# Communication System

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# COMMUNICATION SYSTEM

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# 1 AMPLITUDE MODULATION

## 1.1. Introduction

**Band limiting :** Bandun limited to bandlimited (LPF)

**Base band signal :** Message signals, low cut off *fre* = 0 Hz or very close to 0 Hz.

**Bandpass signal :** By shifting baseband signal to very high freq.

- Wideband signal : $\frac{f_H}{f_L} >>> 1$ (Base band signal)
- Narrowband signal : $\frac{f_H}{f_L} \approx 1$ (Bandpass signal)

**Modulated Signal:**

$$C(t) = A_c \cos(\omega_c t + \phi) = A_c \cos \omega_c t$$

Carrier signal

(carrier before modulation)

$S(t) = A(t)\cos[\omega_c t + \phi(t)]$ Modulated signal

$A(t)$ → Instantaneous amplitude; $\omega_c t$ → Instantaneous frequency; $\phi(t)$ → Instantaneous phase

**Amplitude Modulation:**

DSB-FC (Double side band full carrier)

$C(t) = A_c \cos \omega_c t$ carrier before modulation

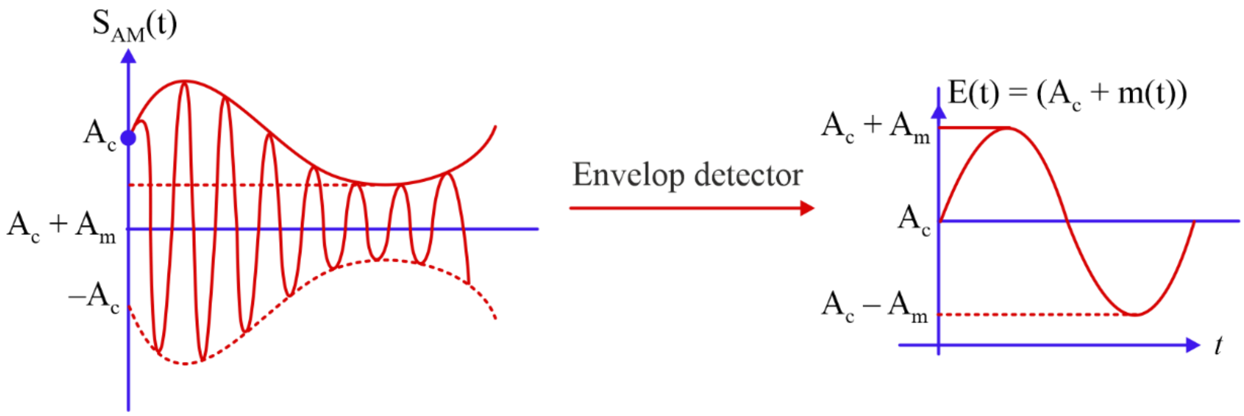
$S_{AM}(t) = A_c \cos \omega_c t + m(t) \cos \omega_c t \Rightarrow S_{AM}(t) = [A_c + m(t)] \cos \omega_c t$ carrier after modulation

Modulation Index $\mu = \frac{[m(t)]}{A_c}$max

(1) $\mu < 1$ (under modulation)

$$\mu = \frac{A_m}{A_c} < 1$$

$$\mu = \frac{[E(t)]_{max} - [E(t)]_{min}}{[E(t)]_{max} + [E(t)]_{min}}$$




$\mu > 1,\ A(t) > 0,\ E(t) = A(t)$

Recovery through E, D possible.

$S(t)_{max} = E(t)|_{max} = A_c(1+\mu)$

$S(t)_{min} = E(t)|_{min} = A_c(1-\mu)$

(2) Critical Modulation:- $\mu = 1, A(t) \geq 0$, $E(t) = A(t)$, m(t) can be recovered with envelope detector .

(3) Over modulation: $\mu > 1, A(t) \ngtr 0, E(t) = |A(t)|$, not possible by E.D

## Frequency Related Parameters

(1) m(t) → B.W $= f_m$ ; $f_{max} = f_m$

(2) $C(t) \rightarrow f_{max} = f_m$

(3) S(t) → B.W = $2f_m$ (2 times of freq. of m(t)), $f_{max} = f_c + f_m$, $f_{min} = f_c - f_m$

➢ $P_{AM} = P_C + P_{SB}$

$$P_{USB} = P_{LSB} = \frac{P_m}{4}$$

## Modulation efficiency

$$\eta = \frac{P_{SB}}{P_{AM}} = \frac{P_m/2}{P_c + \frac{P_m}{2}}$$

Share of sideband power in total power

➢ $k_a$ [Amplitude sensitivity of amplitude modulator]

$k_a = \frac{1}{A_c}$ (per volt),

$A(t) = A_c[1 + k_a m(t)]$

A (t) > 0, E. D. Applicable

DSB – FC →
- $[A_c + m(t)]\cos 2\pi f_c t \rightarrow \mu = \dfrac{|m(t)|_{max}}{A_c}$
- $A_c[1 + k_a m(t)]\cos 2\pi f_c t \rightarrow \mu = k_a\,|m(t)|_{max}$

**For single tone sinusoidal signal**

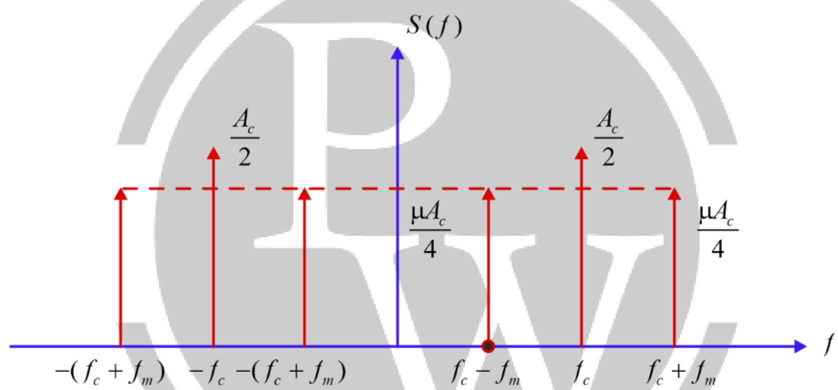
$$f_{max} = f_m, BW = 0\text{Hz}, P_m = \frac{A_m^{\,2}}{2} \longrightarrow \text{for message signal}$$

$$\mu = \frac{A_m}{A_c}$$

$$S_{AM}(t) = [A_c + A_m \cos 2\pi f_m t]\cos 2\pi f_c t$$

$$= A_c[1 + \mu\cos 2\pi f_m t]\cos 2\pi f_c t$$

$$S_{AM}(t) = \underset{\text{carrier}}{\underbrace{A_c \cos 2\pi f_c t}} + \underset{\text{USB}}{\underbrace{\frac{\mu A_c}{2}\cos[2\pi(f_c + f_m)t]}} + \underset{\text{LSB}}{\underbrace{\frac{\mu A_c}{2}\cos[2\pi(f_c - f_m)t]}}$$




$$P_{AM} = P_c + \frac{P_m}{2} = \frac{A_c^2}{2}\left[1 + \frac{\mu^2}{2}\right] = \underset{P_c}{P_c} + \frac{P_c\mu^2}{2} \rightarrow P_{SB}$$

$$\eta = \frac{\frac{P_c\mu^2}{2}}{P_c + \frac{P_c\mu^2}{2}} \Rightarrow \%\eta = \frac{\mu^2}{2 + \mu^2} \times 100\%$$

- If $k_a$ given $\longrightarrow \mu = k_a\,|m(t)|_{max}$

  If $k_a$ not given $\longrightarrow \mu = \dfrac{|m(t)|_{max}}{A_c}$

**Important Points:**

| $\mu = 0$ | $\mu = 1$ | % Change |
|---|---|---|
| $P_{AM} = P_c$ | $P_{AM} = 1.5P_c$ | 50 % |
| $\eta = 0$ | $\eta = \frac{1}{3} = 33.33\ \%$ | 0 % to 33.33 % |

(2) $\mu \uparrow \rightarrow \eta \uparrow$

(3) $P_{AM} \rightarrow$ Will be constant if $P_c \uparrow$ and $\mu \downarrow$

If $m(t)$ is multiple single tone signal-

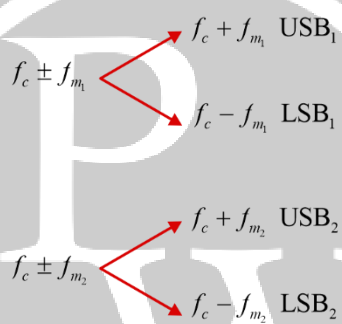
$$S(t) = A_c\left[1 + \frac{A_{m_1}}{A_c}\cos 2\pi f_{m_1} t + \frac{A_{m_2}}{A_c}\cos 2\pi f_{m_2} t\right]\cos 2\pi f_c t$$

$$\mu_1 = \frac{A_{m_1}}{A_c}, \mu_2 = \frac{A_{m_2}}{A_c} - \mu_1 > \mu_2$$

$$m(t) \rightarrow f_{m_1}, f_{m_2} \rightarrow f_{\max} = f_{m_2}$$

$f_{m_2} > f_{m_1}$ $\quad BW = f_{m_2} - f_{m_1}$

$S(t) = f_c$ $\quad$ BW = 2 × Max. Freq. component of $m(t)$




## Power Related Parameters

$$P_m = \frac{A_{m_1}^2}{2} + \frac{A_{m_2}^2}{2}$$

$$P_{AM} = P_c + \frac{P_m}{2} = \frac{A_c^2}{2}\left[1 + \frac{\mu_1^2}{2} + \frac{\mu_2^2}{2}\right]$$

$$P_{USB_1} = P_{LSB_1} = \frac{P_c\mu_1^2}{2}, P_{USB_2} = P_{LSB_2} = \frac{P_c\mu_2^2}{2}$$

$$P_{USB_1} = P_{LSB_1} = \frac{P_c\mu_1^2}{4}, P_{USB_2} = P_{LSB_2} = \frac{P_c\mu_2^2}{4}$$

$$\eta = \frac{P_{SB}}{P_{AM}} = \frac{\mu_T^2}{2 + \mu_T^2}$$

$$\mu_T = \mu_1^2 + \mu_2^2 + \mu_3^2 + ----$$

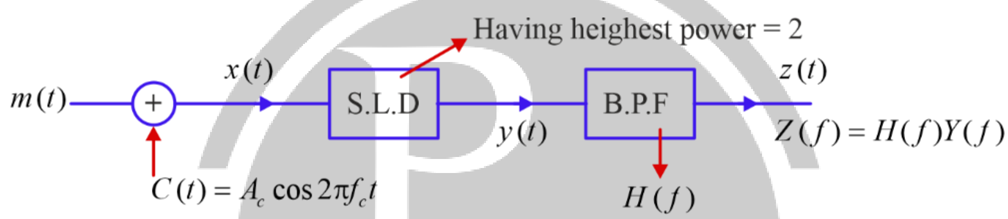
**Important Points:**

| $m(t)$ (volt) | | $P_{AM}$ | $P_{rod}$ |
|---|---|---|---|
| (1) | Sinusoidal | $P_c\left(1+\frac{\mu^2}{2}\right)$ | $\frac{P_c}{R}\left(1+\frac{\mu^2}{2}\right)$ |
| (2) | Square wave | $P_c(1+\mu^2)$ | $\frac{P_c}{R}(1+\mu^2)$ |
| (3) | Triangular wave | $P_c\left(1+\frac{\mu^2}{3}\right)$ | $\frac{P_c}{R}\left(1+\frac{\mu^2}{3}\right)$ |

$$V_{AM}=V_c\sqrt{1+\frac{\mu^2}{2}},\ I_{AM}=I_c\sqrt{1+\frac{\mu^2}{2}}\ \text{for sinusoidal}$$

## DSB- FC [AM] Modulator

### (1) Square law Modulator:




$$y(t)=a_0m(t)+a_0A_c\cos 2\pi f_ct+a_1m^2(t)+\frac{a_1A_c^2}{2}+\frac{a_1A_c^2}{2}\cos 4\pi f_ct+2m(t)A_c\cos 2\pi f_ct$$

(1) (2) (3) (4) (5) (6)

- Only (2) and (6) are desirable

$$Z(t)=a_0A_c\cos 2\pi f_ct\left[1+\frac{2a_1}{a_0}m(t)\right]\ \text{DSB –FC}$$

$$Z(t)=A_c'[1+k_am(t)]\cos 2\pi f_ct\ \text{only when}\ f_c >>> 3f_m\,f_c >>> (2+1)f_m$$

$$A_c'=a_0A_c,\ k_a=\frac{2a_1}{a_0},\ \mu=k_a|m(t)|_{\max}$$

### (2) Switching Modulator:

$$Z(t)=\frac{A_c}{2}\left[1+\frac{4}{\pi A_c}m(t)\cos 2\pi f_ct\right]DSB-FC$$

$$Z(t)=A_c'[1+k_am(t)]\cos 2\pi f_ct$$

$$A_c'=\frac{A_c}{2},\ k_a=\frac{4}{\pi A_c}\qquad \mu=k_a|m(t)|_{\max}$$

## DSB- FC Demodulator-

### (1) Square law demodulator-

$$Y(t) = a_1 A_c A_m \cos 2\pi f_m t + \frac{a_1 A_m^2}{4} + \frac{a_1 A_m^2}{4} \cos 4\pi f_m t$$

$$Y(t) = B_0 + B_1 \cos \omega_0 t + B_2 \cos 2\omega_0 t$$

- $2^{nd}$ harmonic distortion $D_2 = \left|\frac{B_2}{B_1}\right| = \frac{\mu}{4}$

  $(D_2)_{max} \% = 25\%$
- Practically not used
- $\left(\frac{S}{I}\right)_{min} = \frac{2}{\mu}$

## Envelope Detector:

$$\sqrt{A^2 + B^2} \cos \omega_c t \rightarrow \boxed{E.D} \rightarrow \sqrt{A^2 + B^2}$$

(1) $x(t) = A\cos \omega_0 t + B\sin \omega_0 t \rightarrow E(t) = \sqrt{A^2 + B^2}$

(2) $x(t) = A\cos(\omega_0 t + \theta) + B\sin \omega_0 t \rightarrow E(t) = \sqrt{A^2 + B^2 - 2AB\sin\theta}$

(3) $x(t) = A(t)\cos \omega_c t \rightarrow E(t) = |A(t)|$

(4) $x(t) = (A_c + m(t))\cos \omega_c t \rightarrow E(t) = |A_c + m(t)|$

## Important Points:

- Used only when $\mu \le 1$
- $T_c = R_S C <<, \frac{1}{f_c}$ (charging time constant)
- $T_c = R_S C >>> \frac{1}{f_c} \rightarrow$ Peaks are not detected.
- Diagonal clipping $\rightarrow R_L C = \frac{1}{f_m}$
- To avoid diagonal clipping $R_L C <<< \frac{1}{f_m}, R_L C \le \frac{\sqrt{1-\mu^2}}{\omega_m \mu}$
- $T_a = R_L C \approx \frac{1}{f_c}$ fluctuation is output
- To remove fluctuation $R_L C >>> \frac{1}{f_c}$
- Proper choice of discharging time constant $R_L C$ -

No fluctuation $\frac{1}{f_c} <<< R_L C <<< \frac{1}{f_m}$ No diagonal clipping

(7) m(t): Multitone $f_m \to f_{max}$ = Max. freq. component of $m(t)$

## 1.2. Synchronous Detector

| $\Delta\omega$ | $\Delta\phi$ | $\Delta\phi$ | **Recovery** |
|---|---|---|---|
| = 0 | $\neq 0$ | $\pm(2n+1)\frac{\pi}{2}$ | ✓ |
| = 0 | $\neq 0$ | $=(2n+1)\frac{\pi}{2}$ | Q.N.E |
| $\neq 0$ | = 0 | = 0 | × |
| = 0 | = 0 | = 0 | ✓ |

**DSB-SC :**

$$S_{DSB-SC}(t) = m(t)A_c(\cos 2\pi f_c t) \quad E(t) = |A(t)| = |A_c m(t)|$$

- $B.W = 2 \times$ max. freq. component of $m(t)$
- $P_{DSB} = P_m P_c = P_{SB} \to P_{USB} = P_{LSB} = \frac{P_{SB}}{2} + \frac{P_m P_c}{2}$

  $P_{DSB} = \frac{P_c \mu^2}{2} = P_{SB}$
- Single tone modulation.

  $S(t) = \frac{A_c A_m}{2}\cos[2\pi(f_c + f_m)t] + \frac{A_c A_m}{2}\cos[2\pi(f_c - f_m)t]$

  $P_{DSB} = P_m P_c = \frac{A_c^2 A_m^2}{4}$

  Multiline $P_{DSB} = P_c P_m = \frac{A_c^2}{2}\left[\frac{A_{m_1}^2}{2} + \frac{A_{m_2}^2}{2}\right]$

  Square wave - $P_{DSB} = P_c P_m = \left(\frac{A_c^2}{2}\right) A_m^2$

  Triangular wave $P_{DSB} = P_c P_m = \left(\frac{A_c^2}{2}\right)\left(\frac{A_m^2}{3}\right) a$

  Saw-toothed wave- $P_{DSB} = P_c P_m = \left(\frac{A_c^2}{2}\right)\left(\frac{A_m^2}{3}\right)$

(1) Balanced Modulator- $S_{DSB}(t) = 2A_c k_a m(t)\cos f_c t$

(2) Ring Modulator- $y(t) \propto m(t)\cos\omega_c t$

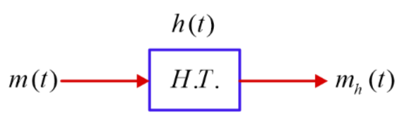
$$\Delta\omega = 0,\ \Delta\phi \neq 0,\ \ y(t) = 0 \quad \text{QNE}$$

$$\Delta\omega \neq 0,\ \Delta\phi = 0,\ \ y(t) = \frac{A_c A_C'}{2} m(t)\cos(\Delta\omega t) \longrightarrow \text{distorted } m(t)$$

$$\Delta\omega = 0,\ \Delta\phi = 0,\ \ y(t) = \frac{A_c A_C'}{2} m(t) \longrightarrow \text{Attenuated}$$

## Hilbert Transformation.

$$h(t) = \frac{1}{\pi t},$$




$$mh(t) = m(t) * \frac{1}{\pi t}$$

$$H(\omega) = -j\,\text{sgn}(\omega) = \begin{cases} -j & \omega > 0 \\ 0 & \omega = 0 \\ j & \omega < 0 \end{cases}$$

- $M_h(f) = M(f)[-j\,\text{sgn}(f)]$
- $H.T[\cos\omega(t)] = \sin\omega(t) \xrightarrow{H.T} -\cos\omega(t)$
- Non causal LTI system.
- $x(t) \xleftrightarrow{H.T.} x_h(t)$

  $x_h(t) \xleftrightarrow{H.T.} -x(t)$
- Magnitude spectrum of x(t) and $x_h(t)$ will be same
- If x(t): Band limited then $x_h(t)$ is also bandlimited.
- If x(t) is non periodic then $x_h(t)$ is also non periodic
- x(t) and $x_h(t)$ are orthogonal signal.

## Drawback of DSB-SC

- 2 sideband Txed.
- If receiver is designed in such a way that it may recover the complete message signal from single SB then DSB-SC S/S becomes impractical.

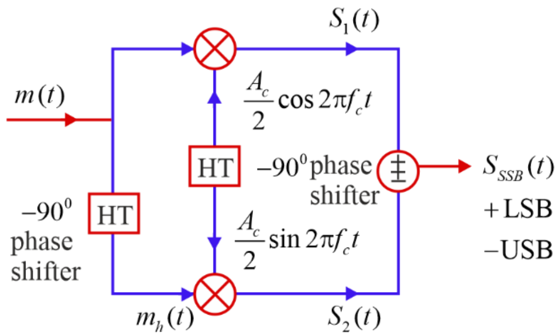
## SSB- SC (Single sideband suppressed carrier)

(1) Point to point communication

(2) Two methods of generation → Phase derserimination / Frequency discrimination

## (a) Phase Discrimination:

$$S(t)_{SSB} = \frac{A_c m(t)}{2}\cos 2\pi f_c t \pm \frac{A_c m_h(t)}{2}\sin 2\pi f_c t \quad \begin{matrix} + \Rightarrow LSB \\ - \Rightarrow USB \end{matrix}$$

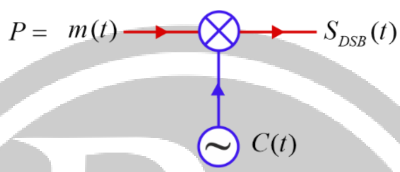



### Problem Solving

(1) Identify the phase descrimination setup

(2) Phase descrimination setup

(3) Phase descrimination setup:




$+ \Rightarrow S_{DSB}(t) \rightarrow LSB$

$- \Rightarrow S_{DSB}(t) \rightarrow USB$

| Spectral gap in D.S.B | BPF | Signal |
|---|---|---|
| 0 Hz | Ideal | SSB-SC |
| 0 Hz | Practical | VSB-SC |
| $\neq$ 0Hz | Ideal | SSB -SC |
| $\neq$ 0Hz | Practical | depends on practical BPF → $V_{SB}-SC$ / $SSB-SC$ |

➢ SSB- SC can be demodulated by Synchronous detection.

(1) $\Delta\omega = 0, \Delta\phi \neq 0, m(t)$ recovery not possible $\rightarrow$ freq. synchronization

(2) $\Delta\omega \neq 0 \;\; \Delta\phi = 0, m(t)$ recovery not possible $\rightarrow$ Phase synchronization

(3) Perfect syne, $\Delta\phi = 0, \Delta\omega = 0$ can be recovered

(4) $\Delta\omega = 0, \Delta\phi = \frac{\pi}{2} \rightarrow$ No QNE

**Note:**

(1) When video signal is transmitted through SSB- SC modular VSB- SC is generated.

(2) Synchronous detector can not recover $m(t)$ video signal from the above generated VSB- SC.

## Percentage Power Saved

(1) % power saved in DSB- SC as compare to DSB-FC.

$$\% \ P_{saved} = \frac{P_{saved}}{P_{Total}} \times 100\%$$

$$\% \ P_{saved} = \frac{P_c}{P_c\left[1+\frac{\mu^2}{2}\right]} = \frac{2}{2+\mu^2} = (1-\eta)$$

(2) % power saved in SSB- SC as compare to DSB-FC-

$$\% \ P_{saved} = \frac{4+\mu^2}{4+2\mu^2}$$

(3) % power saved in SSB-SC as compared to DSB-SC.

$$\% \ P_{saved} = 50\%$$

| Modulation | | B.W | Power | Application |
|---|---|---|---|---|
| (1) | DSB-FC | $2f_{\max}$ | $P_C + P_{SB}$ | Broadcatting |
| (2) | DSB- SC | $2f_{\max}$ | $P_{SB}$ | × |
| (3) | SSB-SC | $f_{\max}$ | $\frac{P_{SB}}{2}$ | Point to point voice communication |
| (4) | VSB-SC | $f_{\max} < f < 2f_{\max}$ | $\frac{P_{SB}}{2} < P_{VSB} < P_{SB}$ | Point to point video communication. |

## Pre envelope and Complex Envelope

(1) Pre Envelope calculated for both baseband and bandpass signal.

Let $x(t)$ is real signal.

$$x_+(t) = \text{Pre envelope of } x(t)$$

$$x_+(t) = x(t) + j\,\hat{x}(t)$$

$$\hat{x}(t) = HT\,[x(t)]$$

$$x_+(f) = x(f)[1+\text{sgn}(f)]$$

**Complex Envelope:** For bandpass only but result in low pass only

$x(t) \rightarrow$ Bandpass signal.

**Step-1.** Calculate $x_+(t) = x(t) + j\,\hat{x}(t)$

**Step 2.** $\dfrac{x_c(t) = x_+(t)e^{-j\omega_c t}}{X_c(f) = X_+(f+f_c)}$ left shift of pre envelope by $f_c$

❑❑❑

# 2 ANGLE MODULATION

## 2.1. Introduction

| | Signal = $x(t)$ | $\|x(t)\|_{max}$ |
|---|---|---|
| (1) | $A\cos\omega_0 t + B\cos\omega_0 t$ | $\|A+B\|$ |
| (2) | $A\sin\omega_0 t + B\sin\omega_0 t$ | $\|A+B\|$ |
| (3) | $A\sin\omega_0 t + B\cos\omega_0 t$ | $\sqrt{A^2+B^2}$ |
| (4) | $A\cos\omega_1 t + B\cos\omega_2 t$ | $\|A+B\|$ |
| (5) | $A\sin\omega_1 t + B\sin\omega_2 t$ | $\|A+B\|$ |
| (6) | $A\cos\omega_1 t + B\sin\omega_2 t$ | → $\|A+B\|$ if $A=B$<br>→ $<\|A+B\|$ if $A\neq B$ |

### 2.1.1. Instantaneous Angle and Instantaneous frequency-

$$S(t) = A_c \cos[\theta_i(t)]$$

$\theta_i(t) \rightarrow$ Instantaneous angle (rad)

$$\frac{d\theta_i(t)}{dt} = \omega_i(t) \rightarrow \text{instantaneous angular frequency.}$$

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt} \text{ or } f_i(t) = \frac{\omega_i(t)}{2\pi}$$

$$\theta_i(t) = \int_{-\infty}^{t} \omega_i(t)$$

- **Angle Modulation :**
  - Frequency Modulation
  - Phase Modulation
- **Frequency Modulation :**

  $S_{angle}(t) = A_c \cos[\omega_c t + \Delta\phi(t)]$

If angle Modulation is FM, $\frac{d\Delta\phi(t)}{dt} \propto m(t)$

$$\frac{d\Delta\phi(t)}{dt} = K_f m(t),\ K_f = \text{ frequency sensitivity of frequency modulator}$$

$$\omega_i(t) = \omega_c + K_f m(t) \quad \Rightarrow \quad \omega_i(t) = \omega_c + \underbrace{\Delta\omega(t)}_{\text{frequency deviation}}$$

$$\theta_i(t) = \theta_c + \int_{-\infty}^{t} K_f m(t)\,dt \qquad \Delta\omega(t) = K_f m(t)$$

$$\Delta\omega(t) = \frac{d\Delta\phi(t)}{dt}$$

## Few Important Results

| For Important Results | For $K_f : \frac{\text{rad}}{\text{V-sec}}$ | $K_f : \frac{\text{Hz}}{\text{Volt}}$ |
|---|---|---|
| 1. Instantaneous frequency | $\omega_i(t) = \omega_c t + K_f m(t)$ | $f_i(t) = f_c + K_f m(t)$ |
| 2. Instantaneous frequency deviation | $\Delta\omega(t) = K_f m(t)$ | $\Delta f(t) = K_f m(t)$ Hz |
| 3. Frequency deviation in +ve direction | $[\Delta\omega(t)]_{max} = K_f [m(t)]_{max}$ | $[\Delta f]_{max} = K_f [m(t)]_{max}$ |
| 4. Frequency deviation in –ve direction | $[\Delta\omega(t)]_{min} = K_f [m(t)]_{min}$ | $[\Delta f(t)]_{min} = K_f [m(t)]_{min}$ |
| 5. Maximum value of instantaneous frequency | $[\omega_i(t)]_{max} = \omega_c + [\Delta\omega(t)]_{max}$ | $[f_i(t)]_{max} = f_c + [\Delta f(t)]_{max}$ |
| 6. Minimum value of instantaneous frequency | $[\omega_i(t)]_{min} = \omega_c + [\Delta\omega(t)]_{min}$ | $[f_i(t)]_{min} = f_c + [\Delta f(t)]_{min}$ |
| 7. Peak to peak frequency deviation | $[\Delta\omega]_{p-p} = [\omega_i(t)]_{max} - [\omega_i(t)]_{min}$ | $[\Delta f]_{p-p} = [f(t)]_{max} - [f(t)]_{min}$ |
| 8. Maximum frequency deviation | | |
| 9. Modulation index or deviation ratio of FM | $\lvert\Delta\omega(t)\rvert_{max} = K_f [m(t)]_{max}$ | $\lvert\Delta f(t)\rvert_{max} = K_f [m(t)]_{max}$ |
| $B_{FM} = \frac{\text{Maximum frequency deviation}}{\text{Maximum frequency component of } m(t)}$ | $B_{FM} = \frac{K_f \lvert m(t)\rvert_{max}}{\omega_{max}}$ | $B_{FM} = \frac{K_f \lvert m(t)\rvert_{max}}{f_{max}}$ |

| Important Phase Calculation | $K_f \left(\frac{\text{rad}}{\text{V-sec}}\right)$ | $K_f : \frac{\text{Hz}}{\text{Volt}}$ |
|---|---|---|
| 1. Instantaneous phase deviation in FM | $\Delta\phi(t) = K_f \int_{-\infty}^{t} m(\tau)\,d\tau$ | $\Delta\phi(t) = 2\pi K_f \int_{-\infty}^{t} m(\tau)\,d\tau$ |
| 2. Maximum phase deviation in FM | $\lvert\Delta\phi(t)\rvert_{max} = K_f \left\lvert \int_{-\infty}^{t} m(\tau)\,d\tau \right\rvert$ | $2\pi K_f \left\lvert \int_{-\infty}^{t} m(\tau)\,d\tau \right\rvert_{max}$ |

## General expression for FM

$K_f : \frac{\text{rad}}{\text{V-sec}}$

$$S_{angle}(t) = A_c \cos\left[\omega_c t + \int_{-\infty}^{t} K_f m(\tau)\, d\tau\right]$$

For $K_f : \frac{\text{Hz}}{\text{Volt}}$

$$S_{FM} = A_c \cos\left[\omega_c t + 2\pi K_f \int_{-\infty}^{t} m(\tau)\, d\tau\right]$$

For $m(t) = A_m \cos 2\pi f_m t$ –

$$f_{\max} = f_m, [m(t)]_{\max} = +A_m,\ [m(t)]_{\min} = -A_m,\ |m(t)|_{\max} = A_m$$

$$S_{FM}(t) = A_c \cos\left[\omega_c(t) + B_{FM} \sin(2\pi f_m t)\right]$$

For $m(t) = A_{m_1} \cos 2\pi f_{m_1} t + A_{m_2} \cos 2\pi f_{m_2} t$ –

$$f_{\max} = (f_{m_1}, f_{m_2})_{\max}$$

$$S_{FM}(t) = A_c \cos\left[\omega_c(t) + B_1 \sin 2\pi f_{m_1} t + B_2 \sin 2\pi f_{m_2} t\right]$$

$$B_1 = \frac{K_f A_{m_1}}{f_{m_1}}, B_2 = \frac{K_f A_{m_2}}{f_{m_2}}$$

## Phase Modulation –

$$\Delta\phi(t) \propto m(t)$$

$$\underset{\text{rad}}{\underset{\downarrow}{\Delta\phi(t)}} = K_p \underset{\text{Volt}}{\underset{\downarrow}{m(t)}}$$

$K_p$ : Phase sensitivity of phase modulator

$$K_p = \frac{\text{rad}}{\text{Volt}}$$

## Phase Calculation :

$K_p$ : rad/Volt

$$\theta_i(t) = \omega_c t + K_p m(t)$$

1. Instantaneous phase deviation $= \Delta(t) = K_p m(t)$
2. Maximum phase deviation $= |\Delta\phi(t)|_{\max} = K_p |m(t)|_{\max}$

## Frequency Calculation

$\omega_i(t) = \omega_c + \Delta\omega t$

- $\Delta\omega(t) = K_p \dfrac{dm(t)}{dt}$
- $|\Delta\omega(t)|_{\max} = K_p \left|\dfrac{dm(t)}{dt}\right|_{\max}$
- $|\Delta\omega(t)|_{\min} = K_p \left|\dfrac{dm(t)}{dt}\right|_{\min}$
- $|\Delta\omega_i(t)|_{\max} = \omega_c + [\Delta\omega(t)]_{\max}$
- $|\Delta\omega_i(t)|_{\min} = \omega_c + [\Delta\omega(t)]_{\min}$
- $\Delta\omega_{p-p} = [\omega_i(t)]_{\max} - [\omega_i(t)]_{\min}$
- $|\Delta\omega(t)|_{\max} = K_p \left|\dfrac{dm(t)}{dt}\right|_{\max}$
- $\beta_{FM} = \dfrac{|\Delta\omega(t)|_{\max}}{\omega_{\max}} = \dfrac{K_p \left|\dfrac{dm(t)}{dt}\right|_{\max}}{\omega_{\max}}$

$S_{FM}(t) = A_c \cos\left[\omega_c(t) + K_{PM}(t)\right]$

When $m(t) = A_m \cos 2\pi f_m t$

$|m(t)|_{\max} = A_m,\ f_{\max} = f_m,\ \Delta\omega(t) = -K_p A_m \omega_m \sin\omega_m t$

- $[\Delta\omega(t)]_{\max} = K_p A_m \omega_m$
- $\{\omega(t)\}_{\min} = -K_p A_m \omega_m$
- $[\omega_i(t)]_{\max} = \omega_c + K_p A_m \omega_m,\ [\omega_i(t)]_{\min} = \omega_c - K_p A_m \omega_m$
- $(\Delta\omega)_{p-p} = 2K_p A_m \omega_m$
- $|\Delta\omega(t)|_{\max} = K_p A_m \omega_m$
- $\beta = K_p A_m = |\Delta\phi(t)|_{\max}$

$S_{PM}(t) = A_c \cos\left[\omega_c(t) + \beta_{PM} \cos 2\pi f_m t\right]$

$S_{PM}(t) = A_c \cos\left[\omega_c(t) + \beta_1 \cos 2\pi f_{m_1} t + \beta_2 \cos 2\pi f_{m_2} t\right]$

## Types of FM –

- Narrow Band $(\beta <<< 1)$
- Wide Band

$$S_{FM}(t) = \underset{\text{Carrier}}{\underset{\downarrow}{A_c \cos 2\pi f_c(t)}} + \frac{A_c\beta}{2}\cos[2\pi(\underset{\text{USB}}{\underset{\downarrow}{f_c}} + f_m)t] - \frac{A_c\beta}{2}\cos[2\pi(\underset{\text{LSB}}{\underset{\downarrow}{f_c}} - f_m)t]$$

$$S_{FM}(t) = S_{NBFM}(t)$$

- B.W $= 2f_m$
- $P_{NBFM} = P_C\left(1+\frac{\beta^2}{2}\right) \quad \beta <<< 1, \beta^2 <<< 1$

$$P_{NBFM} \approx P_C = \frac{A_c^2}{2}$$

## Relation between DSB-FC and NBFM –

$$S_{AM}(t) = A_c \cos 2\pi f_c(t) + \frac{A_c\mu}{2}\cos[2\pi(f_c + f_m)t] + \frac{A_c\mu}{2}\cos[2\pi(f_c - f_m)t]$$

$$S_{NBFM}(t) = A_c \cos 2\pi f_c(t) + \frac{A_c\beta}{2}\cos[2\pi(f_c + f_m)t] - \frac{A_c\beta}{2}\cos[2\pi(f_c - f_m)t]$$

1.

| Frequency Component | Strength AM | Strength NBFM |
|---|---|---|
| $f_c$ | $\frac{A_c}{2}$ | $\frac{A_c}{2}$ |
| $f_c + f_m$ | $\frac{\mu A_c}{4}$ | $\frac{\beta A_c}{4}$ |
| $f_c - f_m$ | $\frac{\mu A_c}{4}$ | $\frac{-\beta A_c}{4}$ |

2. $S_{NBFM}(t) + S_{AM}(t) = \text{SSB-SC} \rightarrow \text{USB-FC}$

   $S_{AM}(t) - S_{NBFM}(t) = \text{SSB-SC} \rightarrow \text{LSB-FC}$

3. LSB in NBFM is $180^0$ inverted w.r.t to LSB in AM

- $S_{FM}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + nf_m)t]$

  For aby value of $\beta$

- $J_n(\beta) = (-1)^n J_n(\beta)$

- $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$
- $J_0(\beta) = 0, \beta = 2.4, 5.5, 8.6, 11.8$
- as $n \uparrow, \rightarrow J_n(\beta) \downarrow$

  $\beta <<< 1 : S(t) \rightarrow$ 1 Carrier + 2 SB NB Angle Modulation

  If $\beta >>> 1 : S(t)$ :1 Carrier + Infinite SB Wide Band Angle Modulation

  Ideal BW of WBFM $= \infty$

**Carson's Rule –**

$$BW = (\beta + 1) 2 f_m \quad \beta_{PM} \text{ for PM}$$

$\beta_{FM}$ for FM

- Power of Carrier before modulation $= \frac{A_c^2}{2} = P_c$
- Power of Carrier after modulation $P = P_c \left[ J_0^2 \beta + 2(J_1^2(\beta) + J_2^2(\beta) + \ldots \right]$
- Power of Carrier component in modulated signal $= P_c J_0^2(\beta)$
- $P_{SB} = 2P_c \left[ J_1^2(\beta) + J_2^2(\beta) + \ldots \right]$
- $\eta = \frac{P_{SB}}{P_{Total} \rightarrow P_c \left[ J_0^2(\beta) + 2(J_1^2(\beta) + J_2^2(\beta) + \ldots) \right]}$

  If $J_0(\beta) = 0$ then $\eta = 100\%$
- For Infinite sidebands $P_{WB} = P_c$

**For Non sinusoidal –**

$$S_{FM} = A_c \sum_{n=-\infty}^{\infty} |C_n| \cos\left[2\pi(f_c + nf_m)t + \angle C_n\right]$$

| $m(t)$ | | BW |
|---|---|---|
| Singletone sinusoidal | $\longrightarrow$ | $(\beta + 1) 2 f_m$ |
| Non sinusoidal periodic signal | $\longrightarrow$ | $(\beta + 1) 2 f_m$, $f_m$ = fundamental frequency |
| Other Cases | $\longrightarrow$ | $(\beta + 1) 2 f_{\max}$ |

$\text{BW} = (1 + \beta) 2 f_{\text{m}}$ or $2(\Delta f + f_{\max})$

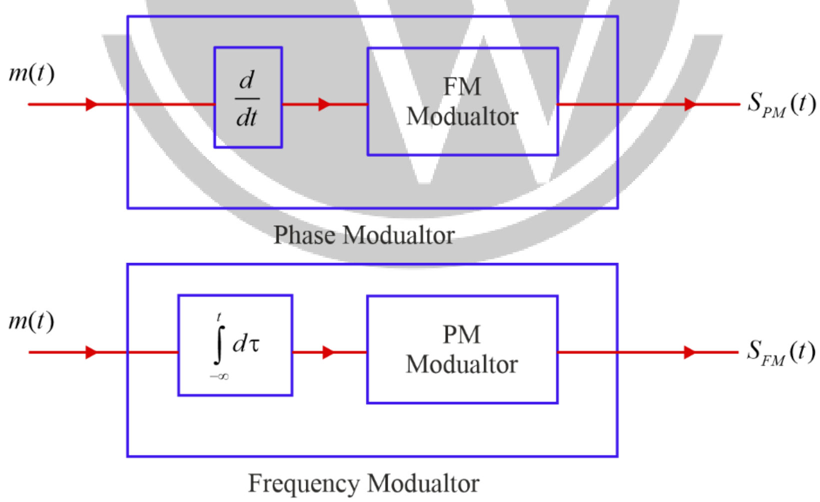
## Frequency Mixture and Multiplier

| Mixture/Multiplier Input | Mixture Output | (Multiplied by *n*) Multiplier Output |
|---|---|---|
| $A_c$ | $A_c'$ | $A_c'$ |
| $f_c$ | $\lvert f_C - f_L \rvert$ or $f_C + f_L = f_C'$ | $nf_c$ |
| $\beta$ | $\beta$ | $n\beta$ |
| $f_m$ | $f_m$ | $f_m$ |
| $\Delta f$ | $\Delta f$ | $n\Delta f$ |
| BW | BW | $(n\beta+1)2f_m$ |
| Spactral spacing | $f_m$ | $f_m$ |
| Frequency components | $f_c', f_c' \pm f_m, f_c' \pm 2f_m$ | $nf_c, nf_c \pm f_m, nf_c \pm 2f_m$ |

## Wideband Angle Modulation generation –

$$PM[m(t)] = FM\left[\frac{dm(t)}{dt}\right] \text{ If } K_p = K_f = K$$

$$FM[m(t)] = PM\left[\int_{-\infty}^{t} m(\tau)\,d\tau\right]$$



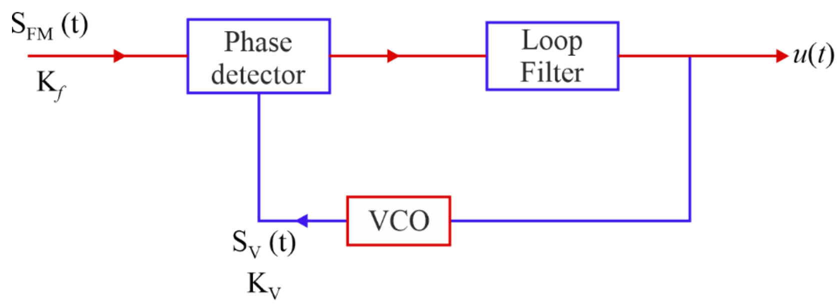

## Wideband FM Generation Methods

1. Armstrong Method (Indirect Method)
2. Direct Method
   - VCO (Voltage Controlled Oscillator is used). It is modified version of Hartley oscillator

$$\frac{\Delta\omega}{\omega_c} = \frac{\Delta C}{2C_0}$$

## FM Demodulator

1. Theoretical method
2. Practical method. PLL (Phase Locked Loop)




(1) $v(t) = \frac{K_f}{K_V} m(t)$

(2) Lock mode $\longrightarrow$ Frequency lock

Capture mode $\longrightarrow$ Phase lock

(3) $\text{L.R} \geq \text{C.R}$

## Super Hetrodyne Receiver

$f_L$ = Local oscillator frequency

$f_S$ = Desired frequency

$f_{Si}$ = Frequency of image station

**Case 1 :** If relation between $f_l$ and $f_s$ is not mentioned.

Assume : $f_l > f_s$

1. $f_l = f_s + IF$
2. $f_{Si} = f_l + IF$
3. $f_{Si} = f_s + 2IF$

**Case 2 :** When relation between $f_i$ and $f_s$ is given

If $f_{Si} < f_l < f_s$ If $f_S < f_s < f_{sl}$, then Case 1

then 1. $f_s = f_l + IF$

2. $f_l = f_{Si} + IF$

3. $f_s = f_{Si} + 2IF$

## Image Rejection Ratio

$$\text{IRR} = \sqrt{1 + P^2Q^2}$$

$Q$ : Quality factor of Oscillator

$$P = \frac{f_{Si}^2 - f_s^2}{f_{Si} f_s} \quad f_{Si} > f_s \quad P^2Q^2 >>> 1$$

$$\text{IRR} = PQ$$

# 3 Random Variable and Random Process

## 3.1. Introduction

**Random variable → Real and complex**

- R.V. is a function performing mapping from sample space of R.E. to real line.
- $X(\lambda)$: Random variable
- Domain of R.V. $\rightarrow \lambda$ (Sample point)
- Range of R.V. $\rightarrow$ Subset of real line
- One to one or many to one mapping
- $P\{X \le a\} \rightarrow$ Probability of set in which all the comes satisfy $x(\lambda) \le a$.

**CDF of R.V.**

Let random variable X, $x \rightarrow$ Values taken by R.V.

(1) $F_X(x) = P\{X \le x\} = 1 - P\{X > x\}$

(2) $F_X(a) = P\{X \le a\} = 1 - P\{X > a\}$

(3) $F_{|X|}(y) = P\{|X| \le y\} = P\{-y \le X \le y\}$

**Properties**

(1) $F_X(\infty) = 1$

(2) $F_X(-\infty) = 0$

(3) $F_X(\infty) + F_X(-\infty) = 1$

(4) $F_X(x) = P\{X \le x\} \Rightarrow 0 \le F_X(x) \le 1$

(a) CDF always non negative.

(b) Lower bound: $F_X(x) = 0$, upper Bound = 1

(5) CDF is monotonically non decreasing function of $x\left(\dfrac{dF_X(x)}{dx} \ge 0\right)$

(6) Graph of CDF is always amplitudes continuous from right.

- **Key point :**

(1) $P\{a < x \le b\} = F_X(b^+) - F_X(a^+)$

(2) $P\{a \le X \le b\} = F_X(b^+) - F_X(a^-)$

(3) $P\{a < X < b\} = F_X(b^-) - F_X(a^+)$

(4) $P\{a \le X < b\} = F_X(b^-) - F_X(a^-)$

$P(x = a)$

- CDF is continuous at $x = a$
  - $P\{x = a\} = F_X(a^+) - F_X(a^-)$
  - $P\{x = a\} = 0$
- CDF is continuousat $x = a$
  - $P\{x = a\} = F_X(a^+) - f_X(a^-)$
  - $\neq 0$
  - $P\{x = a\} =$ Size of Jump

## Probability Density Function

Random variable $X$

$x \rightarrow$ Variable taken by R.V.

$f_X(x) \rightarrow$ Symbol

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$f_X(x) = \int_{-\infty}^{x} f_X(x)\, dX$$

$$f_X(x) = \int_{-\infty}^{a} f_X(\lambda)\, d\lambda$$

## Properties :

(1) $f_X(x) \ge 0 \rightarrow$ Non negative

(2) $0 \le f_X(x) < \infty \longrightarrow$ Upper bound; $0 \downarrow$ Lower bound

(3) $F_X(\infty) = \int_{-\infty}^{\infty} f_X(x)\, dx = 1$

(4) Graph of PDF can be even or NENO but cannot be odd.

(5) $P\{-\infty < X \le x\} = \int_{-\infty}^{x} f_X(\lambda)\, d\lambda$

(6) $P\{a < X \le b\} = \int_{a^+}^{b^+} f_X(x)\, dx$

(7) $P\{a \le X \le b\} = \int_{a^-}^{b^+} f_X(x)\,dx$

(8) $P\{a < X < b\} = \int_{a^+}^{b^-} f_X(x)\,dx$

$P(x = a)$

- Graph having impulse at $x = a$
  - $P\{x = 0\} = F_X(a^+) - F_X(a^-)$
  - $P\{x = a\} = \int_{a^-}^{a^+} f_Y(x)\,dx$
  - = Total area
- No impulse at $x = a$
  - $P\{x = a\} = \int_{a^-}^{a^+} f_Y(x)\,dx$
  - $P\{x = a\} = 0$

## Discrete Random Variable:

(1) PDF should have impulses only.

(2) CDF should have staircase only.

(1) **Probability mass function of DRV :** Let $X$ is D.R.V.

$P_X(x) = P(X = x)$ probability such that $X = x$

➢ $0 \le P_X(x) = 1$

➢ $\sum_x P_X(x) = 1$

(2) **PDF of a D.R.V :** Let X is O.R.V.

$$f_X(x) = \sum_i P_X(x_i)\delta(x - x_i) = \sum_i P(x = x_i)\delta(x - x_i)$$

(3) **CDF of a D.R.V. :** Let X is D.R.V.

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$F_X(x) = \sum_x P\{X = x_i\}u\{x - x_i\}$$

(4) $P\{X = a\}$ may or may not be zero.

## Continuous Random Variable

Maps sample point to continuous range of values on real axis.

(1) PDF of C.R.V should not contain impulses at all.

(2) CDF of C.R.V

➢ Should not contain jump type discontinuity

➢ It should be amplitude continuous every where

(3) PMF not defined for C.R.V because for CRV $P\{X = a\}$ will always be zero.

$$P(A/B)=\frac{P(A\cap B)}{P(B)}\rightarrow \text{Conditional probability of A given B.}$$

$$P(A\cap B)=P(B)P(A/B)=P(A)P\left(\frac{B}{A}\right)=\text{Joint probability.}$$

**Expectation operator :** Performs operations on R.V. only.

## Linear Operator

$$E[C]=C,\ E[C^2]=C^2 \qquad E(X)=\begin{cases}\int_{-\infty}^{\infty} xf_X(x)\,dx & X:CRV\\ \sum_i x_i P\{X=x_i\} & X:DRV\end{cases}$$

$$E[aX]=aE[X]$$

$$E[aX+b]=aE[X]+E[b]$$

$$E[ag(X)+bH(y))=aE[g(x)]+bE[H(y)]$$

$$E[X^2]=\int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$E[g(x)]=\int_{-\infty}^{\infty} g(x)\,f_X(x)\,dx$$

## Gaussian Random Variable

CRV X is having Gaussian or random distribution.

X is having Gaussian PDF, X is called G.R.V.

$$E[X]=\mu_X,\ E[(X-\mu_x)^2]=\text{Variance }=\sigma_X^2$$

$$X\sim N\{\mu_X,\sigma_X^2\} \qquad f_X(x)=\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}} \qquad -\infty<x<\infty$$

**Key Point :**

(1) $$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}}dx=1$$

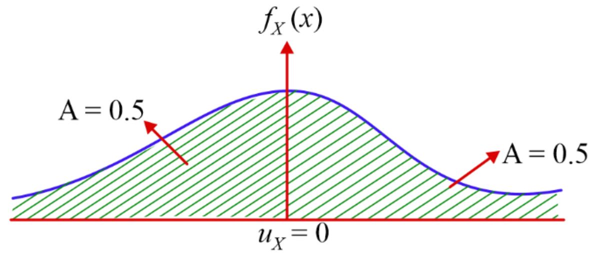
(2) $$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}}dx=\mu_X=E[X]$$

(3) $$\int_{\mu_X}^{\infty}\frac{1}{\sqrt{2\pi\sigma_X^2}}e\frac{-(x-\mu_X)^2}{2\sigma_X^2}dx=\int_{-\infty}^{\mu_X}\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}}dx=\frac{1}{2}$$

## Zero mean Gaussian distribution-

$$X \neg N(\mu_{X,}\sigma_X^2) \Rightarrow X \sim N[0,\sigma_X^2] \Rightarrow E[X]=0$$

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma_X^2}}e^{\frac{-x^2}{2\sigma_X^2}}$$




**Zero Mean, unit variance :**

$$X \sim N(0.1) \quad f_y(x)=\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}, \quad \int_0^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx=\frac{1}{2}$$

Q- function :

$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-z^2/2}dz \qquad \text{as } x\uparrow, Q(x)\downarrow$$

$$Q(\infty)=0,\ Q(-\infty)=1,\ Q(0)=0.5,\ Q(x)+Q(-x)=1$$

$$P[X>z]=Q(P)=Q\left[\frac{z-\mu_X}{\sigma_X}\right]$$

$$f_X(z)=P(X\le z)=1-P(X>z)=1-Q\left[\frac{z-\mu_X}{\sigma_X}\right]$$

## Statistical averages of a R.V.

$n^{th}$ order moment about origin-

$$E[(X-0)^n]=E[X^n]=\begin{cases}\int_{-\infty}^{\infty}x^n f_X(x)dx & X:CRV\\ \sum_i x_i^n P\{X=x_i\} & X:DRV\end{cases}$$

### 1st order moment about origin

$$E[X]=\int_{-\infty}^{\infty}xf_X(x)dx=\sum_i x_i P\{X=x_i\}$$

$$E[X]=\bar{X}=\mu_X=m_i \rightarrow dc \text{ value, avg. value Mean value}$$

$[E[X]]^2 \rightarrow d.c.$ power

## $2^{nd}$ order moment about origin-

$$E[(X-0)^2]=E[X^2]=\overline{X}^2\begin{cases}\int_{-\infty}^{\infty}x^2 f_X(x) & X:CRV\\ \sum_i x_i P\{X=x_i\} & X:DRV\end{cases}$$

$E[X^2]=$ Mean square value of R.V. X = Total power of R.V. $x$

$1^{st}$ order moment about mean - $E[(X-\mu_X)]=0$

$2^{nd}$ order moment about mean $-E[(X-\mu_X)^2]=E[X^2]-\mu_X^2$

$$\sigma_X^2 = E[X^2]-u_X^2$$

$\sigma_X^2$ → A.C. Power; $E[X^2]$ → Total Power; $u_X^2$ → dc Power

**Important point:**

(1) $\sigma_X^2 \geq 0, E[X^2]\geq \mu_X^2$

(2) If X is zero mean R.V.

$$E[X^2]=\sigma_X^2, \text{MSV}(X)=Var(X)$$

(3) Standard deviation

$$\sqrt{\text{Variance}}=\sqrt{\sigma_X^2}=\pm\sigma_X$$

(4) $Y=aX+b$

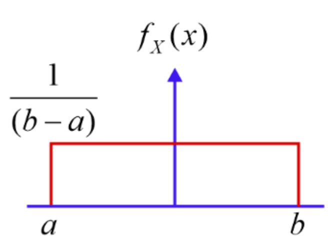
$$E[Y^2]=a^2E[X^2]+b^2+2ab\,E[X]$$

$$\sigma_Y^2=a^2\sigma_X^2$$

## Standard Distribution of R.V.

(1) Uniform distribution $X\sim U[a,b]$

$$f_X(x)=\begin{cases}\dfrac{1}{(b-a)} & a\leq X\leq b\\ 0 & \text{otherwise}\end{cases}$$

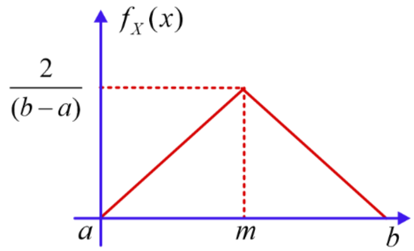


$$E[X]=\frac{a+b}{2},\ E[X^2]=\frac{a^2+b^2+ab}{3},\quad \sigma_X^2=\frac{(b-a)^2}{12}$$

(2) Triangular distribution

$X\sim$ tri $(a,m,b)$

$$E[X]=\frac{a+m+b}{3}$$




(3) Rayleigh Distribution

$X\to CRV$

$$f_X(x)=\begin{cases}\frac{x}{\sigma_X^2}e^{\frac{-x^2}{2\sigma_X^2}} & x\ge 0\\ 0 & \text{else}\end{cases}$$

$$\int_0^\infty \frac{x}{\sigma_X^2}e^{\frac{-x^2}{2\sigma_X^2}}dx=1$$

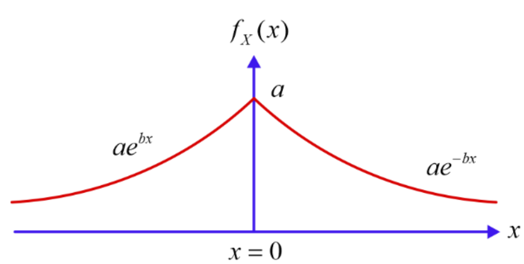
If X and Y are two G.R.V. Then $Z=\sqrt{X^2+Y^2}$ will have reyleigh distribution.

**(4) Exponential Distribution :** If CRV has exponential distribution then it will have PDF

$$f_X(x)=\begin{cases}\lambda e^{-\lambda x} & x\ge 0\\ 0 & x>0\end{cases}\qquad \int_0^\infty \lambda e^{-\lambda x}dx=1$$

Laplacian Distribution

$X\to CRV$




$$f_X(x)=ae^{-b|x|}\quad -\infty<x<\infty$$

If $\frac{2a}{b}=1,\quad a>0,b>0$

$$f_X(x) = \begin{cases} ae^{bx} & x < 0 \\ ae^{-bx} & x > 0 \end{cases}$$

## Discrete Random variable-Binomial, Position distribution

### Binomial distribution necessary condition:-

(1) The no of trials $n$ showed be finite.

(2) Trials are independent

(3) Each trials should result in 2 outcomes success or failure.

(4) Prob of success in each trial should be constant.

**PMF:** $$P\{X = r \text{ success}\} = n_{c_r} p_r q^{n-r}$$

$$E[X] = \sum_i x_i p\{X = x_i\} = n_p \quad \sigma_X^2 = npq$$

$$E[X^2] = npq + (np)^2$$

Std deviation $\sigma_X = \pm\sqrt{npq}$

## Position Distribution

Specific type of binomial distribution where $n \to \infty$

$$n \to \text{very large}, \; p \to \text{very small}, \; np \to \text{ finite } \lambda = np$$

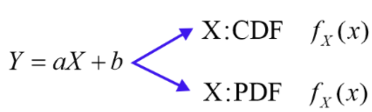
$$p\{X = r\} = \frac{\lambda^r e - \lambda}{r!} \text{ probability of } X = r \text{ (success)}$$

$$E[X] = \lambda, \sigma_X^2 = \lambda$$


If $Y = g(X)$ is having monotonic $T_X$ .

Given $X \xrightarrow{PDF} f_X(x)$,

$$f_Y[y] = \left\{ f_X[x] \left| \frac{dx}{dy} \right| \right\} \text{function of y}$$




(1) case $a > 0$

$$F_Y[y] = F_X\left(\frac{y-b}{a}\right), f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

(2) $Y = -aX + b \quad a > 0$

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{-a}\right),\ f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{-a}\right)$$

**Monotonic linear $T_X$:**

$$y = aX + b$$

$$X \sim U[m_1, m_2] \to Y \sim U[am_1 + b, am_2 + b]$$

$$X \sim \Delta[m_1, m_2, m_3] \to Y \sim \Delta[am_1 + b, am_2 + b, am_3 + b]$$

$$X \sim N[\mu_X, \sigma_X^2] \to Y \sim N[\mu_Y, \sigma_y^2]$$

$$Y \sim N[a\mu_X + b, a^2\sigma_X^2]$$

**Monotonic Non-Linear $T_X$:**

$$X \to f_X(x)$$

$$Y \to X^3,\ f_Y(y) = ?$$

$$f_Y(y) = \left\{ f_X(x)\left|\frac{dx}{dy}\right| \right\}$$

$$f_Y(y) = \frac{1}{3y^{2/3}} f_X(y^{1/3})$$

**Non - Monotonic $T_X$:**

$$Y = y, g(X) = y, X = g^{-1}(y)$$

$$\begin{cases} \to x_1 \\ \to x_2 \\ \to x_3 \end{cases}$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + ----$$

**2D Random variable :**

$$(X, Y) \to 2\,\text{DR.V.} \begin{cases} \to F_{X,Y}(x, y) = \text{Joint CDF} \\ \to f_{X,Y}(x, y) = \text{Joint PDF} \\ \to P_{XY}(x_i, y_i) = \text{Joint PMF} \end{cases}$$

If A and B are independent

$$P\left(\frac{A}{B}\right) = P(A),\ \ P\left(\frac{B}{A}\right) = P(B),\ \ P(A \cap B) = P(A)P(B)$$

$$F_{X,Y}(x,y) = F_X(x)F_{\frac{Y}{X}}\left(\frac{y}{x}\right) = F_Y(y)F_{\frac{X}{Y}}\left(\frac{x}{y}\right)$$

Marginal CDF of $X$ — $F_X(x)$

Conditional CDF of $Y$ given $X$ — $F_{\frac{Y}{X}}\left(\frac{y}{x}\right)$

Marginal CDF of $Y$ — $F_Y(y)$

Conditional CDF of $X$ given $Y$ — $F_{\frac{X}{Y}}\left(\frac{x}{y}\right)$

### If X and Y are independent R.V.

$$F_{XY}(x,y) = f_X(x) f_Y(y)$$

- $f_{XY}(x,y) = f_X(x) f_{\frac{Y}{X}}\left(\frac{y}{x}\right) = f_Y(y) f_{\frac{X}{Y}}\left(\frac{x}{y}\right)$

If X and Y are independent R.V.

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

- $P_{XY}(x_i, y_j) = P_X(x_i) P_{\frac{Y}{X}}\left(\frac{y_j}{x_i}\right) = P_Y(y_j) P_{\frac{X}{Y}}\left(\frac{x_i}{y_j}\right)$

If $X$ and $Y$ are independent R.V.

$$P_{XY}(x_i, y_j) = P_X(x_i) P_Y(y_j)$$

Joint CDF = Let $(X, Y)$ are BIVARIATE R.V.

$$F_{XY}(x,y) = P\{X \le x) \cap (Y \le y) = P\{X \le x; Y \le y\}$$

### Properties:

(1) $0 \le F_{XY}(x,y) \le 1$

(2) $F_{XY}(-\infty, y) = P\{(X \le -\infty) \cap (Y \le y)\} = 0$

(3) $F_{XY}(x, -\infty) = 0$

(4) $F_{XY}(-\infty, -\infty) = 0$

(5) $F_{XY}(\infty, \infty) = 1$

(6) $F_{XY}(x_1, y_1) = P\{(X \le x_1) \cap (Y \le y_1)\}$

(7) $P\{(x_1 < X \le x_2) \cap (y_1 < Y \le y_2)\}$

$$= F_{XY}(x_1^+ . y_1^+) + F_{XY}(x_2^+, y_2^+) - F_{XY}(x_1^+, y_2^+) - F_{XY}(x_2^+, y_1^+)$$

(8) $F_{XY}(x,y) = F_X(x) F_{\frac{Y}{X}}\left(\frac{y}{x}\right) = F_Y(y) F_{\frac{X}{Y}}\left(\frac{x}{y}\right)$

(9) $X$ and $Y$ are independent R.V.

$$F_{XY}(x,y) = F_X(x) F_Y(y)$$

(10) $F_X(x,y) = F_{XY}(x,\infty), \quad F_Y(y) = F_{XY}(\infty, y)$

## Conditional CDF

$$F_{\frac{X}{Y}}\left(\frac{x}{y}\right) = \frac{F_{XY}(x,y)}{F_Y(y)} \quad \text{of} \quad F_Y(y) \neq 0$$

$$F_{\frac{X}{Y}}\left(\frac{x}{y}\right) = \frac{P[(X \le x) \cap (Y \le y)]}{P[(X \le \infty) \cap (Y \le y)]}$$

## Joint PDF

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial X \partial Y}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv$$

$$F_{XY}(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

## Marginal PDF

(1) $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy, \;\; f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$

If X and Y are independent $f_{XY}(x,y) = f_X(x)\, f_Y(y)$

$$f_{XY}(x,y) = f_X(x)\, f_{\frac{Y}{X}}\left(\frac{y}{x}\right) = f_Y(y)\, f_{\frac{X}{Y}}\left(\frac{x}{y}\right)$$

## Conditional PDF

$$f_{XY}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)dxdy}{\int_{-\infty}^{\infty} f_{XY}(x,y)dx}$$

## Probability Calculation in 2-D region

### Given Joint PDF

$R_2$
Region in which PDF if defined

$$f_{XY}(x,y) = \begin{cases} \text{define} & [(x_1 < X \le x_2) \cap (y_1 < Y \le y_2)] \\ 0 & \text{else} \end{cases}$$

$P\{(a < X \le b) \cap (c < y \le d)\} = ?$

$R_1$ : Region in which probability has to be calculated.

## Method:

$$P(X, Y \in R_1) = \iint_R f_{XY}(x, y)dxdy \qquad (R = R_1 \cap R_2)$$

(1) X and Y not independent R.V.

$$P(X, Y \in R_1) = \iint_{R_r} f_{X,Y}(x) f_Y(Y)dxdy \qquad R = (R_1 \cap R_2)$$

(Central Limit Theorem)

If $X$ and $Y$ are D.R.V

$$\sum_i \sum_j P_{XY}(x_i, y_j) = 1$$

$$P_{XY}(x_i, y_j) = P\{(X = x_i) \cap (Y = y_j)\}$$

Joint PMF

## Marginal PMF :

$$P_X(x_i) = \sum_j P_{XY}(x_i, y_j)$$

$$P_Y(y_j) = \sum_i P_{XY}(x_i, y_j)$$

## Minimum of 2 independent R.V.

X, Y are two I.R.V

$$\min(X, Y) > Z = (X > Z) \cap (Y > Z)$$

$$P[\min(X, Y) > Z] = P[X > Z]P[Y > Z] = \iint_R f_{XY}(x, y)dxdy$$

$$P[\min(X, Y) \le Z] = 1 - P[\min(X, Y) > Z]$$

$$\underbrace{P[\min(X, Y) > Z]}_{R} = \iint_R f_X(x) f_Y(y)dxdy$$

Let $Z = \text{Max }(X, Y) \rightarrow$ R.V.

CDF of Z $\quad F_Z(Z) = F_X(Z) \cdot F_Y(Z)$

PDF of Z $\quad f_Z(Z) = F_X(Z) f_Y(Z) + F_Y(Z) f_X(Z)$

Let $Z = \min[X, Y] \rightarrow$ R.V.

CDF of Z $\quad F_Z(Z) = f_X(Z) + g_Y(Z) + f_Y(Z)F_X(Z)$

PDF of Z $\quad f_Z(Z) = f_X(Z) + f_Y(Z) - F_X(Z) f_Y(Z) - F_Y(Z) f_X(Z)$

## Statistical parameters of 2D R.V.

(1) $(k, r)^{th}$ order joint moment about origin $E[X^k Y^r]$ $(1,1)^{st}$ order joint moment about origin.

$E[X^1, Y^1] = E[XY] = R_{XY} \rightarrow$ Cross correlation between R.V. X and V.

- $E[XY] = R_{XY} = 0 \rightarrow$ R.V. X and Y are orthogonal.

(2) $(k,r)^{th}$ order joint moment about mean-

$E[(X-\bar{X})^k(Y-\bar{Y})^r]$

$(1,1)^{st}$ order joint Moment about mean-

$E[(X-\bar{X})(Y-\bar{Y})] = E[XY] - \bar{X}\bar{Y} = \text{cov}(X,Y)$

$\text{cov}(X,Y) = \sigma_{XY} = E[XY] - E[X]E[Y] = R_{XY} - \mu_x \mu_y$

When 2 R.V. X and Y are uncorrelated-

$\text{cov}(X,Y) = 0$, $E[X,Y] = E[X]E[Y]$

- $E[X^k Y^r] = E[X^k]E[Y^r] = X, Y$ are independent.
- If 2 R.V. are independent then they has to be uncorrelated but converse is not necessarily true.

## One function of two R.V.

$$W = aX + bY$$

(1) $E[W] = aE[X] + bE[Y]$

(2) $E[W^2] = a^2 E[X^2] + b^2 E[Y^2] + 2ab\, R_{XY}$

(3) $\sigma_W^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\text{cov}(X,Y)$

One function of Three R.V. $W = aX_1 + bX_2 + cX_3$

(1) $E[W] = a\mu_{X_1} + b\mu_{X_2} + a\mu_{X_3}$

(2) $E[W^2] = a^2 X_1^{\,2} + bX_2^2 + c^2 X_3^2 + 2abX_1X_2 + 2bcX_2X_3 + 2caX_1X_3$

(3) $\sigma_W^2 = a^2\sigma_{X_1}^2 + b^2\sigma_{X_2}^2 + c^2\sigma_{X_3}^2 + 2ab\,\text{cov}(X_1, X_2) + 2bc\,\text{cov}(X_2, X_3) + 2ca\,\text{cov}(X_1, X_3)$

Var (X + Y) = Var (X – Y)

Only when X, Y are $\rightarrow$ uncorrelated and independent

## Correlation coefficient

$$\rho(X,Y) = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{\text{cov}(X,Y)}{(\text{Std.dev.of } X) \times (\text{std.dev.of } Y)}$$

- $-1 \le \rho \le 1$
- $\rho(X,X) = 1, \quad \rho(X,-X) = -1$
- $X$, $Y$ are independent $\rho(X,Y) = 0$

Let $X$, $Y$ are two R.V.

$$E\left[\frac{g(Y)}{X=x}\right]=\int_{-\infty}^{\infty} g(y)\,f_{\frac{Y}{X}}\left(\frac{y}{x}\right)dy$$

$$E\left[\frac{g(X)}{Y=y}\right]=\int_{-\infty}^{\infty} g(x)\,f_{\frac{X}{Y}}\left(\frac{x}{y}\right)dx$$

## Calculation of probability in n-D region

### Theorem -1

If $X_1, X_2, X_3 --- X_n$ are statistically independent random variables.

Let $Z = X_1 + X_2 ---- X_n$

$$\underset{f_{X_1}(z)}{\uparrow} \quad \underset{f_{X_2}(z)}{\uparrow} \qquad\qquad \underset{f_{X_n}(z)}{\uparrow}$$

$$f_Z(z)=f_{X_1}(z)*f_{X_2}(z)----*f_{X_n}(z)$$

When and only when all the R.V. are statistically independent.

- R.V. are linearly combined.

### Theorem-2

$X_1, X_2, X_3 --- X_n$ are statically independent non Gaussian R.V.

$Z = X_1 + X_2 + X_3 ---- X_n$

$f_Z(z)=f_{X_1}(z)*f_{X_2}(z)----*f_{X_n}(z)$

If $n\to\infty$ $f_Z(z)=$ Gaussian irrespective of nature of $(X_i)_{i=1}^{n}$

### Theorem-3

$X_1, X_2, X_3 --- X_n$ are statistically independent G.R.V.

$Z = X_1 + X_2 --- + X_n$

$f_Z(z)=f_{X_1}(z)+f_{X_2}(z)+----f_{X_n}(z)$

$n \to$ Finite| infinite, $\to$ Z: GRV

## Problem Solving Technique :

**Case 1 :** $X_1, X_2 ---- X_n$ are statistically independent G.R.V.

$$P[X_1+X_2+X_3>a]=\underset{\text{Non GRV}}{\underset{\downarrow}{P(Z>a)}}=\int_{a}^{\infty} f_Z(z)dz=1-\int_{-\infty}^{a} f_Z(z)dz$$

Where $Z = X_1 + X_2 + X_3$

$$f_Z(z) = f_{X_1}(z) \times f_{X_2}(z) \times f_{X_3}(z)$$

**Case 2 :** If $X_1, X_2, X_3$ are statistically independent G.R.V.

$$P(X_1 + X_2 + X_3 > a) = P[Z > a] = Q\left[\frac{a - \mu_z}{\sigma_z}\right]$$

$$Z = X_1 + X_2 + X_3$$

$$\mu_2 = \mu_{X_1} + \mu_{X_2} + \mu_{X_3}, \quad \sigma_z^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2$$

> **Note :** If $X_1, X_2, X_3 ---- X_n$ are I.I.D. random variables
>
> P (one of them is largest) $= \frac{1}{n}$
>
> P (one of them is smallest) $= \frac{1}{n}$

## Random Process

$X(\lambda, t) = \{X(\lambda_1, t), X(\lambda_2, t)\}$ $\longrightarrow$ Collection of sample function of Euemble of sample function

$\downarrow$

Random process or Random signal or stochastic signal

$X(\lambda_1, t_1) \rightarrow$ sample value, values taken by R.V. When R.P. is observed at $t = t_1$

C.T.R.P $\rightarrow$ It maps the sample points onto continuous time sample function, collection of continuous time sample function.

$$\underset{\text{C.T.R.P}}{X(t) = A\cos(\omega_0 t + \phi)} \xrightarrow{t = t_1} \underset{\text{R.V., D.R.V.}}{X(t_1) = A\cos(\omega_0 t_1 + \phi)}$$

t $\rightarrow$ Continuous time, A $\rightarrow$ Constant, $\omega_0 \rightarrow$ constant

$\phi \sim U[-\pi, \pi] \rightarrow$ CRV

Any typical R.P can be understood as $x(t) = f(t, \phi)$

(Function of time and R.V.) $x(n) = f(n, \phi)$

## Statistical parameter of R.P.

**Case 1 :** $X(t) \xrightarrow[t = t_0]{} X(t_0) - \text{CRV}$

$$E[X(t_0)] = \int_{-\infty}^{\infty} x\, f_{X(t_0)}(x)\, dx$$

$$E[x^2(t_0)] = \int_{-\infty}^{\infty} x^2\, f_{X(t_0)}(x)\, dx, \qquad \sigma_{X(t_0)}^2 = E[X^2(t_0)] - 9E(X(t_0))^2]$$

**Case 2 :** $\underset{\text{CTRP}}{X(t)} \xrightarrow[t = t_0]{} X(t_0) - \text{DRV}$

$$E[X(t_0)] = \sum_i x_i P_{X(t_0)}(x_i) = \sum_i x_i P\{X(t_0) = x_i\}$$

$$E[x^2(t_0)] = \sum_i x_i^2 P_{X(t_0)}(x_i) = \sum_i x_i^2 P\{X(t_0) = x_i\}$$

$$\sigma^2_{X(t_0)} = E[X^2(t_0)] - (E[X[t_0])^2$$

**Case 3 :** DTRP $\rightarrow$ CRV

$E[X(n_0)], E[x^2(n_0)], \sigma^2_{X(n_0)}$ $\rightarrow$ Some as case 1, replace to by $n_0$

**Case 4 :** DTRP $\rightarrow$ DRV

$E[X(n_0)], E[x^2(n_0)], \sigma^2_{X(n_0)}$ $\rightarrow$ Same as case-2, Replace to by $n_0$

## CTR.P

$X(t) \xrightarrow{t=t_1} X(t_1)$

$X(t) \xrightarrow{t=t_2} X(t_2)$

$$E[X(t_1)X(t_2)] = R_{X(t_1)X(t_2)} = R_{XX}(t_1, t_2)$$

Auto correlation of RP X(t)

$X(t) \rightarrow X(t)$

$X(t) \rightarrow X(t+\tau)$

Then $E[X(t_1)X(t+\tau)] = R_{XX}(t, t+\tau)$

Cov $(X(t_1)X(t_2)) = E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)]]$

$\sigma_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_{X(t_1)}\mu_{X(t_2)}$

Auto covariance of R.P. $X(t)$

$\sigma_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - \mu_{X(t)}\mu_{X(t+\tau)}$

## Cross Correlation

$$\underset{R.P}{X(t)} \xrightarrow[t=t_1]{} \underset{R.V}{X(t_1)},\ \underset{R.P}{Y(t)} \xrightarrow[t=t_2]{} \underset{R.V}{Y(t_2)}$$

$$E[X(t_1)Y(t_2)] = R_{XY}(t_1, t_2)$$

(1) If $R_{XY}(t_1, t_2) = 0\ \forall\ t_1 \in \text{TR}$ $X(t)$ and $Y(t)$ $R.P.$ will

$t_2 \in \text{TR}$ Become orthogonal.

(2) $\text{Cov}[X(t_1), Y(t_2)] = R_{XY}(t_1, t_2) - \mu_{X(t_1)}\mu_{X(t_2)}$

$= 0\ \forall\ t_1 \in \text{TR}$

$t_2 \in \text{TR}$

RP $X(t)$ and $Y(t)$ are uncorrelated.

If $X(t_1)$ and $X(t_2)$ are independent

$$E[X(t_1)X(t_2)]\begin{cases} E[X(t_1)E[X(t_2)] & t_1 \neq t_2 \\ E[X^2(t_1)] & t_1 = t_2 \end{cases}$$

Same for DRV, replace t by n.

Types of R.P.

(1) Strict sense stationary R.P. → R.P. should be independent of time shift

$$X(t) \to \frac{X(t_1)X(t_2)---X(t_k)}{kR.V.}$$

## K[th] order Joint PDF-

$$(x_1, x_2 ---x_k) \quad \text{...(i)}$$

$$f_X(t_1)X(t_2)---X(t_k)$$

$$X(t): \quad \frac{X(t_1+\tau), X(t_2+\tau)---X(t_k+\tau)}{k \text{ R.V.}}$$

## K[th] order Joint PDF-

$$(x_1, x_2 ---x_k) \quad \text{...(ii)}$$

$$f_{X(t_1+\tau)X(t_2+\tau)---X(t_k+\tau)}$$

(i) = (ii) → $X(t)$ is solid to be SSSRP.

$$f_{X(t_1)}(x) = f_{X(t_2+\tau)}(x) \quad \text{independent of time}$$

2[nd] order joint PDF is independent of time shift.

$$f_{X(t_1)X(t_2)}(x_1, x_2) \to f_{X(0)X(t_2-t_1)}(x_1, x_2)$$

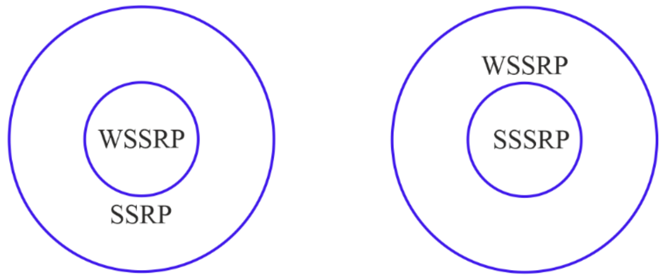
- Does not depend on individual sampling instances $t_1$ and $t_2$
- Depends on time difference between sampling instances $t_1$ and $t_2$

$$E[X(t_1)X(t_2)] = E[X(0)X(t_2-t_1) = R_{XX}(t_1, t_2)$$

$$E[X(0)X(t_2-t_1)] = R_{XX}(0, t_1-t_2) = R_{XX}(t_1 \sim t_2)$$

$$\sigma_{XX}(t_1 \sim t_2) = R_{XX}(t_1 \sim t_2) - \mu_X^2$$

WSSRP → There are stationary RP which are stationary at least upto 2[nd] order.




(1) ✓ (2) → Not necessarly ×

(1) $E[X(t)] = \mu_X$ Constant

(2) $E[X^2(t)] = \text{Constant}$

(3) $\sigma^2_{X(t)} = \text{Constant}$

$\text{E[RP]} = \text{E[RV]}$

$\text{MSV(RP)} = \text{MSV(RV)}$

$\text{Var(RP)} = \text{Var(RV)}$

(4) $E[X(t_1)X(t_2)] = R_{XX}(t_1 \sim t_2)$

$E[X(t+\tau)X(t)] = R_{XX}(-\tau)$

$E[X(t)X(t+\tau)] = R_{XX}(\tau)$

$R_{X_X}(\tau) = R_{XX}(-\tau)$

$\sigma^2_{X(t)} = R_{XX}(0) - \mu_X^2$

Cov $[X(t)X(t+\tau)] = R_{X_X}(\tau) - \mu_X^2$

(5) $E[X^2(t)] = R_{XX}(0) \qquad \tau = 0$

(6) $E[X(t)X(t+\tau)] = R_{XX}(\tau)\text{ACF} = \begin{cases} E[X(t)]E[X(t+\tau)] = \mu_X^2 & (\tau \neq 0) \\ E[X^2(t)] = R_{XX}(0) & (\tau = 0) \end{cases}$

(7) $\text{cov}[X(t)X(t+\tau)] = R_{XX}(t, t+\tau) - \mu_X(t)\mu_X(t+\tau)$

$C_{XX}(\tau) = R_{XX}(\tau) - \mu_X^2 = \begin{cases} 0 & \tau \neq 0 \\ R_{XX}(0) - \mu_X^2 & \tau = 0 \end{cases}$

$R_{X_X}(\tau) = \begin{cases} \mu_X^2 & \tau \neq 0 \\ R_{X_X}(0) & \tau = 0 \end{cases}$

**Important point :**

(1) If X(t) is zero mean WSSRP.

E [X (t) = 0

$\sigma^2_{X(t)} = E[X^2(t)]$

Var $[X(t)] = \text{MSV}\{X(t)\}$

Var $\{X(t = t_1)\} = \text{MSV}\{X(t = t_1)\}$

(2) If X (k) is zero mean WSSRP

$E[X(k)] = 0 \quad \sigma^2_{X(k)} = E[X^2(k)]$

Var $[X(k)] = \text{MSV}[X(k)]$

➢ $X(t): \text{WSSRP} + \text{IIDRP}$

$E[X(t)] = \mu_X,\ E[X(t+\tau)] = \mu_X,\ E[X^2(t)] = R_{XX}(0) = \text{Constant}$

$\sigma^2_{X(t)} = \text{Constant}$

Let $Y(t) = X(at + b)$

$E[Y(t) = \mu_X,\ E[y^2(t)] = R_{XX}(0)$

$\sigma^2_{Y(t)} = R_{XX}(0) - \mu_X{}^2$

$E[Y(t)Y(t+\tau)] = R_{XX}(a\tau) = R_{YY}(\tau)$

Cov $[Y(t)Y(t+\tau)] = C_{YY}(\tau) = R_{XX}(a\tau) - \mu_X^2$

For $Y(t) \to \left[\begin{array}{l} \to \mu_Y = \mu_X = \text{Constant} \\ R_{YY}(\tau) = R_{XX}(a\tau) \end{array}\right., Y(t) \to \text{WSSRP}$

➢ Time shift, Time reversal, time scaling does not affect stationary nature of R.P.

Let $Y(t) = aX(t) + b,\ X(t)$ is WSSRP

$E[Y(t)] = a\mu_X + b = \text{Constant}$

$E[Y^2(t)] = a^2 R_{XX}(0) + b^2 + 2ab\mu_X = \text{Constant}$

$\sigma^2_{Y(t)} = a^2 \sigma^2_{X(t)}$

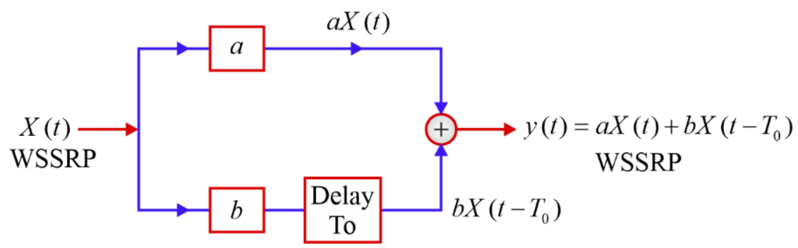
Cov $[Y(t)Y(t+\tau)] = R_{YY}(\tau) - \mu_Y^2$

$E[Y(t)Y(t+\tau)] = a^2 R_{XX}(\tau) + 2ab\mu_X + b^2 = R_{YY}(\tau)$

$y(t) \to \text{WSSRP}$

➢ Linear transformation of WSSRP does not change its stationarity.

➢ If WSSRP passed through LTI system, output is also a WSSRP.




$E[Y(t)] = (a+b)\mu_X$

$E[Y^2(t)] = a^2 R_{XX}(0) + b^2 R_{XX}(0) + 2abR_{XX}(\tau_0)$

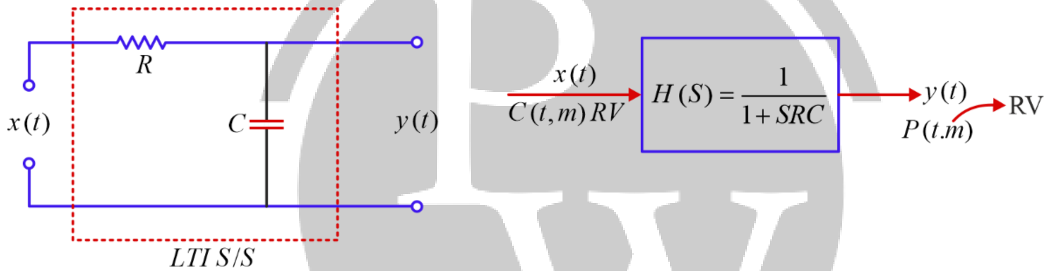
$\sigma^2_{Y(t)} = (a^2+b^2)\sigma^2_{X(t)} + 2ab[R_{XX}(T_0) - \mu_X^2]$

$R_{YY}(\tau) = (a^2+b^2)R_{XX}(\tau) + abR_{XX}(\tau - T_0) + abR_{XX}(\tau + T_0)$

$C_{YY}(\tau) = a^2 C_{XX}(\tau) + b^2 C_{XX}(\tau) + abR_{XX}(\tau - T_0) + abR_{XX}(\tau + T_0) - 2ab\mu_X^2$

$E[X(n)X(n+k)] = \delta[k] = R_{XX}(k)$ IIDRP

$E[X(n)X(n+k)] = E[X^2(n)](k=0)$




A, $\omega_0 \rightarrow$ constant, $\theta \sim U(0, 2\pi)$ OR $\theta \sim U[-\pi, \pi]$

$$X(t) = A\cos(\omega_0 t + \theta)$$

$E[A\cos(\omega_0 t + \theta)] = 0$

$E[A\cos(\omega_0 t + \theta + \propto)] = 0$

$E[X^2(t)] = \frac{A^2}{2}, \sigma^2_{X(t)} = \frac{A^2}{2}$

$E[X(t)X(t+\tau)] = \frac{A^2}{2}\cos\theta\omega_0\tau = R_{XX}(\tau)$

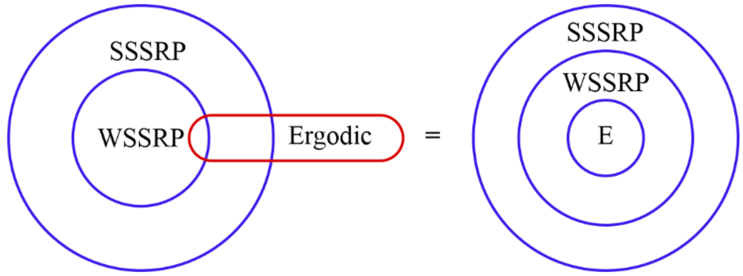
Cov $[X(t)X(t+\tau)] = \frac{A^2}{2}\cos\omega_0\tau$

$X(t)$: WSSRP + periodic with $\tau_0 \rightarrow R_{XX}(\tau)$ will also be periodic with same T.P.

## ERGODIC Random Process :

Time Avg = Statistical Aug.

$$\frac{1}{T}\int X(t)\,dt = E[X(t)]$$




**Auto Correlation and its properties**

Similarity between 2 Samples

Let $X(t)$ is WSSRP, $X(t)$ is observed $\tau$ duration apart

(1) $E[X(t)X(t+\tau) = R_{XX}(\tau)$

(2) $R_{XX}(-\tau) = R_{XX}(\tau)$ : Even

(3) $|R_{XX}(\tau)| \le R_{XX}(0)$

(4) $\{R_{XX}(\tau)\}_{\max} = R_{XX}(0) =$ Maximum similarity

(5) $\int_{-\infty}^{\infty} R_{XX}(\tau)d\tau = 2\int_{0}^{\infty} R_{XX}(\tau)d\tau = 2\int_{-\infty}^{0} R_{XX}(\tau)d\tau$

(6) $x(t)$ → Penodic → $R_{xx}(\tau)$ : Periodic
$x(t)$ → Non Penodic → $R_{xx}(\tau)$ : Non Periodic

(7) $E[x^2(t)] = MSV[x(t)] = R_{XX}(0)$ → Energy of R.P. → Energy Signal $x$(t)
→ Power of R.P → Power Signal $x$(t)

(8) $X(t)$ is power signal

$$R_{XX}(0) = E[X^2(t)] = \sigma^2_{X(t)} + \sigma^2_{X(t)}$$

$R_{XX}(0)$ ↓ Total power of R.P

$\sigma^2_{X(t)}$ ↓ A.C power of R.P

$\sigma^2_{X(t)}$ ↘ D.C power of R.P

(9) If $X(t)$ is ergodic and WSSRP, it has no periodic component

$$E[X(t) = \mu_X \ne 0$$

$$\mu_X^2 = \lim_{\tau\to\infty} R_{XX}(\tau) = \lim_{|\tau|\to\infty} R_{XX}(\tau)$$

If not ergodic but WSSRP then $R_{XX}(0) = E[X^2(t)]$, $R_{XX}(\infty) \ne \mu_X^2$

**Important point:**

$X(t) \leftrightarrow X(\omega)$ $\qquad$ $X(t) \leftrightarrow X(f)$

$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $\qquad$ $X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}dt$ $\qquad$ $x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}dt$

$X(0) = \int_{-\infty}^{\infty} x(t)dt$ $\qquad$ $x(0) = \int_{-\infty}^{\infty} X(f)df$

$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$ $\qquad$ $X(0) = \int_{-\infty}^{\infty} x(t)dt$

## 3.2. Parserval Theorem

$$E_{x(t)} = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Density Function

- Energy spectral Density (ESD): $G_{XX}(f)$, $G_{XX}(\omega)$ $\frac{\text{Joule}}{\text{Hz}}$
- Power spectral Density (PSD): $S_{XX}(f)$, $G_{XX}(\omega)$ $\frac{\text{Watt}}{\text{Hz}}$

Energy spectral density $X(t) \rightarrow$ WSSRP, Engery

ESD

$x(t) \xrightarrow{F.T} X(\omega) \Rightarrow |X(\omega)|^2 = G_{XX}(\omega)$

$x(t) \longrightarrow X(f) \Rightarrow |X(f)|^2 = G_{XX}(f)$

ESD of $x$(t)

$$E[X(t)X(t+\tau)] = R_{XX}(\tau) \xrightarrow{F.T.} G_{XX}(\omega)$$

$$R_{XX}(\tau) \xrightarrow{F.T.} G_{XX}(f)$$

$$\text{ACF}(X(t)) \xleftarrow{F.T} \text{ESD}[x(t)]$$

$$G_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)d\tau = 2\int_{0}^{\infty} R_{XX}(\tau)d\tau$$

Zero freq. value of ESD = Area under ACF

$$\underset{E[X^2(t)]}{\underset{\downarrow}{R_{XX}}}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{XX}(\omega)d\omega = \frac{\text{Area under ESDG}_{XX}(\omega)}{2\pi}$$

$$\int_{-\infty}^{\infty} G_{XX}(f)df = \text{Area under ESD} \qquad \text{G}_{XX}(f)$$

## Energy Calculation :

$$E_X(t) = \int_{-\infty}^{\infty} |X(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{XX}(\omega)d\omega = \int_{-\infty}^{\infty} G_{XX}(f)df$$

$$G_{XX}(\omega) = G_{XX}(-\omega)$$

## Power spectral density – (PSD)

$X(t) \to$ Power signal, WSSRP

$$X(t) \xleftrightarrow{PSD} S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{T}|X_T(\omega)|^2$$

$$X(t) \xleftrightarrow{PSD} S_{XX}(f) = \lim_{T\to\infty} \frac{1}{T}|X_T(f)|^2$$

(1) $E[X(t)X(t+\tau)] = R_{XX}(\tau)$ $\qquad$ $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau$

(2) $\text{ACF}[X(t)] \xleftrightarrow{F.T.} \text{PSD}[X(t)]$

$R_{XX} \xleftrightarrow{F.T.} \tau_{XX}(\omega)$ $\qquad$ $S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j2\pi f\tau}d\tau$

(3) $S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)d\tau = 2\int_{0}^{\infty} R_{XX}(\tau)d\tau$

Zero freq. value of = Area under ACF

PSD

(4) $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = \int_{-\infty}^{\infty} S_{XX}(f)e^{j\omega\tau}df$

$$R_{XX}(0) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega \\ \int_{-\infty}^{\infty} S_{XX}(f)df \end{cases}$$

(5) $R_{XX}(\tau) = R_{XX}(-\tau),\ S_{XX}(\omega) = S_{XX}(-\omega)$

(6) Calculation of power

$$E[X^2(t)] = R_{XX}(0) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = \dfrac{\text{Area under PSD}}{2\pi} \\ \displaystyle\int_{-\infty}^{\infty} S_{XX}(f)df = \text{Area under PSD} \end{cases}$$

$$E[X^2(t)] = R_{XX}(0) = \frac{1}{\pi}\int_0^{\infty} S_{XX}(\omega)d\omega = 2\int_0^{\infty} S_{XX}(f)df$$

## Total power

$$\text{A.C. Power} = \sigma^2 X(t), \text{D.C.Power} = \mu^2{}_X(t)$$

## Mean or Aug value

$$E[X(t)] = \sqrt{\frac{1}{2\pi}\int_{0^-}^{0^+} S_{XX}(\omega)d\omega}$$

$$E[X(t)] = \mu^2_{X(t)} = \begin{cases} \dfrac{1}{2}\displaystyle\int_{0^-}^{0^+} S_{XX}(\omega)d\omega \\ \displaystyle\int_{0^-}^{0^+} S_{XX}(f)df \rightarrow \end{cases}$$

Is non-zero only when impulse is not prelent at zero frequency

$$\sigma^2_{X(t)} = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{0^-} S_{XX}(\omega)d\omega + \dfrac{1}{2\pi}\int_{0^+}^{\infty} S_{XX}(\omega)d\omega \\ \displaystyle\int_{-\infty}^{0^-} S_{XX}(f)df + \int_{0^+}^{\infty} S_{XX}(f)df \end{cases}$$

- If X (t) is real, PSD is also real.
- PSD is even.
- PSD is non-negative, $S_{XX}(\omega) \geq 0;\ \ S_{XX}(f) \geq 0$

## ESD of Modulated Signal (Band Pass Signal)

$$\underset{\text{Base band R.P.}}{\underset{\downarrow}{X(t)}} \xleftrightarrow{ESD} G_{XX}(f)$$

$$\underset{\text{Band pass R.P.}}{\underset{\downarrow}{Y(t)}} = X(t).A_c\cos 2\pi f_c t \qquad Y(f) = \frac{A_c}{2}[X(f+f_c) + X(f-f_c)]$$

Or $\xleftrightarrow{ESD} G_{YY}(f) = |Y(f)|^2$

$$X(t).A_c \sin 2\pi f_c t \xleftrightarrow{ESD} G_{YY}(f) = |Y(f)|^2$$

$$G_{YY}(f) = \frac{A_c^2}{4}[G_{XX}(f - f_c) + G_{XX}(f + f_c)] \qquad f_c >>> f_m$$

$$R_Y(\tau) = \frac{A_c^2}{2} R_X(\tau)\cos 2\pi f_c T$$

## PSD of Modulated Signal (Bandpass Signal)

$X(t) \rightarrow$ power single

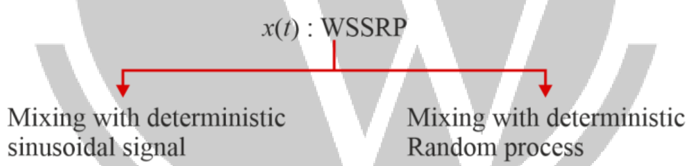
$$Y(t) = A_c \cos(2\pi f_c t).X(t) \rightarrow S_{YY}(f) \rightarrow \text{PSD}$$

$$S_{YY}(f) = \lim_{T\to\infty} \frac{|Y_T(f)|^2}{T}$$

$$S_{YY}(f) = \frac{A_c^2}{4}\left\{\lim_{T\to\infty} \frac{|X_T(f - f_c)|^2}{T} + \lim_{T\to\infty} \frac{|X_T(f + f_c)|^2}{T}\right\}$$

$$S_{YY}(f) = \frac{A_c^2}{4}[S_{XX}(f - f_c) + S_{XX}(f + f_c)]$$

$$R_{YY}(\tau) = \frac{A_c^2}{4} R_{XX}(\tau)\cos 2\pi f_c \tau$$




$$R_Y(\tau) = \frac{A_c^2}{2} R_X(\tau)\cos\omega_c \tau \qquad R_Y(\tau) = \frac{A_c^2}{2} R_X(\tau)\cos\omega_c \tau$$

$$S_Y(f) = \frac{A_c^2}{2}[S_X(f - f_c) + S_X(f + f_c)] \qquad S_Y(f) = \frac{A_c^2}{4}[S_X(f - f_c) + S_X(f + f_c)]$$

## 3.3. Cross Correlation

$X(t)$; WSSRP, $Y(t)$ : WSSRP

$$E[X(t)Y(t+\tau)] = R_{XY}(t, t+\tau) = R_{XY}(\tau)$$

$$E[Y(t+\tau)X(t)] = R_{YX}(-\tau)$$

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

- $R_{XY}(\tau) \pm R_{YX}(-\tau)$

- $R_{XY}(\tau) = R_{XY}(f) \to$ May/may not be
- $R_{XY}(\tau) \le \sqrt{R_{XX}(0)R_{YY}(0)}$
- $|R_{XY}(\tau)| = \frac{1}{2}R_{XX}(0) + R_{YY}(0)$

## Cross Covariance

$$C_{XY}(\tau) = R_{XY}(\tau) - \mu_X \mu_Y$$

$$C_{YX}(\tau) = R_{YX}(\tau) - \mu_Y \mu_X$$

## Cross Spectral Density

$$R_{XY}(\tau) \xleftrightarrow{F.T} S_{XY}(f)$$

$$R_{YX}(\tau) \xleftrightarrow{F.T} S_{YX}(f)$$

If RP X (t) and Y (t) are orthogonal-

$$E[X(t)Y(t+\tau)] = R_{XY}(\tau) = 0 = R_{YX}(-\tau)$$

$$E[Y(t)X(t+\tau)] = R_{YX}(\tau) = 0 = R_{XY}(-\tau)$$

- X(t) and Y(t) are uncorrelated and atleast one of them have zero mean-

  $E[X(t)] = 0 \text{ or } E[Y(t)] = 0$

  $\text{cov}[X(t)Y(t+\tau)] = 0$

  $C_{XY}(\tau) = R_{XY}(\tau) - \mu_X \mu_Y = 0$

  $R_{XY}(\tau) = 0$

  $R_{XY}(\tau) = R_{YX}(\tau) = 0$

- X(t) and Y(t) are independent R.P. and atleast one of them have zero mean.

  Cov $[X(t)Y(t+\tau)] = 0$

  $C_{XY}(\tau) = R_{XY}(\tau) - \mu_X \mu_Y = 0$

  $R_{XY}(\tau) = 0$

  $R_{YX}(\tau) = 0 = R_{XY}(\tau)$

## Combination of WSSRP

$$Z(t) = X(t) \pm Y(t)$$

$$R_{ZZ}(\tau)=R_{XX}(\tau)+R_{XY}(\tau)\pm R_{YX}(\tau)\pm R_{YY}(\tau)$$

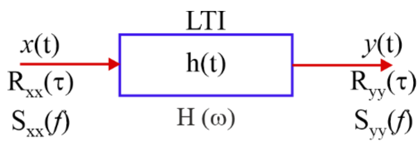
$$S_{ZZ}(f)=S_{XX}(f)+S_{YY}(f)\pm S_{XY}(f)\pm S_{YX}(f)$$

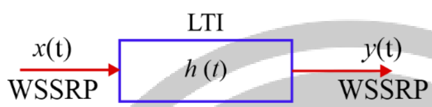
If orthogonal [X (t) and Y (t)] then

$$R_{YX}(\tau)=R_{XY}(\tau)=0$$

$$S_{YX}(f)=S_{XY}(f)=0$$

## Transmission of WSSRP through in LTI system




$$E[X(t)]=\mu_X \qquad E[Y(t)]=\mu_X[H(\omega)]_{\omega=0}$$

$$\mu_Y=\mu_X H(0)$$

If *x*(t) is zero mean WSSRP then *y*(t) is also zero mean WSSRP

$$R_{XY}(\tau)=R_{XX}(\tau)*h(\tau) \text{ and } S_{XY}(f)=S_{XX}(f)H(f)$$

$$R_{YX}(\tau)=R_{XX}(\tau)*h(-\tau) \text{ and } S_{YX}(f)=S_{XX}(f)H(-f)$$

$$R_{YY}(\tau)=R_{XX}(\tau)*h(\tau)*h(-\tau)$$

$$S_{YY}(t)=S_{XX}(f)H(f)H(-f) \qquad \text{if h(t) = real}$$

$$\underset{\text{PSD of O/P}}{S_{YY}(f)}=\underset{\text{PSD of i/p}}{S_{XX}(f)}|H(f)|^2 \qquad H(f)=H(-f)$$

## Power of Y(t)

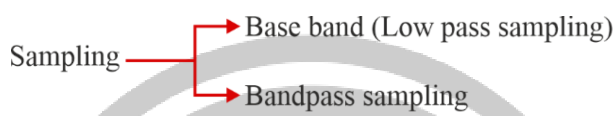
$$E[Y^2(t)]=\int_{-\infty}^{\infty}S_{YY}(f)df=\int_{-\infty}^{\infty}|H(f)|^2 S_{XX}(f)df$$

❒❒❒

# 4 DIGITAL COMMUNICATION

## 4.1. Sampling

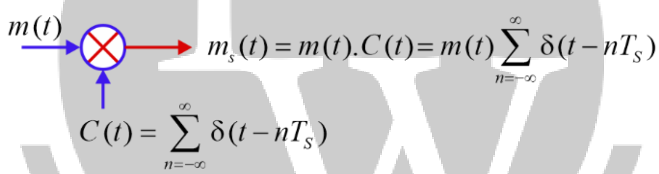
Sampling converts C.T.S into D.T.S, it retains analog or digital nature of signal.




$C(t)$: Impulse Train – Instantaneous or Ideal sampling

$C(t)$: Rectangular Pulse Train : Natural sampling or Flat Top sampling.

Ideal instantaneous sampling




$$m_s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t-nT_s)$$

$$m(t) = M(f) \,|\, M(\omega)$$

$$M_s(\omega) = f_s \sum_{n=-\infty}^{\infty} M(\omega - n\omega_s)$$

$$M_s(f) = f_s \sum_{n=-\infty}^{\infty} M(f - nf_s)$$

1. If a low pass signal is sampled at $f_s > 2f_m$ then it can be recovered from its samples, when
   $(f_s > 2f_m) \cap (f_m \le f_c \le f_s - f_m)$
   $(PBG = T_s)$
   $f_c =$ cut off free of ideal LPF at RX
2. $f_s = 2f_m$, the sampled signal $m_s(t)$ can be recovered into $m(t)$ if,
   $(f_s = 2f_m) \cap (f_c = f_m)$ Ideal LPF
   $(PBG = T_s)$

3. $f_s < 2f_m$, under sampling,

   $T_X$ : Replace generation with ALIASING

   $R_X$ : Recovery not possible

   ➢ ALIASING is overlapping of adjacent replica's in sampled signal.

### 4.1.1. Low pass Sampling Theorem

A low pass sampling signal band limited to $f_{\max}$ Hz, can be sampled and recovered from its samples when and only when

$f_s \geq 2f_{\max}$ at $T_X$ Proper LPF at $R_X$

No Aliasing Recovery

**Nyquist Rate and Nyquist Internal**

Let $m(t)$ is lowpass signal bandlimited to $f_{\max}$ Hz.

$$f_{NY} = 2f_{\max} \qquad T_{NY} = \frac{1}{f_{NY}} = \frac{1}{2f_{\max}}$$

$(f_s)_{\min} = 2f_{\max}$ min $\rightarrow$ sampling rate which ensure to aliasing

$$S(t) = m(t)\cos\omega_c(t) = (f_c + f_m)$$

↑
$f_s(\max)$

$$N_R = 2(f_c + f_m) = f_{NY}$$

$$N_J = \frac{1}{2(f_c + f_m)} = T_{NY}$$

**Combination of Two signals –**

$$x_1(t) \rightarrow f_{\max} \rightarrow f_1, x_2(t) \rightarrow f_{\max} \rightarrow f_2$$

(1) $\pm x_1(t) \pm x_2(t)$
- $f_{\max} = \max(f_1, f_2)$
- $f_{Ny} = 2f_{max} = \max \text{ of } (2f_1, 2f_2)$

(2) $x_1(t) \cdot x_2(t)$
- $f_{\max} = (f_1 + f_2)$
- $f_{Ny} = 2f_{max} = (2f_1 + 2f_2)$

(3) $x_1(t) * x_2(t)$
- $f_{\max} = \text{Min}(f_1, f_2)$
- $f_{Ny} = 2f_{max} = \text{Min}(2f_1, 2f_2)$

$$m(t): A_m\cos\omega_m t, A_m\sin\omega_m t \rightarrow f_m$$

$$c(t): \sum_{n=-\infty}^{\infty}\delta(t - nT_s) \rightarrow 0, f_s, 2f_s, 3f_s$$

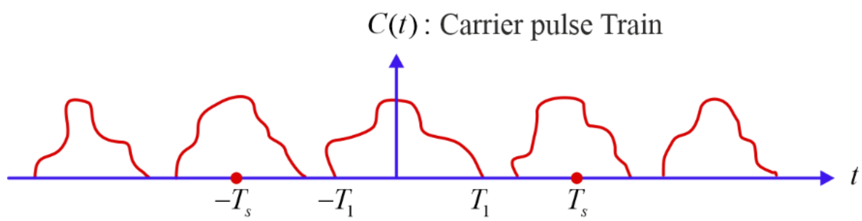
$$m_s(t) = m(t)c(t) = |0 \pm f_m|, |f_s \pm f_m|, |2f_s \pm f_m|, |3f_s \pm f_m| \ldots$$

## Sampling of signal by using general carrier pulse train

$$m(t) \longrightarrow M(f) \text{ or } M(\omega)$$

$$c(t) \longrightarrow c(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s), C(\omega) = \sum_{n=-\infty}^{\infty} 2\pi C_n \delta(\omega - n\omega_s)$$

$$M_s(t) = M_s(f) = \sum_{n=-\infty}^{\infty} C_n M(f - nf_s),\ M_s(\omega) = \sum_{n=-\infty}^{\infty} C_n M(\omega - n\omega_s)$$

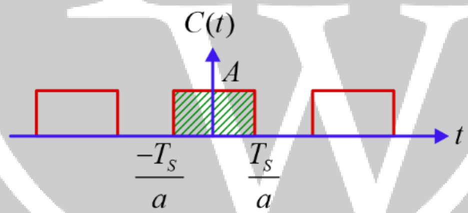



$$C_0 = \frac{\text{Area in } T_s}{T_s}$$

Recovery – $T_X$ : $f_s \geq 2 f_m$ $R_X$ : LPF = Proper $f_c$

$$y(t) = m(t) \qquad PBG = \frac{1}{C_0}$$

If $c(t)$ is rectangular pulse




- $m(t)$ is lowpass –

$$m_s(t) = c(t)m(t) \rightarrow M_s(f) = \sum_{n=-\infty}^{\infty} \frac{2A}{a} \sin C\left(\frac{2n}{a}\right) M(f - nf_s)$$

Recovery- $(f_s > 2 f_m) \cap (f_m \leq f_c \leq f_s - f_m)$

| PBG of LPF | $y(t)$ |
|---|---|
| 1 | $C_0 m(t)$ |
| $\frac{1}{C_0}$ | M(t) |
| $K$ | $KC_0\, m(t)$ |

$$C_0 = \frac{\text{Area in } T_s}{T_s}$$

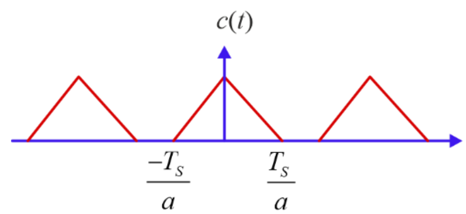
- $m$(t) is sinusoidal –

$$m(t) = A_m \cos 2\pi f_m t \longrightarrow f_m$$

$$c(t) = 0,\ f_s, 2f_s, \ldots \text{ except } \ n = \frac{Ka}{2} \quad K \in I, K \neq 0$$

$$m_s(t) = 0 \pm f_m, f_s \pm f_m, 2f_s \pm f_m, \ldots$$

If c(t) is Triangular-Frequency absent $n = Ka \quad K \neq 0, K \in I$




- Rest of the things same as Rectangular pulse

## Bandpass Sampling :

*m*(t) is lowpass signal.

$$f_s = \frac{2f_H}{K} \qquad K = \frac{f_H}{f_H - f_L} \qquad NR = 2f_H$$

## Previous Integer

$f_H$ = Maximum frequency component of Bandpass signal

## Natural Sampling

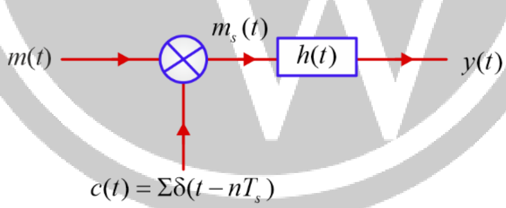
***m*(t):** Low pass sampling

**c(t):** Train of finite duration pulse or rectangular pulse

$$M_s(f) = \sum_{n=-\infty}^{\infty} C_n M(f - nf_s)$$

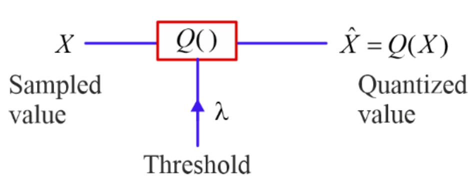
## Flat Top Sampling

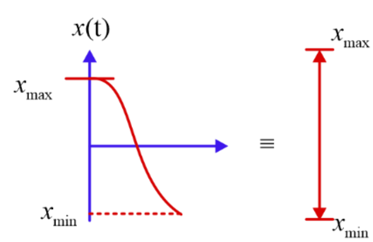
Instantaneous sampling followed by a filter.




$$y(t) = \sum_{n=-\infty}^{\infty} m(nT_s) h(t - nT_s)$$

## Quantizer

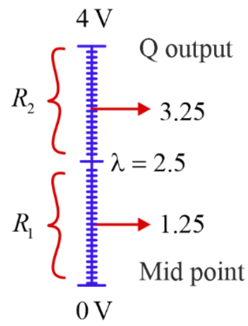



Discretizes amplitude axis, analog to digital signal.

Dynamic Range of $x(t) = (x_{max} - x_{min})$

$Q$ input : $\qquad$ $Q$ : output : $\hat{x}(t)$




$$\hat{x}(t) = \begin{cases} 1.25 & 0 \le x(t) \le 2.5 \\ 3.25 & 2.5 \le x(t) \le 4 \end{cases}$$

- Many to one circuit.

## Uniform Quantizer

$$\Delta = \frac{\text{DR of } Q}{L} = \frac{m_{L+1} - m_1}{L}$$

$\Delta_1 = \Delta_2 = \Delta_3 \ldots$

L = Number of quantization level of $Q$.

## Non uniform Quantizer

1. $\Delta_1 = \Delta_2 \ne \Delta_3 \ldots \Delta_i = \Delta_{i+1} \ldots \ne \Delta_L$
2. $\Delta_1 \ne \Delta_2 \ne \Delta_3 \ldots \Delta_i \ne \Delta_{i+1} \ldots \ne \Delta_L$

## Quantizer Error –

$$X \longrightarrow Q(\,) \longrightarrow \hat{X} = Q(X)$$

$$Q \cdot E = X_{QE} = X - \hat{X} = X - Q(X)$$

R.V Sampled Value Quantized Value

**Q.E.P** $E\left[X_{QE}^2\right] = \int_{-\infty}^{\infty} x_{QE}^2 f_{QE}[qE]\,dqE$

When PDF of $QE$ is given.

$$E\left[X_{QE}^2\right] = E\left[(X - \hat{X})^2\right] = \int_{-\infty}^{\infty} (x - \hat{x})^2 f_X(x)\,dx$$

When PDF of R.V at input of Quantizer ($X$) is given.

(a) PDF is uniform

$$E\left[X_{QE}^2\right] = \frac{\Delta_1^2}{12} \times A_1 + \frac{\Delta_2^2}{12} \times A_2 + \frac{\Delta_3^2}{12} \times A_3 + \ldots$$

$$= \frac{\Delta_1^2}{12} \text{ [Area of region in which step size is 1]} + \frac{\Delta_2^2}{12} \text{ [Area of region in which step size is 2]} + \ldots$$

If quantization is uniform $E\left[X_{QE}^2\right] = \frac{\Delta^2}{12}$

(b) PDF is stair case $E\left[X_{QE}^2\right] = \frac{\Delta_1^2}{12} \times A_1 + \frac{\Delta_2^2}{12} \times A_2 + \ldots$

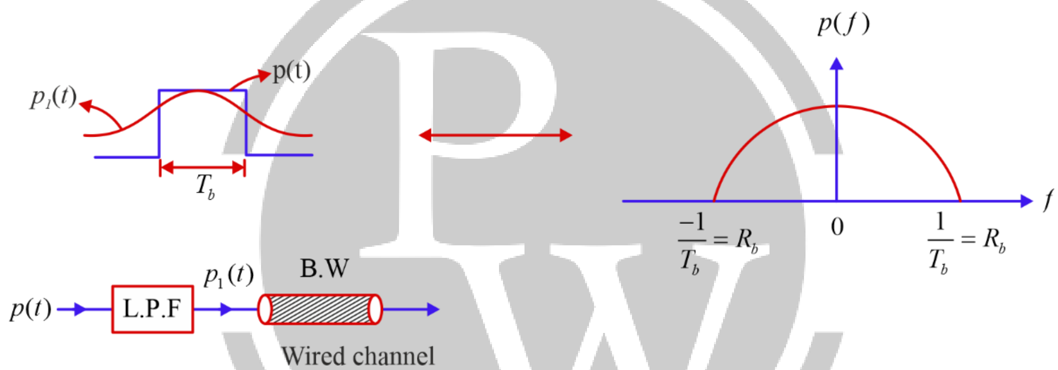
(c) PDF is non uniform $E\left[X_{QE}^2\right] = \frac{\Delta_1^2}{12} A_1 + \frac{\Delta_2^2}{12} A_2 + \ldots$

$$SQNR = \frac{\text{Signal power}}{Q \cdot E \cdot P} = \frac{E[X^2]}{E\left[X_{QE}^2\right]}$$

$$(SQNR)_{dB} = 10 \log_{10} SQNR$$

## 4.2. Pulse Transmission

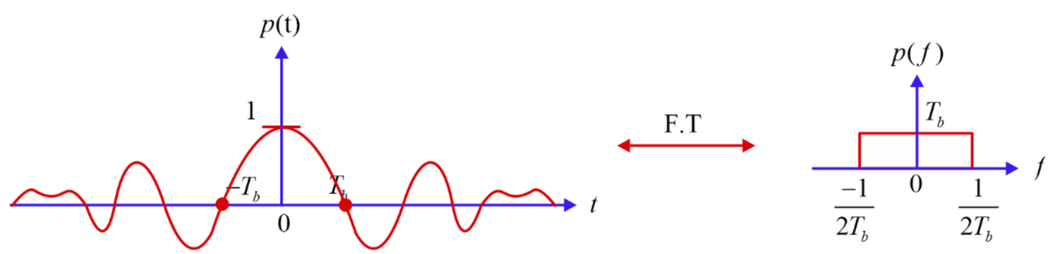
### 1. Rectangular Pulse




- Bandwidth of wired channel for $T_X$ of single pulse of duration $T_b$ :

  $(BW) \geq R_b$

### 2. Since pulse




- BW of wired channel for $T_X$ of single $\sin C$ pulse having zero cross over or integer multiple of $T_b$.

  $(BW) \geq \frac{R_b}{2}$

  Tb: Bit interval

  $R_b$ : Bit rate $\rightarrow$ Bit/sec

➢ Minimum transmission BW of a wired channel for baseband transmission is $= \frac{R_b}{2}$

Number of levels $L \le 2^n$ $\quad L_{\max} = 2^n$

$n$ is used to represent binary power quantization level.

## M-ary Scheme

$$M = 2^N$$

M ⇒ Number of different symbols of duration $NT_b$ each.

$T_s = NT_b$ : Symbol duration

$N =$ Number of bits combined in binary sequence at a time

## Pulse Code Modulation

Bit rate, $\quad R_b = nf_s$ bits/sec

$$\frac{-\Delta}{2} \le Q \cdot E \le \frac{\Delta}{2}, \quad E(y)^2 = \frac{\Delta^2}{12}$$

If Mid point Mapping is used.

$$\Delta = \frac{DR \text{ of signal}}{L} = \frac{DR \text{ of } Q}{L}$$

$$L \le 2^n \qquad \frac{-\Delta}{2} \le Q_e \le \frac{\Delta}{2} \qquad Q_e\big|_{\max} = \frac{\Delta}{2} \qquad \Delta_{\min} = \frac{DR \text{ of signal}}{2^n}$$

$$P_{QE} = E[X_{QE}^2] = E[y^2] = \int y^2 f_Y(y)\,dy$$

When PDF of Q. Eis given.

$$P_{QN} = P_{QE} = \sum_{i=L}^{L} \int_{m_i}^{m_{i+1}} (x - \hat{x}_i)^2 f_X(x)\,dx$$

When PDF of $X$ is given.

If $Q_e \sim U\left[\frac{-\Delta}{2}, \frac{\Delta}{2}\right] = P_{QE} = \frac{\Delta^2}{12}$

$$SQNR = \frac{12}{\Delta^2} P_s$$

Bit Interval $\quad T_b = \frac{1}{R_b}$

$\text{B.W} \ge R_b \rightarrow$ Rectangular Pulse

$\text{B.W} \ge \frac{R_b}{2} \rightarrow$ Since Pulse

$(\text{B.W})_{\min} = (\text{BW})_{PCM} = \frac{R_b}{2}$

## Signal to Quantization Noise Power

$m(t) \rightarrow$ Single tone sinusoidal

$m(t) = A_m \cos\omega_m t$

1. $P_s = \overline{m^2(t)} = \frac{A_m^2}{2}$
2. $P_s = \overline{m^2(t)} = \frac{A_m^2}{3L^2}$
3. $\Delta = \frac{2A_m}{L}$
4. $SQNR = \frac{3}{2}L^2$
5. $(SQNR)_{dB} = (1.76 + 20\log_{10} L)\, dB$
6. $(SQNR)_{\max} = \frac{3}{2}4^n$
7. $(SQNR)_{\max}\, dB = (1.76 + 6n)\, dB$

### $x(t)$ is uniformly distributed $[-A_m, A_m]$

1. $P_s = \frac{A_m^2}{3}$
2. $P_{QE} = \frac{A_m^2}{3L^2}$
3. $\Delta = \frac{2A_m}{L}$
4. $SQNR = L^2$
5. $(SQNR)_{dB} = 20\log_{10} L$
6. $(SQNR)_{dB} \leq 6n\,\text{dB}$
7. $(SQNR)_{\max}\,\text{dB} = 6n\,\text{dB}$

## Key point:

1. $SQNR = \frac{3}{2}L^2$
2. $(SQNR)_{\max} = \frac{3}{2}4^n$

   If $n \rightarrow n \pm k$

   $(SQNR)_{\max} \rightarrow 4^{\pm k}$ $\qquad R_b = nf_s$

3. $(SQNR)_{\max} = (1.76 + 6n)\,dB$

   $n \to n \pm k$

   $(SQNR)_{\max} \to \pm 6\,dB$

4. $n$ given : $L = 2^n$

5. L given:
   - L = Binany Power : $L = 2^n$
   - L ± Binany Power : $L \le 2^n$

6. $SQNR$:
   - L=Binany Power : $(SQNR) = \frac{3}{2}L^2 = (SQNR)_{\max}$
   - L ≠ Binany Power : $(SQNR)_{\max} = \frac{3}{2}4^n$

7. $n$ calculation : $n_{\min}$

8. Default $m(t) =$ Sinusoidal

**Drawback of PCM**

$$\text{BW} = \frac{nf_s}{2}, P_{QE} = \frac{\Delta^2}{12}$$

$$n\uparrow \longrightarrow L\uparrow \longrightarrow \Delta\downarrow \longrightarrow PQE\downarrow \longrightarrow BW\uparrow$$

$$n\downarrow \longrightarrow L\downarrow \longrightarrow \Delta\uparrow \longrightarrow \underbrace{PQE\uparrow \longrightarrow BW\downarrow}$$

# 4.3. DPCM (Differential Pulse Code Modulation)

## 4.3.1. PCM vs DPCM

1. $\Delta$ fix for Both $Q$ –

   $(\text{BW})_{PCM} > (\text{BW})_{DPCM}$

   $(SQNR)_{PCM} = (SQNR)_{DPCM}$

   D.R at input of $Q$ of PCM is greater than DPCM.

2. $L$ fix for Both $Q$ –

   $(\text{BW})_{PCM} = (\text{BW})_{DPCM}$

   $(SQNR)_{PCM} < (SQNR)_{DPCM}$

3. In case of DPCM the difference between current sample and predicted value of current sample is Quantized, Encoded, Line Coded and Wired $T_{Xed}$.

## Delta Modulator

- The recovered signal is "stair-case" approximation of $t_{Xed}$ original analog message signal.
- Stairs are added or substracted of sampling instance.
- Size of each stair is $\Delta$ = Step size of stairs

## Tracking Error in DM.

### 1. Slope overload Error

$$\left|\frac{dm(t)}{dt}\right|_{max} >> \frac{\Delta}{T_s} \quad \text{Occurance of S.O.E}$$

- To avoid SOE, $\Delta\uparrow\uparrow\uparrow$ by keeping $T_s$ constant such that –

$$\left|\frac{d}{dt}m(t)\right|_{max} \le \frac{\Delta}{T_s} \Rightarrow \text{For sinusoidal } m(t) = A_m \cos\omega_n t$$

$$A_m \le \frac{\Delta f_s}{\omega_m}$$

### 2. Granular Error

It occurs when $\Delta$ is large.

- To remove it $\Delta \rightarrow$ small

  If $SOE\uparrow$, G.E $\downarrow$ and vice versa.

## *SQNR* in DM

$$P_{QE} = E\left[X_{QE}^2\right] = \frac{\Delta^2}{3}$$

1. $SQNR = \frac{P_s}{P_{QE}} = \frac{3P_s}{\Delta^2}$ $\qquad f_H$ = cut off frequency of LPF

2. $(SQNR)_D = \frac{3P_s}{\Delta^2} \times \frac{f_s}{f_m} = \frac{3P_s}{\Delta^2} \times \frac{f_s}{f_H}$

3. $(SQNR)_{max} + m(t)$ is sinusoidal + SOE avoid $= \frac{3}{80}\left(\frac{f_s}{f_m}\right)^2$

4. $[(SQNR)_D]_{max} + m(t)$ is sinusoidal + SOE avoid $= \frac{3}{80}\left(\frac{f_s}{f_m}\right)^3 \rightarrow$ By default.

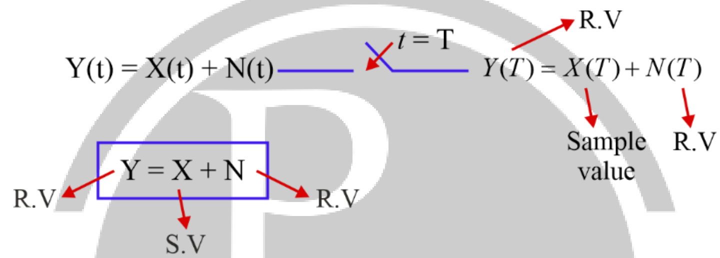
❑❑❑

# 5 DIGITAL RECEIVER

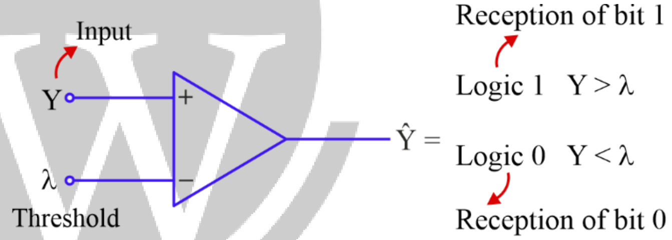
## 5.1. Introduction

$X(t) \rightarrow$ Deterministic signal process

$N(t) \rightarrow$ random signal process




| N | Y = N + X |
|---|---|
| Uniform PDF | Uniform PDF |
| Gaussian PDF | Gaussian PDF |
| Triangular PDF | Triangular PDF |




1. If but 1 is taxed $\rightarrow$ Receiver must recover but '1'.
2. If but 0 is taxed $\rightarrow$ Receiver must recover but '0'.

**Output of Sampler:**

$$Y = S_{01} + N_0 = \begin{cases} S_{01} + N_{01} & 1\,T_X \\ S_{02} + N_{02} & 0\,T_X \end{cases}$$ Channel noise is signal dependent

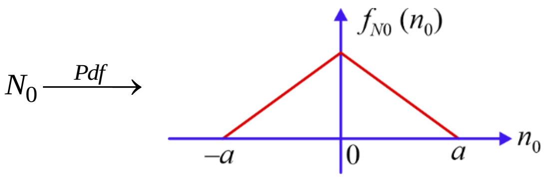
$$Y = S_0 + N_0 = \begin{cases} S_{01} + N_0 & 1\,T_X \\ S_{02} + N_0 & 0\,T_X \end{cases}$$ Channel noise is signal independent

**BER Calculation :**

$$P(1\,T_X) \Rightarrow \quad P(S_1(t):T_X) = P(S_{01}(t):\text{Reception}) = p$$

$$P(0\,T_X) \Rightarrow \quad P(S_2(t):T_X) = P(S_{02}(t):\text{Reception}) = (1-p)$$

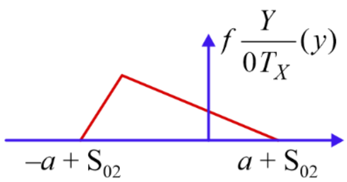
At the input of decision device a condition R.V. is obtained



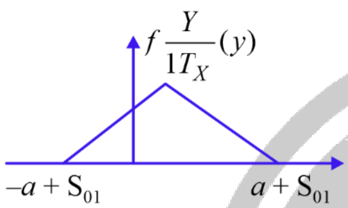

$Y = S_{02} + N_0$

$Y = S_0 + N_0$

$Y = S_{01} + N_0$




→ Conditional PDF of $Y$ when 0 is Txed by $T_x$




→ Conditional PDF of $Y$ when 1 is Txed by $T_x$

$Y > \lambda$: Decide in favour of 1, or decides that bit 1 would have been Txed by Txer

$Y < \lambda$: Decide in favour of 0, or decides the bit 0 would have been Txed by Txer.

### Average Bit Error Rate :

$$P_e = P(1\ T_X \cap 0) + P(0\ T_X \cap 1)$$

- $1\ T_X$ → 1 Txed
- $0$ → decide in favour of "0"
- $0\ T_X$ → 0 Txed
- $1$ → decide in favour of "1"

$$P_e = P(1\ T_X)\, P\left(\frac{0}{1\ T_X}\right) + P(0\ T_X)\, P\left(\frac{1}{0\ T_X}\right)$$

- $P\left(\frac{0}{1\ T_X}\right)$ → decides in favour of "0" provided "1" was txed
- $P\left(\frac{1}{0\ T_X}\right)$ → decides in favour of "1" provided "0" was txed

### Problem Solving Technique:

**Case 1:** When PDF of Noise [noise R.V. at i/p of D.D] is given

- (1) $\lambda$ : Given
  (2) Calculate $P_e$ =?
- (1) Calculate optimum threshold ($\lambda_{opt}$)
  (2) Calculate min prob. of error
  → Min probability of error is calculated at $\lambda_{opt}$.

(a) $\lambda$ is given

$$P_e = P(0\,T_X)P\left(\frac{1}{0\,T_X}\right) + P(1\,T_X)P\left(\frac{0}{1\,T_X}\right)$$

$$P\left(\frac{1}{0\,T_X}\right) = P[Y(0\,T_X) > \lambda] = P(S_{01} + N_0 > \lambda) = P(N_0 > \lambda - S_{01}) = \int_{\lambda - S_{01}}^{\infty} f_{N_0}(no)dno$$

$$P\left(\frac{0}{1\,T_X}\right) = P[Y(1\,T_X) < \lambda] = P(S_{02} + N_0 < \lambda) = P(N_0 < \lambda - S_{02}) = \int_{-\infty}^{\lambda - S_{02}} f_{N_0}(no)dno$$

(b) $\lambda_{opt} \rightarrow$ calculate, $P_{e\min} \rightarrow$ calculate

**Steps 1 :**

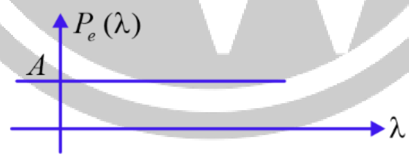
1. Identify conditional PDF of conditional R.V. from the noise R.V. pdf (pdf of No.)

$$Y = \begin{cases} S_{01} + N_0 & 1\,T_X \\ S_{02} + N_0 & 0\,T_X \end{cases}$$

2. Plot the conditional PDF one over another
3. Identify the overlapping or common region and decide range of $\lambda$

   $(\lambda_1 \le \lambda \le \lambda_2)$
4. Choose any arbitrary $\lambda$ in the above range and calculate $P_e = P_e(\lambda)$
5. $P_e(\lambda)$ Vs $\lambda$

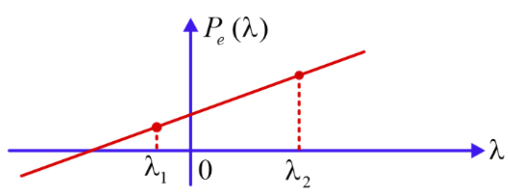
   (i) $P_e(\lambda)$ Vs $\lambda$ : Independent of $\lambda$




   (a) $\lambda_1 \le \lambda \le \lambda_2 \rightarrow$ optimum $\lambda$ is every $\lambda \Rightarrow \lambda \in (\lambda_1 : \lambda_2)$

   (b) $P_{e(\min)} = A$

   (ii) $P_e(\lambda)$ Vs $\lambda$: Linear




$\lambda_1 \le \lambda \le \lambda_2 \rightarrow \lambda_{opt} = \lambda_1$

$P_e(\lambda)_{\min} = P_e(\lambda_{opt})$

(iii) $Pe(\lambda)$ Vs $\lambda$: Non linear

$$\boxed{\frac{d}{d\lambda}P_e(\lambda)=0} \rightarrow \lambda_{\text{opt}} \qquad \boxed{P_e(\lambda=\lambda_{\text{opt}})=P_e\big|_{\min}}$$

$P_e(\lambda=\lambda_{opt})=P_e\big|_{\min}$

(iv) If no overlapping region b/w conditional PDF

$P_e(\lambda)=0 \rightarrow$ BER is 0

**Case 2:** When PDF of conditional R.V. $Y$ is gives at the input of the decision device

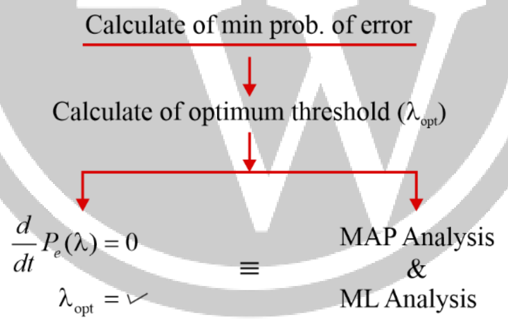
| | |
|---|---|
| $\lambda \rightarrow$ Given | $\lambda_{\text{opt}} \rightarrow$ Calculate |
| $P_e \rightarrow$ Calculate | $P_e\big\|_{\min} \rightarrow$ Calculate |

(a) $P_e = P(0\,T_X)P\left(\frac{1}{0\,T_X}\right)+P(1\,T_X)P\left(\frac{0}{1T_X}\right)$

$$P\left(\frac{1}{0\,T_X}\right)=P(Y(0\,T_X)>\lambda)=\int_{\lambda}^{\infty}\frac{f_Y}{0\,T_X}(y)dy$$

$$P\left(\frac{0}{1\,T_X}\right)=P[(1\,T_X)<\lambda]=\int_{-\infty}^{\lambda}\frac{f_Y}{1T_X}(y)dy$$

(b) Same as case 1 (b)




## MAP Analysis (Maximum A posteriori Analysis)

- MAP receiver always calculate min $P_e$.
- Calculation of $\lambda_{opt} \Rightarrow$ using

$$\frac{d}{d\lambda}P_e(\lambda)=0$$

## MAP Analysis

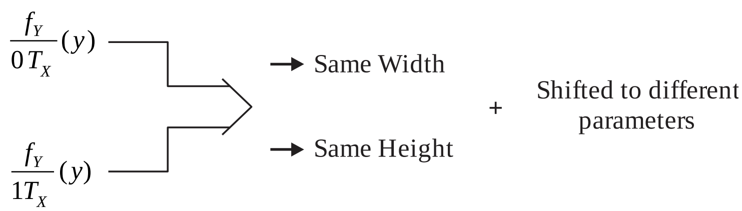
$$P\left(\frac{1\,T_X}{Y}\right)\underset{\text{"0"}}{\overset{\text{"1"}}{\gtrless}} P\left(\frac{0\,T_X}{Y}\right)$$

$$P(1\,T_X)\,f\frac{y}{1\,T_X}(y)\underset{\text{"0"}}{\overset{\text{"1"}}{\gtrless}} P(0\,T_X)\,f\frac{y}{0\,T_X}(y)\equiv y\underset{\text{"0"}}{\overset{\text{"1"}}{\gtrless}}\lambda \nearrow \lambda_{opt}=\lambda$$

## ML Analysis : (Maximum Likelihood Analysis )

It is same as MAP analysis When $P(0\,T_X) = P(1\,T_X) = \frac{1}{2}$

If noise is independent of signal then PDF of conditional R.V at input of decision device will have.




**Key point :** $P(0\,T_X) \neq P(1\,T_X)$

1. Noise is signal dependent/independent $\lambda_{\min} \le \lambda \le \lambda_{\max}$,

   $$\bar{\lambda} = \frac{\lambda_{\min} + \lambda_{\max}}{2}$$

   $P(0\,T_X) < P(1\,T_X) \qquad \lambda_{opt} < \bar{\lambda}$

   $P(1\,T_X) < P(0\,T_X) \qquad \lambda_{opt} > \bar{\lambda}$

2. If noise is signal dependent

   $$Y = \begin{cases} S_{01} + N_0 & 1\,T_X \\ S_{02} + N_0 & 0\,T_X \end{cases}$$

   $P(0\,T_X) = P(1\,T_X) = \frac{1}{2};\ \lambda_{opt} = \bar{\lambda}$

   Only when $Y$ is having Non uniform PDF.

   $P(0\,T_X) \neq P(1\,T_X) = \frac{1}{2};\ \lambda_{opt} < \bar{\lambda};\ P(0\,T_X) < P(1\,T_X)$

   $\lambda_{opt} > \bar{\lambda};\ P(0\,T_X) > P(1\,T_X)$

## When channel noise is Gaussian Random Process

$$P_e = P(0\,T_X)P\left(\frac{1}{0\,T_X}\right) + P(1\,T_X)P\left(\frac{0}{1\,T_X}\right)$$

### Method 1 :

$$P\left(\frac{1}{0\,T_X}\right) = P[Y[0\,T_X] > \lambda] = Q\left[\frac{\lambda - \mu_y[0\,T_X]}{\sigma_y[0\,T_X]}\right]$$

$$P\left(\frac{0}{1\,T_X}\right) = 1 - Q\left[\frac{\lambda - \mu_y(1\,T_X)}{\sigma_y[1\,T_X]}\right]$$

**Method 2 :**

$$P\left(\frac{1}{0\,T_X}\right) = P\left[y[0\,T_X] > \lambda\right] = P\left[S_{02} + N_0 > \lambda\right] = Q\left[\frac{(\lambda - S_{02}) - \mu_{N_0}}{\sigma_{N_0}}\right]$$

$$P\left(\frac{0}{1\,T_X}\right) = P\left[y[1\,T_X] < \lambda\right] = P\left[S_{01} + N_0 < \lambda\right] = 1 - Q\left[\frac{(\lambda - S_{01}) - \mu_{N_0}}{\sigma_{N_0}}\right]$$

If PDF of noise at the input of D.D is given along with $\lambda$–

$$P\left(\frac{1}{0\,T_X}\right) = P\left[y[0\,T_x] > \lambda\right] = \int_{\lambda}^{\infty} \frac{f_Y}{0\,T_X}(y)dy = Q\left[\frac{y - \mu_2}{\sigma}\right]$$

$$P\left(\frac{0}{1\,T_X}\right) = P\left[y[1\,T_x] < \lambda\right] = \int_{-\infty}^{\lambda} \frac{f_Y}{1\,T_X}(y)dy = 1 - Q\left[\frac{y - \mu_2}{\sigma}\right]$$

$\lambda$ optimum

1. **Differentiation :** $P_e = Q(\lambda)$, $\frac{d}{d\lambda}P_e(\lambda) = 0 \rightarrow \lambda_{opt}$

2. **Map Analysis :** $\frac{d}{d\lambda}P_e(\lambda) = 0 \rightarrow \lambda_{opt}$

$$\lambda_{opt} = \left(\frac{\mu_1 + \mu_2}{2}\right) + \frac{\sigma y^2}{(\mu_1 - \mu_2)}\ln\frac{P(0\,T_X)}{P(1\,T_X)}$$

Channel noise is Gaussian, signal and channel noise are independent

$$P_e(\lambda) = P_e(\lambda_{opt}) = P_e\big|_{\min}$$

3. **ML Analysis :** $P(0\,T_X) = P(1\,T_X) = \frac{1}{2}$

$$\lambda_{opt} = \frac{\mu_1 + \mu_2}{2}$$

$$P_e\big|_{\min} = Q\left[\frac{\mu_1 - \mu_2}{2\sigma_y}\right]$$

## Schwartz Inequality

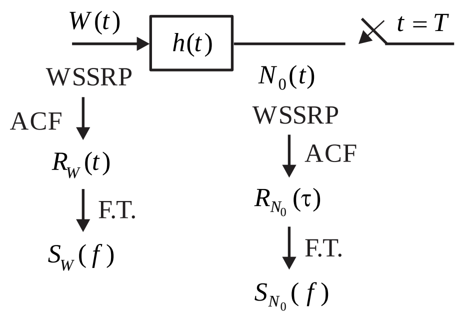
$$\left|\int_{-\infty}^{\infty} X_1(f)X_2(f)df\right|^2 \le \int_{-\infty}^{\infty} |X_1(f)|^2\,df \int_{-\infty}^{\infty} |X_2(f)|^2 df$$

$$\boxed{P(S_0)_{\max} = E_{s(t)} \times E_{h(t)}}$$

- $P(S_0)_{\max}$ → Max. signal power at sampling instance
- $E_{s(t)}$ → Signal energy at input of filter
- $E_{h(t)}$ → Energy of $h(t)$

$$E_{s(t)} = \int_{-\infty}^{\infty} |S(f)|^2 \, df$$

$$\int_{-\infty}^{\infty} |H(f)|^2 \, df = E_{h(t)}$$




$$P_{N_0(t)} = E\left[N_0^{\,2}(t)\right] = \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df$$

$$E\left[N_0^2(t)\right] = \frac{N_0}{2} \times E_{h(t)}$$

$$(SNR)_{\max} = \frac{E_{s(t)}}{(N_0/2)}$$

Only when $H(f) = [S(f)e^{j2\pi fT}]^*$

$$(NSR)_{\max} \text{ at } t = T = \frac{\text{Energy of i/p pulse}}{\text{PSD of i/p white noise}}$$

**For General Noise**

$$H(f) = \left[S(f)e^{j2\pi fT}\right]^*$$

When $$(SNR)_{\max} = \frac{[P_{S0}]_{\max}}{P_{N_0}} = \frac{E_{s(t)} \times E_{h(t)}}{\int_{-\infty}^{\infty} S_{N_0}(f)\,df}$$

$$S_{N_0}(f) = |H(f)|^2 \, S_N(f)$$

## 5.2. Optimum Filter

$$H(f) = e^{-j2\pi fT} S^*(f) = e^{-j2\pi fT} S(-f)$$

$$h(t) = S(T - t)$$

$T$ = Sampling instance

= Duration of incoming pulse

## Unit impulse response of optimum filter

Optimum filter = matched filter $\Rightarrow$ Maximizes signal power at sampling instances.

## Properties of MF

S(t) is an energy pulse of duration *T*.

1. $h(t) = S(T-t)$
2. $S(t) = h(T-t)$
3. $S_0(t) = S(T) * h(t)$
4. $S(t),\ h(t),\ S_0(t)$ are energy signal
5. $E_S(t) = E_h(t) = |S_0(t)|_{\max}$
6. $(SNR)_{\max}$ at $t = T = \dfrac{E_s(t)}{(N_0/2)}$
7. $E_{S_1}(t) > E_{s_2}(t)$ then $Pe_1 < Pe_2$
8. $S_0(f) = |S(f)|^2 e^{-j2\pi fT}$

   $S(t) \leftrightarrow S(f) \leftrightarrow |S(f)|^2 = G_s(f)$

   $R_s(\tau) \xleftrightarrow{F.T.} G_s(f)$

   $ACF[S(t)] \xleftrightarrow{F.T.} PSD[S(t)]$

   $R_s(\tau) \xleftrightarrow{F.T.} |S(f)|^2$

   $S_0(\tau) = R_s(\tau - T) \xleftrightarrow{F.T.} |S(f)|^2 e^{-j2\pi fT} = S_0(f)$

## Matched Filter Output

$$y(t) = S_0(t) + N_0(t) = \begin{cases} a_1(t) + N_0(t) & : 1\ T_X \\ a_2(t) + N_0(t) & : 0\ T_X \end{cases}$$

## Channel Noise is White

Let $P(0\ T_X) = P(1\ T_X) = \dfrac{1}{2}$

$$[Pe]_{\min} = Q\left[\frac{a_1 - a_2}{2\sigma_y}\right] = Q\left(\frac{x}{2}\right) \quad x \uparrow \Rightarrow Q\left(\frac{x}{2}\right)\downarrow$$

Maximization of $|x|^2$

$$|x|^2_{\max} = \frac{\int_{-\infty}^{\infty} |S_1(f) - S_2(f)|^2\, df}{(N_0/2)}$$

When $H(f)=\left[[S_1(f)-S_2(f)]e^{j2\pi fT}\right]^*,\ E_d=\int_{-\infty}^{\infty}|S_1(f)-S_2(f)|^2 df$

$$x_{\max}=\sqrt{\frac{2E_d}{N_0}}$$

$$h(t)=S_1(T-t)-S_2(T-t)\rightarrow x : x_{\max}=\sqrt{\frac{2E_d}{N_0}}$$

$$P_e|_{\min}=Q\left(\frac{a_1-a_2}{2\sigma_y}\right)$$

$$P_e|_{\min}=Q\left(\frac{x}{2}\right) \longrightarrow \boxed{M.F} \longrightarrow P_e|_{\min}|_{\min}=Q\left(\frac{x_{\max}}{2}\right)$$

$h(t)$

$$P_e|_{\min}|_{\min}=Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

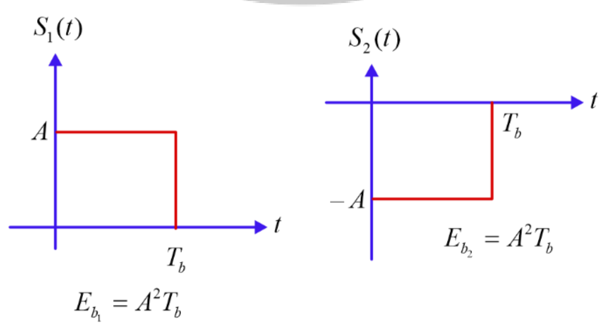
- Only when, $P(0\ T_X)=P(1\ T_X)=1/2$
- AWGN, $\lambda\rightarrow\lambda_{opt}$
- M.F

**For K noise R.V**

$$P_e|_{\min}=Q\left[\sqrt{K}\left\{\frac{(a_1-a_2)}{2\sigma}\right\}\right]$$

$$P_e|_{\min}|_{\min}=Q\left[\sqrt{K}\sqrt{\frac{E_d}{2N_0}}\right]$$

**1.**




$$E_d=\int_{-\infty}^{\infty}d^2(t)dt=4A^2T_b$$

$$d(t)=S_1(t)-S_2(t)=2\text{ A}$$

$$0\le t\le T_b$$

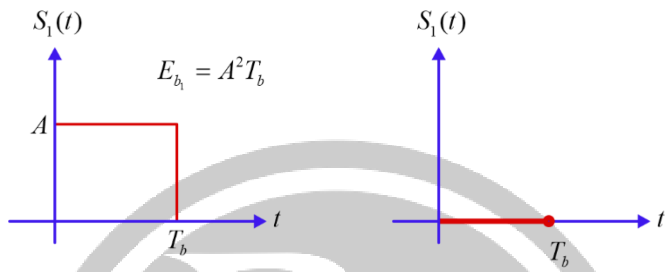
$$P_e = Q\left[\sqrt{\frac{2A^2T_b}{N_0}}\right]$$

$$P_e = Q\left[\sqrt{\frac{2(E_b)_{avg}}{N_0}}\right]$$

$$(E_b)_{avg} = A^2T_b$$

$$E_b = A^2T_b$$

**2.**




$$P_e = Q\left[\sqrt{\frac{E_b}{N_0}}\right]$$

$$(E_b)_{avg} = \frac{E_b}{2}$$

$$P_e = Q\left[\sqrt{\frac{A^2T_b}{2N_0}}\right]$$

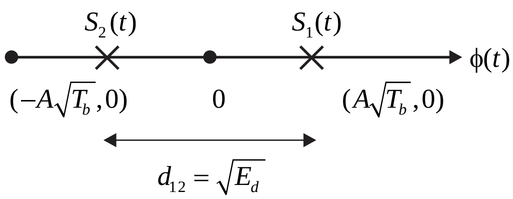
## M-ary Base Bond Signaling

1. Bit rate : $R_b$
2. Bit interval $= T_b = 1/R_b$
3. Symbol duration $= T_s = NT_b$
4. Symbol rate or Baud or Baud rate $R_s = \frac{1}{T_s} = \frac{1}{NT_b} = \frac{R_b}{N}$
5. $T_X$ Bandwidth $(BW) \geq R_s \rightarrow$ Rectangular, $(BW) \geq \frac{R_s}{2} \rightarrow \sin C$

$$(BW)_{\min} = \frac{1}{2}\left(\frac{R_b}{N}\right) = \frac{R_s}{2}$$

## M-ary PAM (2-Any PAM)

1. $M = 2, (N \leq M)$, $s(t) = \begin{cases} S_1(t) = A & 0 \leq t \leq T_b \\ S_2(t) = A & 0 \leq t \leq T_b \end{cases}$ NRZ coding.




$$P_e|_{\min}|_{\min} = Q\left[\sqrt{\frac{E_d}{2N_0}}\right]$$

$$E_d = (d_{12})^2$$

$$P_e = Q\left[\sqrt{\frac{2A^2T_b}{N_0}}\right] \text{ for NRZ}$$

$$(E_s)_{avg} = A^2T_b$$

- Distance of each point from origin $= \sqrt{\text{Energy of that point}}$
- Distance between two point $= \sqrt{\text{Difference energy}} = d_{12}$
- $d_{12}\uparrow \rightarrow Q(\ )\downarrow \rightarrow P_e\downarrow$

$$P_e = Q\left[\sqrt{\frac{A^2T_b}{2N_0}}\right] \text{ for RZ}$$

## Bandpass Sampling:

**(a) Binary ASK :** (m-ary ASK), For '1'→ A, '0', →0

$$P_e = Q\left[\sqrt{\frac{E_d}{2N_0}}\right] = Q\left[\sqrt{\frac{A^2T_b}{4N_0}}\right]$$

$$(E_b)_{avg} = p_1E_1 + p_2E_2 = \frac{1}{2}\times\left(\frac{A^2T_b}{2}\right)$$

$$P_e = Q\left[\sqrt{\frac{(E_b)_{avg}}{N_0}}\right]$$

$$P_e = Q\left[\sqrt{\frac{A^2T_b\cos^2\phi}{4N_0}}\right]$$

**Correlator Based :**

$$\phi(t) = \sqrt{\frac{2}{T_s}}\cos 2\pi f_C t$$

$$0 \le t \le T_s$$

$$"1" = A\cos\omega_C t \qquad 0 \le t \le T_b$$

$$"0" = 0 \qquad 0 \le t \le T_b$$

**(b) BPSK :** $p(t)=\begin{cases} A & 0\le t\le T_b \;:\; 1T_X \\ -A & 0\le t\le T_b \;:\; 0\,T_X \end{cases} \rightarrow \text{Baseband}$

$$s(t)=\begin{cases} A\cos 2\pi f_C t & 0\le t\le T_b \;:\; 1T_X \\ A\cos(2\pi f_C t+\pi) & 0\le t\le T_b \;:\; 0\,T_X \end{cases}$$

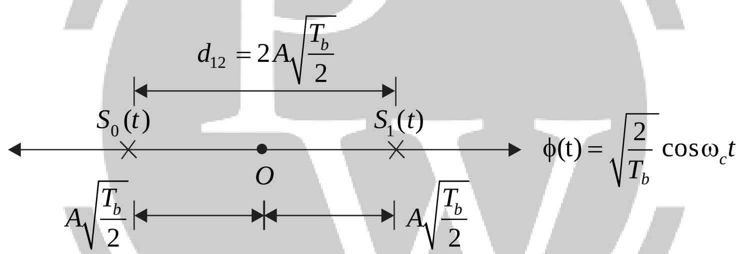
$$P_e=Q\left[\sqrt{\frac{A^2T_b}{N_0}}\right]$$

## Orthonormal Basis Function

$$\phi(t)=\sqrt{\frac{2}{T_b}}\cos 2\pi f_c t \qquad 0\le t\le T_b$$

$$P_e=Q\left[\sqrt{\frac{A^2T_b\cos^2\theta}{N_0}}\right]$$

$$P_e=Q\left[\sqrt{\frac{d_{12}^2}{2N_0}}\right]$$




## M-Ary PSK (Quadrature PSK):

$$M=4 \qquad S_k(t)=A\cos\left(2\pi f_c t+\frac{2\pi}{M}K\right) \qquad 0\le t\le T_s,(T_s=NT_b)$$

$$N=2 \qquad S_k(t)=A\cos\left(2\pi f_c t+\frac{\pi}{2}K\right) \qquad T_s=N\,T_b$$

$K=0,1,2,3$ $\qquad d_{\min}=2d_0\sin\left(\frac{\phi}{2}\right) \qquad d_0=\sqrt{E_s},\quad \phi=\frac{2\pi}{M}$

$$d_{12}=2d_0\sin\left(\frac{\phi}{2}\right)$$

## M-ary PSK

$$M=(2^N)$$

1. Bit Interval $=T_b$, Bit rate $=R_b$, symbol duration $(T_s=NT_b)$
2. Baud rate or symbol rate $R_s=\frac{1}{T_s}=\frac{R_b}{N}$

3. Bit energy $\Rightarrow E_b = \frac{A^2}{2} T_b$

4. Symbol energy $E_s = NE_b$

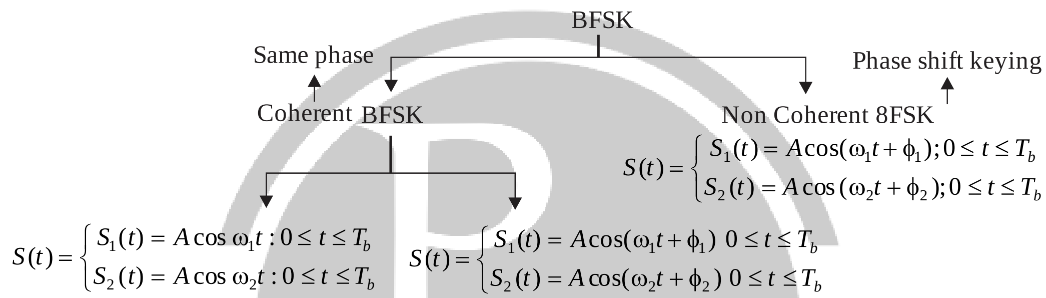
5. Radius of constellation : $d_0 = \sqrt{E_s}$

6. Area of constellation circle $= \pi d_0^2 = \pi E_s$

7. $d_{\min} = 2d_0 \sin\left(\frac{\phi}{2}\right), \left(\phi = \frac{2\pi}{M}\right)$

## Binary FSK

$$SFSK(t) = \begin{cases} A\cos 2\pi f_1 t : 0 \le t \le T_b : 1T_X \\ A\cos 2\pi f_2 t : 0 \le t \le T_b : 0T_X \end{cases} \qquad (f_1 >>> f_2)$$




## Coherent BFSK

1. $\phi = 0,\ R_b = \text{HCF}[mR_b, nR_b] = \text{HCF}[2(f_1 + f_2), 2(f_1 - f_2)]$

2. $(\phi \ne 0)\ R_b = \text{HCF}[mR_b, nR_b] = \text{HCF}[(f_1 + f_2), 2(f_1 - f_2)]$

3. Non-Coherent – $R_b = \text{HCF}[mR_b, nR_b] = \text{HCF}[(f_1 + f_2), (f_1 - f_2)]$

## Condition for Orthogonality

Coherent FSK
- $d = 0 (f_1 + f_2) = \frac{mR_b}{2}, (f_1 - f_2) = \frac{nR_b}{2}$
- $d \ne 0 (f_1 + f_2) = mR_b, (f_1 - f_2) = nR_b$

$R_b = \text{HCF}[mR_b, nR_b]$

## Non-Coherent FSK

$\phi_1, \phi_2$

$(f_1 + f_2) = mR_b, (f_1 - f_2) = nR_b$

$R_b = \text{HCF}(mR_b, nR_b)$

➢ $P(OT_X) = P(1T_X) = \frac{1}{2}$

➢ Channel Noise : White (AWGN)

- $\lambda_{opt}$

- Filter Method → $f_1 = k \mid T_b$ ; $f_2 = m \mid T_b$

$$Pe = Q\left[\sqrt{\frac{E_d}{2N_0}}\right] = Q\left[\sqrt{\frac{A^2 T_b}{2N_0}}\right]$$

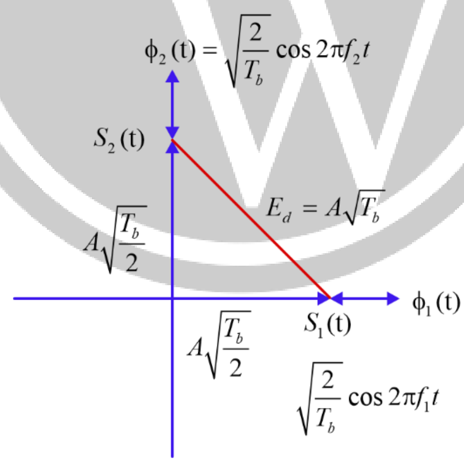
### Orthogonal FSK :

$$Pe = Q\left[\sqrt{\frac{0.5A^2 T_b}{N_0}}\right]$$

### Non-orthogonal FSK :

$$Pe = Q\left[\sqrt{\frac{0.6A^2 T_b}{N_0}}\right]$$

$$Pe = Q\left[\sqrt{\frac{(d_{12})^2}{2N_0}}\right] = Q\left[\sqrt{\frac{A^2 T_b}{2N_0}}\right]$$




### M-ary FSK

N bits are grouped together so that $M = 2^N$ symbols or sinusoids of duration $T_s = NT_b$ are generated having

- Same amplitude, same frequency, different frequency.

$$f_k = \frac{n}{T_s} \quad E_{s_0} = E_{s_1} = \ldots = E_{s_{M-1}} = \left(\frac{A^2}{2} \times T_s\right)$$

$$T_s = NT_b \qquad R_s = \frac{1}{T_s} = \frac{1}{NT_b} = \frac{R_b}{N}$$

| **Scheme** | | ***Pe*** | **For *K*** |
|---|---|---|---|
| **BASK** | $\longrightarrow$ | $P_e = Q\left[\sqrt{\dfrac{A^2T_b}{4N_0}}\right]$ | $\Rightarrow P_e = Q\left[\sqrt{\dfrac{KA^2T_b}{4N_0}}\right]$ |
| **BPSK** | $\longrightarrow$ | $P_e = Q\left[\sqrt{\dfrac{A^2T_b}{N_0}}\right]$ | $\Rightarrow P_e = Q\left[\sqrt{\dfrac{KA^2T_b}{N_0}}\right]$ |
| **BFSK** | $\longrightarrow$ | $P_e = Q\left[\sqrt{\dfrac{A^2T_b}{2N_0}}\right]$ | $\Rightarrow P_e = Q\left[\sqrt{\dfrac{KA^2T_b}{2N_0}}\right]$ |
| | | $P_e = Q\left(\dfrac{\mu_1 - \mu_2}{2N_0}\right)$ | $\Rightarrow Q\left[\sqrt{K}\left(\dfrac{\mu_1 - \mu_2}{2N_0}\right)\right]$ |

For $K$ AWGN identical independent.

## Amplitude phase shift keying (APSK)

$$S_i(t) = r_i \cos[2\pi f_c t + \theta_i] \qquad (0 \le t \le T_s) \qquad i = 0 \text{ to} \ldots M-1$$

$$T_s = NT_b$$

**Case 1:** $r_i =$ constant

$\theta_i =$ variable

**Case 2:** $r_i =$ variable

$\theta_i =$ constant

$$\begin{cases} S_i(t) = r_i \cos(2\pi f_c t + \theta_i) \rightarrow \text{M-Ary PSK} \\ S_i(t) = r_i \cos(2\pi f_c t + \theta_i) \rightarrow \text{M-Ary ASK} \end{cases}$$

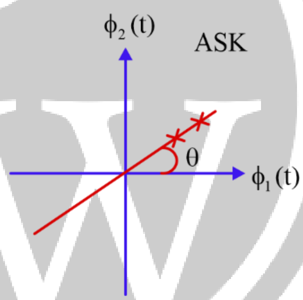



**Fig. ASK**

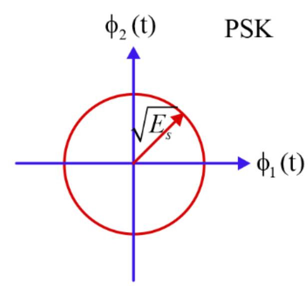
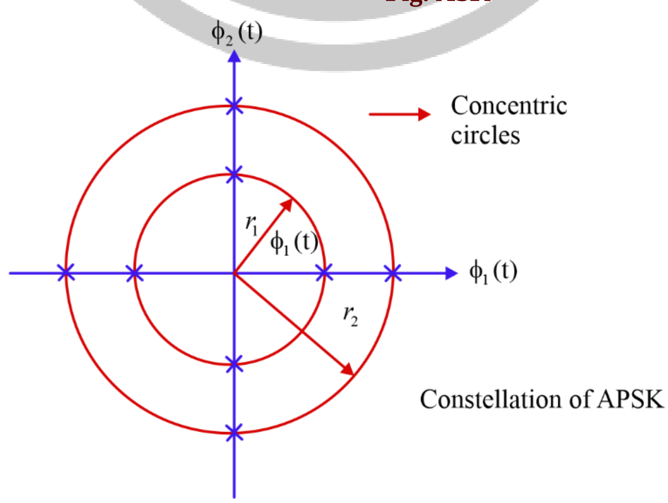



**Fig. PSK**




**8 point APSK = 8 point QAM**

❒❒❒

# 6 INFORMATION THEORY

## 6.1. Introduction

| Information in Event $(X = x_i)$ | Base | Unit |
|---|---|---|
| $I[X = x_i] = -\log_b p\{X = x_i\}$ | 2 | Bits |
| | 10 | Decit |
| | $e$ | Nat |

### 6.1.1. Properties of Digital Information

1. $I[X = x_i] = -\log_2 P[X = x_i]$
2. $P[X = x_i] > P[X = x_2] \Leftrightarrow I[X = x_2] < I[X = x_2]$
3. $P[X = 1] = \log_2 1 = 0$ bits
4. $P[X = 0] = -\log_2 0 = \infty$ bits
5. $0 \le P[X = x_i] \le 1 \;\Leftrightarrow 0 \text{ bits} \le I[X = x_i] < \infty$ bits
6. For any event $[X = x_i], I[X = x_i] \ge 0$
7. $I[(X = x_i) \cap (X = x_2)] = I[X = x_1] + I[X = x_2]$

Average information of source $X$ = Entropy of source $X$

$$H[X] = -\sum_{i=1}^{M} P[X = x_i] \log_2 P[X = x_i] \text{ bits/symbol}$$

$$H[X] = -\sum_{i=1}^{M} P_i \log_2 P_i$$

**Case 1.** All $M$ events are equiprobable –

$H[X] = \log_2 M$ bits/symbol $\Rightarrow$ Maximum entropy

**Case 2.** Out of $M$ events only 1 event is certain.

$H[X] = 0$

$$0 \le H[X] \le \log_2 M \qquad M = \frac{1}{P}$$

Information Rate – Symbol rate = $r$ symbols/sec

Entropy = H(X) bits/symbol

Information Rate R = r H (X) bits/sec

1. If $r = f_s$ and all event equiprobable, $L = 2^n$, $H(X) = \log_2 L$

   $R = nf_s$

## Source Coding

1. Reduces the redundancy of bits.
2. Two types of source coding
   (a) Fixed length source coding
   (b) Variable length source coding
      (i) Shannon Fano coding
      (ii) Huffman coding

**Key Point :**

(a) Average code length $= L_{avg} = \sum_{i=1}^{K} n_i p_i$

(b) Entropy of source $= H(X)$

(c) Code efficiency $\eta = \dfrac{H(X)}{L_{avg}}$

η should be as high as possible.

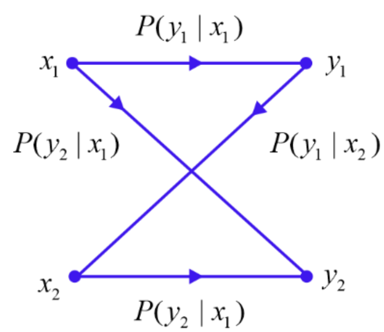
(d) Code redundancy λ = (1 – η)

**Discrete channel :** A channel is called as discrete if $X$ and $Y$ are having finite size.

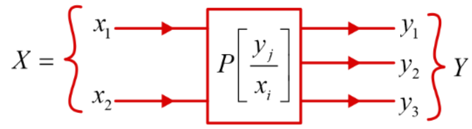
**Memoryless channel :** Each present output symbol depends on present input symbol.

$$x = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \rightarrow \boxed{P(y_j / x_i)} \rightarrow \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = y$$

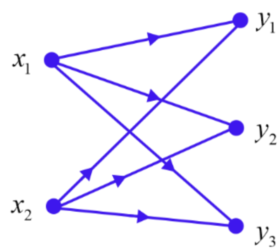
## Binary Channel : (2 input & 2 output)



### Non Binary Channel :




### Binary Channel :




*Sum of elements of row in channel matrix is always ‘1’.

**Joint Channel Matrix**

$$[P(x;y)] = \begin{matrix} & y_1 & \cdots & y_m \\ \begin{matrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{matrix} & \multicolumn{3}{l}{\begin{bmatrix} P(x_1 \cap y_1) & \cdots & P(x_1 \cap y_m) \\ P(x_2 \cap y_1) & \cdots & P(x_2 \cap y_m) \\ & & \\ P(x_n \cap y_1) & \cdots & P(x_n \cap y_m) \end{bmatrix}_{n\times m}} \end{matrix}$$

$$[P(x \cap y)] = P(X)P\left(\frac{Y}{X}\right)$$

**Condition Channel Matrix**

$$[P\left(\frac{y}{x}\right) = \begin{matrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{matrix} \begin{bmatrix} P\left(\frac{y_1}{x_1}\right) & \cdots & P\left(\frac{y_m}{x_1}\right) \\ \vdots & \vdots & \vdots \\ P\left(\frac{y_1}{x_n}\right) & \cdots & P\left(\frac{y_m}{x_n}\right) \end{bmatrix}_{n\times m}$$

$$[P(y)] = P(X)P\left(\frac{Y}{X}\right)$$

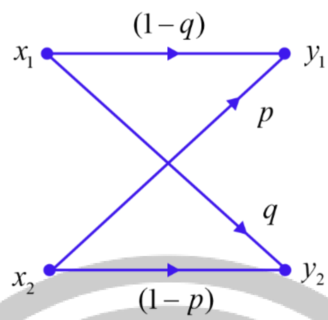
$$P(y_1) = P(x_1 \cap y_1) + P(x_2 \cap y_1) + \ldots + P(x_n \cap y_1)$$

$$[P(y)]_{1\times m} = [P(x_1), P(x_2) \ldots P(x_n)]_{1\times n}$$

$$[P(y)]_{1\times m} = [P(x)]_{1\times n}\left[P\left(\frac{Y}{X}\right)\right]_{n\times m}$$

$$\begin{bmatrix} P\left(\frac{y_1}{x_1}\right) & \cdots & P\left(\frac{y_m}{x_1}\right) \\ \vdots & & \vdots \\ P\left(\frac{y_1}{x_n}\right) & \cdots & P\left(\frac{y_m}{x_n}\right) \end{bmatrix}_{n\times m}$$

## Binary Non-symmetrical channel




## Cross over probabilities are different

$$P\left(\frac{Y}{X}\right) = \begin{array}{c} \quad y_1 \quad\quad y_2 \\ \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix} \begin{array}{c} x_1 \\ x_2 \end{array} \end{array}$$

$$= \begin{bmatrix} P\left(\frac{y_1}{x_1}\right) & P\left(\frac{y_2}{x_1}\right) \\ P\left(\frac{y_1}{x_2}\right) & P\left(\frac{y_2}{x_2}\right) \end{bmatrix}_{2\times 2}$$

- $P(x_1) + P(x_2) = 1$
- $P(y_1) + P(y_2) = 1$
- $P\left(\frac{y_1}{x_1}\right) + P\left(\frac{y_2}{x_2}\right) = 1$
- $P\left(\frac{y_1}{x_2}\right) + P\left(\frac{y_2}{x_2}\right) = 1$
- $P(y_1) = P(x_1 \cap y_1) + P(x_2 \cap y_1)$
- $P(y_2) = P(x_1 \cap y_2) + P(x_2 \cap y_2)$

## Aposteriori Probabilities

$$P\left(\frac{x_1}{y_1}\right) = \frac{P(x_1)P\left(\frac{y_1}{x_1}\right)}{P(y_1)}$$

$$P\left(\frac{x_2}{y_2}\right) = \frac{P(x_2)P\left(\frac{y_2}{x_2}\right)}{P(y_2)}$$

## Map Analysis

### (a) At $r_0$ :

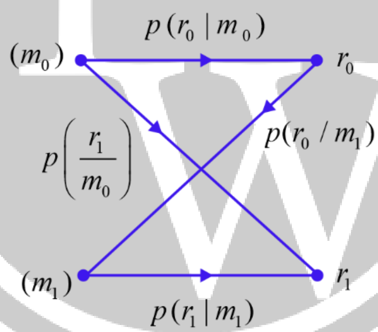
$$P\left(\frac{m_0}{r_0}\right) \underset{m_1}{\overset{m_0}{\gtrless}} P\left(\frac{m_1}{r_0}\right)$$

$$P(m_0)P\left(\frac{r_0}{m_0}\right) \underset{m_1}{\overset{m_0}{\gtrless}} P(m_1)P\left(\frac{r_0}{m_1}\right)$$

### (b) At $r_1$:

$$P\left(\frac{m_0}{r_1}\right) \underset{m_1}{\overset{m_0}{\gtrless}} P\left(\frac{m_1}{r_1}\right)$$

$$P(m_0)P\left(\frac{r_1}{m_0}\right) \underset{m_1}{\overset{m_0}{\gtrless}} P(m_1)P\left(\frac{r_1}{m_1}\right)$$




- After MAP application receiver will be optimum.

## Probability of Correctness

$$P_c = P(m_0 \cap r_0) + P(m_0 \cap r_1)$$

$$P_c = P(m_0)P\left(\frac{r_0}{m_0}\right) + P(m_0)P\left(\frac{r_1}{m_0}\right)$$

$$P_e = 1 - P_c$$

## Binary Symmetrical Channel

Cross over probabilities are same.

- $P\left(\frac{x_1}{y_2}\right)$ = Probability that $x_1$ was transmitted given than $y_2$ received

$$P\left(\frac{x_1}{y_2}\right)+P\left(\frac{x_2}{y_2}\right)=1$$

$$P\left(\frac{x_1}{y_1}\right)+P\left(\frac{x_2}{y_1}\right)=1$$

## Joint Entropy

$$H(XY)=-\sum_{i=1}^{n}\sum_{j=1}^{m}P(x_i,y_j)\log_2 P(x_i,y_j)$$

$$H(XY)=-\sum_{i=1}^{n}\sum_{j=1}^{m}P\{(X=x_i)\cap(Y=y_j)\}\log_2 P\{(X=x_i)\cap(Y=y_j)\}$$

$$H(XY)=H(YX)$$

## Conditional Entropy

$$H\left(\frac{X}{Y}\right)=-\sum_{i=1}^{n}\sum_{j=1}^{m}P\{(X=x_i)\cap(Y=y_j)\}\log_2 P\left\{\frac{X=x_i}{Y=y_j}\right\}$$

Conditional entropy of $X$ given $Y$

- Similarly can write $H\left(\frac{Y}{X}\right)$

**Important point :**

1. $H(XY)=H(X)+H\left(\frac{Y}{X}\right)$
2. $H(XY)=H(Y)+H\left(\frac{X}{Y}\right)$
3. If $X$ and $Y$ statically, independent $H(XY)=H(X)+H(Y)$

$$H\left(\frac{Y}{X}\right)=H(Y), H\left(\frac{X}{Y}\right)=H(X)$$

For B.S.C – $C_s$ = Channel capacity

$$C_s=1+P\log_2 P+(1-P)\log_2(1-P)$$

$$C_s=\{I(X;Y)\}_{\max}$$

$$I(X;Y)=H(Y)-H\left(\frac{Y}{X}\right)$$

$$I(X;Y)=H(X)-H\left(\frac{X}{Y}\right)$$

$$H\left(\frac{Y}{X}\right) = -\sum\sum P(x_i, y_j)\log_2 P\left(\frac{y_i}{x_i}\right)$$

$$C_s = \log_2 n$$

$I(X;Y) = H(X)$ loss less channel

**Lossless Channel :**

1. Single non zero element in each column.
2. Channel matric should be DMC type
3. Summation of each now must be 1.
4. $H\left(\frac{X}{Y}\right) = 0 \quad I(X;Y) = H(X),\ C_s = I[X;Y]_{\max} = [H(X)]_{\max} = \log_2 n$

   $n =$ number of input symbol.

## 6.2. Average Mutual Information

$$I(X;Y) = I(X) - I\left(\frac{X}{Y}\right)$$

$$I(X;Y)_{Avg} = H(X) - H\left(\frac{X}{Y}\right) \text{ bit/symbol}$$

$$I(X;Y)_{Avg} = H(Y) - H\left(\frac{Y}{X}\right)$$

$$I(X;Y) = I(Y;X)$$

$$I(X;Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P\{x_i, y_j\}\log_2\left[\frac{P(x_i \mid y_j)}{P(x_i)}\right]$$

$$I(X;Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\log_2\left[\frac{f_{\frac{X}{Y}}\left(\frac{x}{y}\right)}{f_X(x)}\right]dxdy$$

$$P_{XY}(x_i, y_j) = P_X(y_j)P\left(\frac{x_i}{y_j}\right)$$

$$P_{XY}(x_i, y_j) = P(x_i)P\left(\frac{y_j}{x_i}\right)$$

If R.VS are Independent then $I(x;y) = 0$

## 6.2.1. Channel Capacity

### Maximum Average Mutual Information

$$C_s = \{I(x;y)\}_{\max}$$

$$I(x;y) = H(Y) + P\log_2 P + (1-P)\log_2(1-P)$$

$$C_s = 1 + P\log_2 P + (1-P)\log_2(1-P)$$

### B.S.C

P → Cross over probability

- Input are equiprobable.

### Determine Channel :

1. Number of rows in each row must be single.
   - In each row angle element must be 1.
   - Summation of each row will become 1.
   - $H\left(\frac{Y}{X}\right) = 0, I(X;Y) = H(Y) - H\left(\frac{Y}{X}\right)$

     $I(X;Y) = H(Y)$

     $C_s = [H(Y)]_{\max} = \log_2 m$ bit/symbol

### Noise Less Channel :

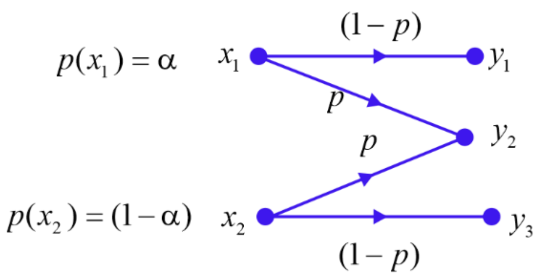
1. Deterministic + Lossless
2. Each row → Single element
3. Each column → Single element

(2 and 3: Mult be 1)

4. $H\left(\frac{X}{Y}\right) = 0, H\left(\frac{Y}{X}\right) = 0$
5. $I(X;Y) = H(X) = H(Y)$
6. $C_s = [I(X;Y)]_{\max} = [H(X)]_{\max} = [H(y)]_{\max} = \log_2 m = \log_2 n$ bits/symbol

   $m = n$

### Binary Erasure Channel



$$I(X;Y) = (1-P)H(X)$$

$$C_s = I[(X;Y)]_{max}$$

$$= (1-P)\log_2 n \qquad n = 2 \text{ for BEC}$$

$$C_s = (1-P)$$

- $C_s = I(X;Y) = \frac{1}{2}\log_2\left[1+\left(\frac{\sigma_x^2}{\sigma_N^2}\right)\right]$ Bits/symbol

$\sigma_N^2 = N_0 B$

(i) $X$ is zero mean R.V $\quad E[X^2] = \sigma_x^2 = S$

(ii) Noise is zero mean R.V $\quad E[N^2] = \sigma_N^2 = N$

**Channel Capacity of AWGN Channel**

$$C_s = \frac{1}{2}\log_2\left(1+\frac{S}{N}\right) \text{ Bit/symbol}$$

$$C = B\log_2\left(1+\frac{S}{N}\right) \text{ bit/sec}$$

**Channel capacity for AWGN**

$$\underset{\text{Channel Capacity}}{C_s} \geq \underset{\text{Information rate}}{R}$$

- For X = zero mean R.V, $N = N_0 B$

  $B \to \infty$

$$C_s = 1.44\frac{S}{N_0} \quad \text{Finite value}$$

For loss less transmission.

## 6.3. Continuous Source and Differential Entropy

X : DRV, $\quad H(X) = -\sum_i p[x = x_i]\log_2 p\{x = x_i\}$

X : CRV, $\quad H(X) = -\int_{-\infty}^{+\infty} Fx(x)\log_2 f_x(x)dx \rightarrow$ Differential entropy

Y : DRV, $\quad H(Y) = -\sum_j p\{y = y_j\}\log_2 p\{y = y_j\}$

Y : CRV, $\quad H(Y) = -\int_{-\infty}^{-\infty} f_y(y)\log_2 f_y(y)dy \rightarrow$ Differential entropy

## 6.3.1. Channel capacity

### (1) For error less | distortion less transmission

(i) If all quantization level are not eauiprobable :

$$C \ge R$$

$$C \ge rH(X)$$

$$C \ge f_S H(X)$$

(ii) If all quantization level are eauiprobable :

$$C \ge R$$

$$C \ge rH(X)$$

$$C \ge f_s \log_2 L$$

$$C \ge nf_s$$

$$\boxed{C \ge R_b}$$

### (2) For AWGN channel- Y: GRV,

Y = X + N, Let $X$ and $N$ are independent

$$\sigma_y^2 = \sigma_x^2 + \sigma_N^2$$

$$H(y) = \frac{1}{2}\log_2[2\pi\sigma_y^2 e]$$

$$\boxed{H(\text{y}) = \frac{1}{2}\log_2[2\pi\text{e}[\sigma_N^2 + \sigma_X^2]} \quad \text{Maximum}$$

$$\boxed{C_S = \frac{1}{2}\log_2\left(1 + \frac{\sigma_X^2}{\sigma_N^2}\right)} \frac{\text{bits}}{\text{symbol}}$$

$\sigma_X^2 \rightarrow P_x$ = Power of signal $X$

$\sigma_N^2 \rightarrow P_x$ = Noise Power

$$C = C_2 \times f_s$$

$$\sigma_N^2 = N_0 B$$

$$\boxed{C = B\log_2\left(1 + \frac{\sigma_X^2}{\sigma_N^2}\right)} \frac{\text{Bits}}{\text{sec}}$$

$\frac{N_0}{2} \rightarrow$ PSD of white Noire

$B \rightarrow$ B.W of channel

$$\boxed{C = B\log_2(1+\text{SNR})}$$
$\downarrow$
Not in dB

$$\boxed{\delta_{NR} = \frac{P_X}{P_N} = \frac{\sigma_X^2}{\sigma_N^2}}$$

$$\boxed{C = B\log_2\left(1+\frac{E_b R_b}{N_0 B}\right)}$$

$$\boxed{C = 1.44\frac{P_X}{N_0}}$$

For infinite Bandwidth $B \to \infty$

## Information in bits | symbol

$$H\left[\frac{y}{x_0}\right] = I\left[\frac{y_0}{x_0}\right]P\left[\frac{y_0}{x_0}\right]P(x_0) + I\left[\frac{y_1}{x_0}\right]P\left[\frac{y_1}{x_0}\right]P(x_0)$$

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# 7 MISCELLANEOUS

## 7.1. FDMA (Frequency Division Multiplexing)

- Multiple signals are multiplexed and simultaneously transmitted through channel.

  $K$ = Number of signals are multiplexed

  B.W ≥ K [B.W of modulation scheme] + ($K$ – 1) [BW of guard Band]

  TDMA (Time division Multiplexing)

  $T_s$ = Frame rate or sampling interval or time taken by commentator to completer its 1 rotation

  (Band limited to same freq.)

  $T_s = nT_b \times N$

  $\boxed{T_s = NnT_b}$ $N$ = Number of signals being multiplexed

  $n$ = of bits/sample

  $T_b = 1$ Bit duration

  $$\boxed{R_b = Nnf_s}$$

  $$\text{Speed of commentator } = f_s \frac{\text{rotation}}{\text{seocnd}} = f_s \times 60\,\text{rpm}$$

  $$(\text{BW})_{\min} = \frac{R_b}{2} = \frac{Nnf_s}{2}$$

- When $x$ number of synchronization $\frac{\text{bits}}{\text{frame}}$ are added – (Band limited to same freq.)

  $$\boxed{T_s = (Nn + x)T_b} \qquad \boxed{R_b = (Nn + x)f_s}$$

- $x\,\text{bit/frame} : Ts = (Nn + x)T_b$

  $x\,\text{bit/2frame} : Ts = \left(Nn + \frac{x}{2}\right)T_b$

- y% (Total of y%) synchronization bits are added –(Band limited to same freq.)

  $$T_s = \left[Nn + \frac{Nn \times y\%}{100}\right]T_b$$

$$R_b = \left[Nn + \frac{Nn \times y\%}{100}\right] f_s$$

- When N signals are band limited to different freq.

$$R_b = nf_{s_1} + nf_{s_2} + ......nf_{s_n}$$

COMA (Code division Multiple Aces)

Processing gain of CDMA $\Rightarrow G = \left(\frac{R_c}{R_h}\right)$

- Each user is assigned with unique code

## Noise

(1) PSD of thermal noise is Gaussian in nature. Also known as Johnson noise

(2) Thermal Noise power $P_n = 4KTBR = \overline{V_n^2} = (\text{V}_n)^2 \text{rms}$

Thermal Noise voltage

$$(V_n)\text{rms} = \sqrt{4KTBR}$$

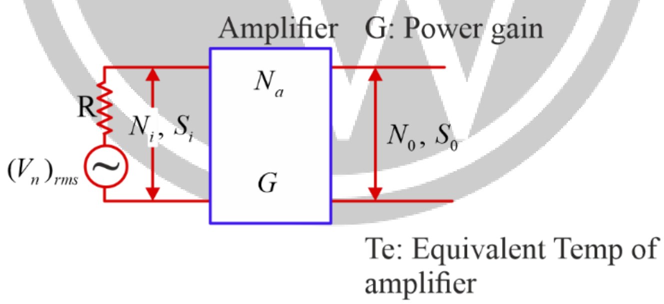
$$(I_n)\text{rms} = \frac{(V_n)\text{rms}}{R}$$

- Max. Power which could be deliver to amplifier = KTB
- Noise figure ($F$) or Noise factor

$$F(dB) = 10\log_{10} F$$

$$N_1 = \text{kTB}$$

$$N_o = N_i G + N_a = KTBG + N_a$$




$$\boxed{F = \frac{(N_i G + N_a)}{GN_i}}$$

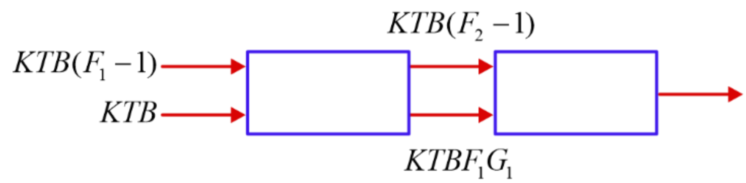
$$\text{F} = \frac{\text{Output Noise including Noisy amplifier}}{\text{Output Noise Considering noiseless amplifier}}$$

$$\boxed{T_e = \frac{Na}{\text{KBG}}} \qquad \boxed{F = \frac{(\text{SNR})_{i/p}}{(\text{SNR})_{o/p}}} = 1 + \frac{T_e}{T}$$

$\boxed{T_e = (f-1)T}$ $N_0$ (output Noise power) = KTBGF

$$N_0 = K\,BG(T + T_e)$$

## Cascaded Amplifier




Output Noise with noisy $amp^r = [KTB(F_2 - 1) + KTBF_1G_1]G_2$

$$F = F_1 + \frac{F_2 - 1}{G_1}$$

$$F = F_1 + \frac{F_2 - 1}{G_1} + \frac{(F_3 - 1)}{G_1G_2} + \text{..........}$$

$$T_e \text{ (equivalent Temp.) } = T_{e_1} + \frac{T_{e_2}}{G_1} + \frac{T_{e_3}}{G_1G_2} + \text{..........}$$

$$xdBW = (x + 30) \text{ dBm}$$

## Noise performance of Analog Signal

$$FOM = \frac{(\text{SNR})_0}{(\text{SNR})_i} = \frac{\text{SNR at the output of } R_X}{(\text{SNR}) \text{ in m}} = \gamma$$

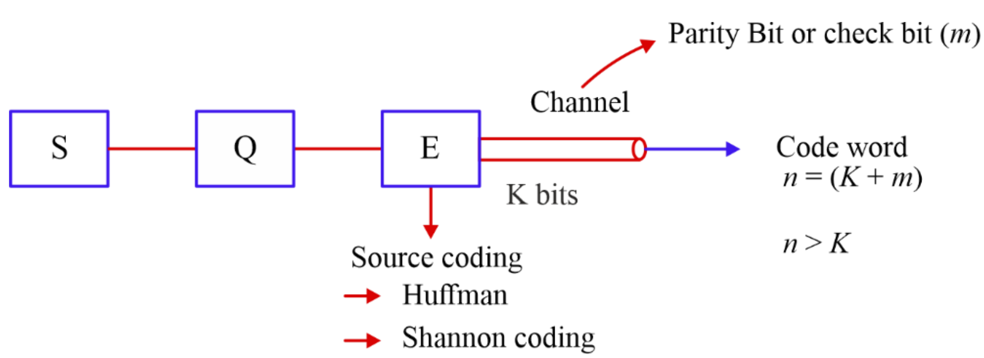
For DSB-Se $\boxed{\gamma = 1}$ , $(SNR)_0 = \frac{P_m}{2N_0B}, (SNR)_i = \frac{P_m}{2N_0B}$

For DCB-FC $\boxed{\gamma = \frac{p_m}{A_c^2 + p_m}} = \eta \rightarrow$ efficiency

For F.M $\gamma = \frac{3}{4\pi^2} \frac{K_f^2 p_m}{B^2} = \frac{3}{2}\beta_{FM}^2$ (For sinusoidal)

For PM $\gamma = K_p^2 P_m = \frac{\beta_{PM}^2}{2}$ (for sinusoidal)

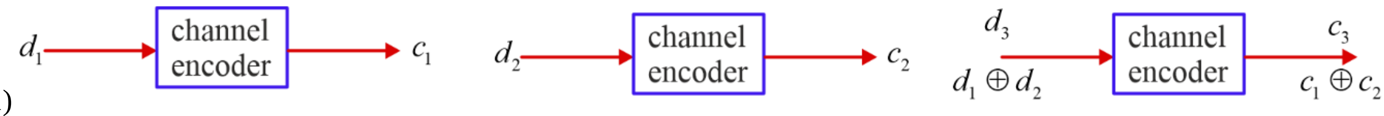
## Channel Coding



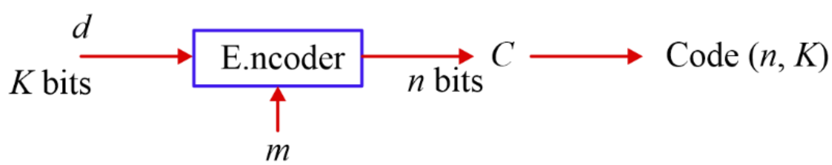
## Linear Block Code

⊕ modulo 2 sum → EXOR

(1)




(2)




(3) Different data words (message word) with $K$ bits = $2^K$

(4) Each data word will have m parity bits attached to generate $2^K$ code words.

(5) Total no of arrangements with n bits at output of encoder will be $\rightarrow 2^n$ out of which only $2^K$ code words are valid.

(6) Rate efficiency = code efficiency = code rate = $K / n$

## Hamming Weight

Number of 1's present in L, B,C

C (7.4)

**Example**: C : 1110001, H.W = 4

## Hamming distance

It represent bit change at respective position

$$X = 1\,1\,0\,1\,0\,1\,1\,1$$
$$Y = 0\,1\,1\,1\,0\,1\,0\,1 \quad d(x, y) = 3$$

## Minimum Hamming Distance ($d_{min}$):

Method 1 $d_{min}$ = Min hamming weight of $2^K$ codes except codes having 0 weight.

Combination: $^{2^K}C_2$ crosscheck

Method 2 $d\min \leq n - K + 1$

Method 3 "Minimum no of columns in parity check matrix [H] Which makes zero sum (module 2)."

## Error detection L.B.C

$\boxed{d_{min} \geq t + 1}$ can detect t errors

## Error correction $\boxed{d_{min} \geq 2t + 1}$

Code Generation at $T_X - [C]_{1\times n} = [D]_{1\times K}[G]_{K\times n}$ $\boxed{C = DG}$

$[G]_{K\times n} \rightarrow$ Generator Matrix

$[G]_{K\times n} = [I_K ; p]_{K\times n}$ or $[p; I_K]_{K\times n}$ $I_k$ = Identity Matrix of order $K$.

Parity check Matrix – $[H] = [P^T;\ I_n - K]_{(n-K)\times n}$

Or

$[H] = [I_{n-K};\ P^T]_{(n-K)\times n}$

Note – $[C]\,[H^T] = 0$

## Correction at Receiver

$c \rightarrow r$

$r = C$ (No error)

$r \neq C$ (Error)

- $r$ will given
- Calculate syndrome : $S = r[H]^T$
- Observe the syndrome: S matches with $i^{th}$ row of $[H]^T$ which Means $i^{th}$ bit from left has error.

Non systematic L.B.C

## Hamming Code

(1) It is a L.B.C

(2) $d_{\min} = 3$

(3) Detect upto 2 bit error

(4) Correct upto 1 bit error

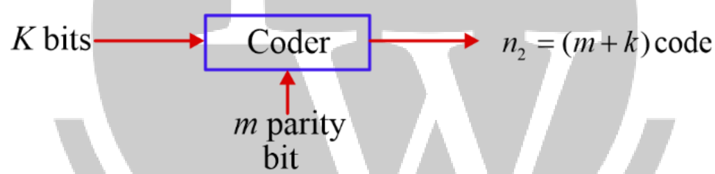
(5) K bit data, m bits parity $\Rightarrow n = (m + K)$ bits code

(6) Parity bit no. is calculated $2^m \geq (m + K + 1)$

$m = ?$

(7) Placing of parity bits ate at $2^0, 2^1, 2^2, ........$ locations

## Cyclic redundancy check code (CRC-Code)




## Problem solving Technique:

(i) $d = K$ bits msg

(ii) divisor polynomial

$x^3 + x + 1 = x^3 + 0x^2 + x + 1 \rightarrow (1011)$

**Step 1.** $K$ msg bits are given

From $(K + m)$ message bits

$\neq m \rightarrow$ addition of m zeros(append)

- Highest order of divisor polynomial (or) (number of bits in divisor polynomial)-1

**Step 2.** Modulo 2 division

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