Date: 03/11/2024

Time: 3 Hours

Number of Questions: 6

Max Marks: 102

Answers & Solutions

for

RMO – 2024-25

Instructions :

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks : 102.
- No marks will be awarded for stating an answer without justification.
- Answer all the questions.
- All questions carry equal marks.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let $n > 1$ be a positive integer. Call a rearrangement $a_1, a_2, \ldots, a_n$ of 1, 2, …, n nice if for every $k = 2, 3, \ldots, n$, we have that $a_1 + a_2 + \ldots + a_k$ is not divisible by k.
   (a) If $n > 1$ is odd, prove that there is no nice rearrangement of 1, 2, …, n.
   (b) If n is even, find a nice rearrangement of 1, 2, …, n.

Sol.

(a)

Let n be $2k + 1$ for some $k > 1$, $k \in Z$

Then for an arrangement to be nice

$n \nmid (a_1 + a_2 + \ldots + a_n)$

But $a_1 + \ldots + a_n = \dfrac{n(n+1)}{2} = \dfrac{(2k+1)(2k+2)}{2} = (k+1)(2k+1) = (k+1)n$

$\Rightarrow \quad n \mid n(k+1) \quad \Rightarrow \quad n \left| \left(\sum_{i=1}^{n} a_i \right) \right.$

Therefore, no nice rearrangement exists for $n \in$ odd

(b)

Let n be $2k$ for some $k \geq 1$, $k \in z$,

Then lets see through some examples

For $n = 2$

$\Rightarrow$ arrangement of {1, 2} such that $2 \mid \Pi(\text{arrangement})$

$\Rightarrow$ 1 at unit place

$\Rightarrow$ {2, 1} is such rearrangement

For $n = 4$

arrangement of {1, 2, 3, 4}

such that $2 \nmid (a_1 + a_2) \Rightarrow \{2, 1\}$

$3 \nmid (a_1 + a_2 + a_3) \Rightarrow \{2, 1, 4\}$

And $4 \nmid (a_1 + a_2 + a_3 + a_4) \rightarrow$ this clearly satisfies as $4 \nmid 10$

$\Rightarrow$ (2, 1, 4, 3) is such rearrangement

$\Rightarrow$ 2, 1, 4, 3, 6, 5 … will be the nice arrangement

$$\{a_i\} \Rightarrow a_i = \begin{cases} i+1, & i \in \text{odd} \\ i-1, & i \in \text{even} \end{cases}$$

$\Rightarrow \quad \sum_{i=1}^{k} (a_i)$ should not be divisible by k then

It would lead to nice rearrangement

For even n,

Let, $k = 2m + 1$, where $m \in I^+$

$$\sum_{i=1}^{k} a_k = \sum_{i=1}^{2m+1} a_i = \left(a_1 + a_2 + \ldots a_{2m} + a_{2m+1}\right)$$

$$= (2+1) + (4+3) + \ldots + (2m + 2m - 1) + (2m + 2)$$

$$= (1 + 2 + \ldots + 2m) + 2(m+1)$$

$$= \frac{(2m)\,(2m+1)}{2} + 2(m+1)$$

$$= 2m^2 + m + 2m + 2 = \left(2m^2 + 3m + 1\right) + 1$$

$$= (2m+1)\,(m+1) + 1$$

Clearly, $(2m+1) \mid (2m+1)(m+1) + 1,\ m \in I^+ \quad \Rightarrow k \nmid \sum_{i=1}^{k} a_i$

Now let $k = 2m$, where $m \in I^+$

$$\sum_{i=1}^{k} = \sum_{i=1}^{2m} a_i = (a_1 + a_2) + (a_3 + a_4) + \ldots + \left(a_{2m-1} + a_{2m}\right)$$

$= 1 + 2 + \ldots + (2m - 1) + (2m)$

$$= \frac{(2m)(2m+1)}{2} = n(2m+1)$$

$2n \nmid 2m^2$ but $2m \nmid m$ for $n \in I^+$

$\Rightarrow 2m \nmid \left(2m^2 + m\right) = m(2m+1)$

$\Rightarrow$ $\{a_i\}$ leads to nice rearrangement.

2. For a positive integer n, let $R(n)$ be the sum of the remainders when n is divided by 1, 2, …, n. For example $R(4) = 0 + 0 + 1 + 0 = 1$, $R(7) = 0 + 1 + 1 + 3 + 2 + 1 + 0 = 8$. Find all positive integers n such that $R(n) = n - 1$.

Sol.

For observation

$R(1) = 0$

$R(2) = 0 + 0 = 0$

$R(3) = 0 + 1 + 0 = 1$

$R(4) = 0 + 0 + 1 + 0 = 1$

$R(5) = 0 + 1 + 2 + 1 + 0 = 4$

$R(6) = 0 + 0 + 0 + 2 + 1 + 0 = 3$

$R(7) = 0 + 1 + 1 + 3 + 2 + 1 + 0 = 8$

$R(8) = 0 + 0 + 2 + 0 + 3 + 2 + 1 + 0 = 8$

$R(9) = 0 + 1 + 0 + 1 + 4 + 3 + 2 + 1 + 0 = 12$

So, for R(even) and R(odd) there is a pattern for second half. Now assuming

Case-I:

$n \in$ even

$$R(n) \geq 1 + 2 + ... + \left(\frac{n}{2} - 1\right) = \frac{n(n-2)}{8}$$

$$n - 1 \geq \frac{n(n-2)}{8}$$

$\Rightarrow \quad n^2 - 10n + 8 \leq 0$

$\Rightarrow \quad n \in (0, 9]$

But we can see that for $n = 2, 4, 6, 8$ will not satisfy the given relation.

Case-II:

$n \in$ odd

$$R(n) \geq 0 + 1 + 2 + 3 + ... + \left(\frac{n-1}{2}\right) = \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2} + 1\right)}{2} = \frac{n^2 - 1}{8}$$

$$n - 1 \geq \frac{n^2 - 1}{8}$$

$n^2 - 8n + 7 \leq 0$

$(n - 1)(n - 7) \leq 0$

$n \in [1, 7]$

$\Rightarrow \quad n \in 1, 3, 5, 7.$

$\therefore \quad$ Only $n = 1$ and 5 satisfies.

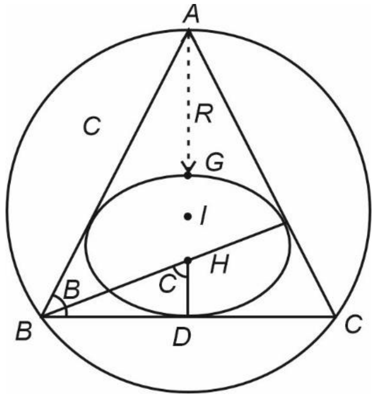
$\therefore \quad$ Positive set of integers $\{1, 5\}$

$\therefore \quad$ Only two integral values satisfies the given statement.

3. Let ABC be an acute triangle with $AB = AC$. Let D be the point on BC such that AD is perpendicular to BC. Let O, H, G be the circumcenter, orthocenter and centroid of triangle ABC respectively. Suppose that $2 \cdot OD = 23 \cdot HD$. Prove that G lies on the incircle of triangle ABC.

Sol.

As ΔABC be acute isosceles Δ



$\Rightarrow$ O, G, I and H will lie on same line, where I is Intentre

Let $AD = x$

$$\Rightarrow \quad GD = \frac{x}{3} = 2GI \quad \text{(as } GD \text{ is diameter)}$$

$$= 2ID$$

$BD = C\sin B$

$$HD = \frac{C\sin B}{\tan C} = 2R\sin^2 B. \ \left(AB = AC \Rightarrow \angle B = \angle C\right)$$

$OD = R\cos A$,

As $2OD = 23HD$ (Given)

$\Rightarrow$ $2R\cos A = (23)(2R\sin^2 B)$

$\Rightarrow$ $2\cos A = 23(1 - \cos^2 B)$, Also $\angle A = 180° - \angle B - \angle C$

$$\Rightarrow \quad \angle A = 180° - 2\angle B \quad \left(\angle B = \angle C\right)$$

$$\Rightarrow \quad \cos A = \frac{23}{25}, \sin A = \frac{4\sqrt{6}}{25}$$

$$\cos B = \frac{1}{5} \text{ and } \sin B = \frac{2\sqrt{6}}{5} \Rightarrow 5\sin A = 2\sin B$$

$$\Rightarrow \quad \frac{\sin A}{\sin B} = \frac{2}{5}$$

If $GD = 2r$ then GD will be diameter and G will lie on in circle

$$\Rightarrow \quad r = \frac{\Delta}{S} = \frac{\frac{1}{2}xa}{\frac{1}{2}(a+b+c)} = \frac{2R\sin A.x}{2R(\sin A + \sin B + \sin B)} = \frac{x\sin A}{6\sin A}$$

$= \frac{x}{6} = \frac{GD}{2}$

$\Rightarrow$ $GD = 2r$

$\Rightarrow$ G lies on in circle of ΔABC.

4. Let a_1, a_2, a_3, a_4 be real numbers such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Show that there exist i, j with $1 \leq i < j \leq 4$, such that $\left(a_i - a_j\right)^2 \leq \frac{1}{5}$.

Sol.

Let's assume that for real numbers

a_1, a_2, a_3, a_4 there exists $\left(a_i - a_j\right)^2 > \frac{1}{5}$

$1 \leq i \leq j \leq 4$

for

without losing the generality

$a_1 \geq a_2 \geq a_3 \geq a_4$, clearly equality doesn't holds otherwise proof will be done

$\Rightarrow$ $a_1 > a_2 > a_3 > a_4$

$\Rightarrow$ $\left(a_1 - a_2\right) > \frac{1}{\sqrt{5}}$ …(i)

$\left(a_2 - a_3\right) > \frac{1}{\sqrt{5}}$ …(ii)

$\left(a_3 - a_4\right) > \frac{1}{\sqrt{5}}$ …(iii)

Adding,

$\Rightarrow$ $a_1 > a_4 + \frac{3}{\sqrt{5}}$

$a_2 > a_4 + \frac{2}{\sqrt{5}}$

If $a_4 > 0 \Rightarrow a_i > 0$ $\forall\, i \in \{1, 2, 3, 4\}$

$\Rightarrow$ $a_2^2 > a_4^2 + \frac{4}{5} + \frac{4}{\sqrt{5}} a_4$

$a_1^2 > a_4^2 + \frac{9}{5} + \frac{6}{\sqrt{5}} a_4$

$a_3^2 > a_4^2 + \frac{1}{5} + \frac{2}{\sqrt{5}} a_4$

$\Rightarrow \quad \sum_{i=1}^{4} a_i^2 > 4a_4^2 + \frac{14}{5} + \frac{12}{\sqrt{5}} a_4 > 4\left(a_4 + \frac{3}{2\sqrt{5}}\right)^2 + \left(\frac{14}{\sqrt{5}} - \frac{9}{20}\right) \Rightarrow \sum_{i=1}^{4} a_i^2 > 1$

We reach to contradiction

If $a_1 < 0 \Rightarrow a_i < 0 \quad \forall\, i \in \{1, 2, 3, 4\}$

Let $b_i = -a_i \Rightarrow b_i > 0 \ \forall\, i \in \{1, 2, 3, 4\}$

$\Rightarrow$ from equation (i), (ii) and (iii)

$$b_2 - b_1 > \frac{1}{\sqrt{5}}$$

$$b_3 - b_2 > \frac{1}{\sqrt{5}}$$

$$b_4 - b_3 > \frac{1}{\sqrt{5}}$$

$\Rightarrow \quad b_1 < b_2 < b_3 < b_4$

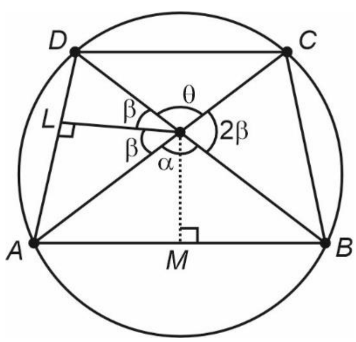
Will lead to similar contraction

$\Rightarrow \quad \sum b_i^2 > 1$

$\Rightarrow$ There must exist such $a_1, a_2, a_3, a_4 \in R$ such that $(a_i - a_j)^2 \le \frac{1}{5}$ $\forall\, i, j \in \{1, 2, 3, 4\}$

5. Let *ABCD* be a cyclic quadrilateral such that *AB* is parallel to *CD*. Let *O* be the circumcenter of *ABCD* and *L* be the point on *AD* such that *OL* is perpendicular to *AD*. Prove that $OB \cdot (AB + CD) = OL \cdot (AC + BD)$.

Sol.



$\because$ $AB \parallel CD$ and *ABCD* is cyclic

$\Rightarrow$ ABCD is cyclic trapezium

$\Rightarrow$ $AD = CB$ and $AC = BD$

Let *r* be the radius of circle

$OB \cdot (AB + CD) = r(AB + CD)$

Let $\angle DOC = \theta$, $\angle AOB = \alpha$

$\angle DOL = \angle AOL = \beta$

$\Rightarrow \quad \angle BOC = 2\beta$

$$AM = r\sin\frac{\alpha}{2}$$

$$AB = 2r\sin\left(\frac{\alpha}{2}\right)$$

Similarly, $CD = 2r\sin\left(\frac{\theta}{2}\right)$

$$AC = 2r\sin\left(\frac{2\beta+\theta}{2}\right)$$

$OL = r\cos\beta$

LHS = $r\left(2r\sin\frac{\alpha}{2} + 2r\sin\frac{\theta}{2}\right)$

$= 2r^2\left[\sin\frac{\alpha}{2} + \sin\frac{\theta}{2}\right]$ … (i)

RHS = $r\cos\beta\left[2r\sin\left(\frac{2\beta+\theta}{2}\right) + 2r\sin\left(\frac{2\beta+\theta}{2}\right)\right]$

$= 2r^2\left(2\sin\left(\frac{2\beta+\theta}{2}\right)\cos\beta\right]$

$= 2r^2\left(\sin\left(2\beta+\frac{\theta}{2}\right) + \sin\frac{\theta}{2}\right]$

Now, $2\beta + 2\beta + \alpha + \theta = 360$

$$2\beta + \frac{\alpha}{2} + \frac{\theta}{2} = 180$$

$= 2r^2\left[\sin\left(\frac{\alpha}{2}\right) + \sin\frac{\theta}{2}\right]$

$\Rightarrow \quad$ LHS = RHS.

6. Let $n \geq 2$ be a positive integer. Call a sequence $a_1, a_2, \ldots, a_k$ of integers an n-chain if $1 = a_1 < a_2 < \ldots < a_k = n$ and a_i divides a_{i+1} for all i, $1 \leq i \leq k - 1$. Let $f(n)$ be the number of n-chains where $n \geq 2$. For example, $f(4) = 2$ corresponding to the 4-chain {1, 4} and {1, 2, 4}.

Prove that $f(2^m \cdot 3) = 2^{m-1}(m + 2)$ for every positive integer m.

Sol.

Lets see via examples

12-chain,

$f(12) = n(S)$, $12 = 2^2 . 3^1$

$S = \{(1, 12), (1, 2, 12), (1, 3, 12), (1, 2, 6, 12), (1, 4, 12), (1, 3, 6, 12) \ldots, (1, 6, 12)\}$

$\Rightarrow$ In another way we need to select divisors of n.

Clearly $f(2k) = f(2^{k-1}) + f(2^{k-2}) + \ldots + f(2^1) + f(2^0)$

$= (2^{k-2} + 2^{k-3} + \ldots + 1) + 1$

$= (2^{k-1} - 1) + 1 = 2^{k-1}$

Let's proceed with induction

$f(2^k \cdot 3) = (k + 2) \cdot 2^{k-1}, k \geq 1$

$\Rightarrow$ Checking the base case

$f(6) = 3 \cdot 2^0 = 3$

$\{(1, 6), (1, 2, 6), (1, 3, 6)\}$

Notice that $(2^k \cdot 3)$ and (2^{k+1}) are proper factor of $(2^{k+1} \cdot 3)$

Also, $f(2^{k+1} \cdot 3^1) = f(2^k \cdot 3) + f(2^{k+1}) + f(2^k \cdot 3)$

$\Rightarrow$ $(k + 2) \cdot 2^{k-1} + 2^k + (k + 2) \cdot 2^{k-1}$

$= (k + 3) \cdot 2^k = [(k + 1) + 1] \cdot 2^{(k-1)+1}$

$\Rightarrow$ Since for $f(k)$, $f(k + 1)$ is also true.

Therefor the proof is done

Alter :

$$f\left(2^m \cdot 3\right) = \sum_k k.\left\lfloor {}^{(m-1)}C_{k-1} + {}^{m-1}C_{k-2} \right\rfloor$$

Using combinotorics argument for selection of factors when 3 is in divisors factors.

$$\Rightarrow \sum_k k\left({}^{m-1}C_{k-2}\right) = \sum_k \left\lfloor (k-1)^m C_{k-1} + {}^mC_{k-1} \right\rfloor$$

$$= \sum_k \left(\frac{m}{k-1}\right)(k-1)\left({}^{m-1}C_{k-2}\right) + \sum_k {}^mC_{k-1}$$

$= m \cdot 2^{m-1} + 2^m$

$= 2^{m-1}(m + 2)$

Hence proved.

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