b

6.7. Taylor Series and Laurentz Series

(i) Taylor series: If the function $f(z)$ is analytic at every point with in a circle with centre at $z = z_0$, then for any point z with in the circle,

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n$$

Where $a_n = \left(\frac{1}{2\pi i}\right) \cdot \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

(ii) Laurentz Series: If the function $f(z)$ is analytic at every point with in a region bounded by two concetric circles C and C_1 with radii r, r_1 respectively $(r > r_1)$ with centre at $z = z_0$, then for any point z with in the region,

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot (z - z_0)^n + \sum_{n=1}^{\infty} b_n \cdot (z - z_0)^{-n}$$

where $a_n = \left(\frac{1}{2\pi i}\right) \cdot \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

and $b_n = \left(\frac{1}{2\pi i}\right) \cdot \oint_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$

Note: All the formulaes above for the cyclic integrals are for counter clockwise case by default, if the questions are asked for clockwise case, the answer evaluted using above formulaes should be written with sign change.

6.8. Residue Theorems

6.8.1. Cauchy Residue Theorem:

If f(z) is analytic in closed curve C except at a finite number of singular points within C,

then $\int_C f(z)dz = 2\pi i \times (\text{Sum of the residue at the singular points within } C)$

Note: In above formula the Contour is anti-clockwise. If the Contour is clockwise then we put a –ve sign in RHS of above equation.

Applying Cauchy theorem we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + ... + \int_{C_n} f(z)dz = 2\pi i[\text{Res}[f(a_1)] + \text{Res}[f(a_2)] + ... + \text{Res}[f(a_n)]]$$

6.8.2 Methods of Evaluating Residues

- If f(z) has a simple pole at z = 0 then $\text{Res}[f(a)] = \lim_{z \to a}(z-a)f(z)$
- If $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\Psi(z) = (z - a) f(z)$, $f(a) \neq 0$

$$\text{Res } f(a) = \frac{\phi(a)}{\Psi'(a)}$$

- If f(z) has a pole of order n at z = a, then Res $f(a) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{Z=a}$

6.9. Singularities of an Analytic function and It's Type

A point at which the function ceases to be analytic is called singular point of functions.

6.9.1. Type of Singularities

- **Removable singularity:** A removable singularity is a singular point z_0 of a function f(z) for which it is possible to assign a complex number in such a way that f(z) becomes analytic. A more precise way of defining a removable singularity is as a singularity z_0 of a function f(z) about which the function f(z) is bounded. For example, the point $x_0 = 0$ is a removable singularity in the sin c function sin c(x) = sin x/x, since this function satisfies sin c(0) = 1.
- **Isolated singularity:** A point a in the domain D of function f(z) is said to be a point of isolated singularity, if f(z) is analytic at each point in some neighbourhood |z – a| < R of a, except not being analytic at a.
- **Pole:** Pole is another kind of singularity defined as if there exist a positive integer m such that $z \to a(z-a)^m f(z) \neq 0$, then z = a is called a pole of order m.

  For example, let $f(z) = 1/(z-5)^3$, then

$$z \to 5(z-5)^3 \frac{1}{(z-5)^3} = 1 \neq 0$$

- **Isolated Essential Singularity:** The definition of isolated essential singularity is, if their does not exist a finite value m such that $z \to (z-a)^m f(z) = k$, where k is a non-zero finite constant. The point z = a is called an isolated essential singularity.

  For example, let f(z) = sin[1/(z – a)] where $\sin[1/(z-a)] = 1/(z-a) - 1/(z-a)^3\, 3! + 1/(z-a)^5 5! - \ldots$

  Here the function has infinite terms in negative power of (z – a), so it is not possible to find a finite value of m.

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