Control Systems

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CONTROL SYSTEMS

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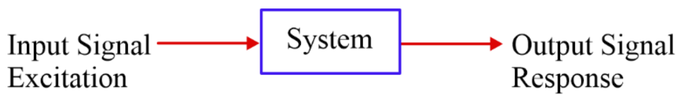
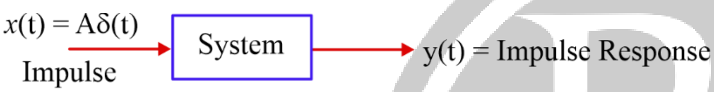
1 BLOCK DIAGRAM REPRESENTATION AND SIGNAL FLOW GRAPH

1.1. Introduction



1. 



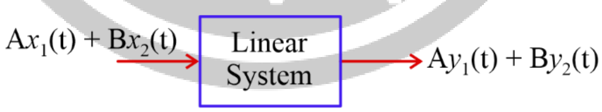
2. 

Similarly if $x(t) = Au(t)$ then $y(t) =$ step Response

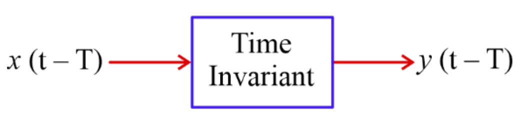
$x(t) = r(t)$ then $y(t) =$ unit ramp Response

$x(t) = Ar(t)$ then $y(t) =$ Ramp Response

Linear System:



Time Invariant:



1.1.1. Linear Time Invariant System

$h(t) \longrightarrow$ unit Impulse Response of L.T.I

$h(t) \xrightarrow{F.T} H(\omega)$ – frequency System

$h(t) \xrightarrow{F.T} H(S)$ – Transfer function of LTI Convolution

$Y(t) = X(t) * h(t)$

$Y(S) = X(S)\, H(S)$

Standard LTI system

$$H(S)=\frac{1/RC}{S+1/RC}=\frac{L.T\ of\ y(t)}{L.T\ of\ x(t)}=\frac{Y(S)}{X(S)}$$

Case 1: T.F to differential equation

$$H(S)=\frac{Y(S)}{X(S)}=\frac{1/RC}{S+1/RC}$$

$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{RC}x(t)$$

Case 2: Differential eq. to T.F

$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{RC}x(t)$$

$$y(s)=\frac{y(0)}{s+\frac{1}{RC}}+\frac{(1/RC)\times(S)}{S+1/RC}$$

Case A Initial condition = 0

$Y(S)=X(S).H(S)$

(a) T.F can be calculated

(b) output can be calculated by using T.F

Case B Initial condition to $\neq 0$

$$Y(S)\neq X(S)H(S)$$

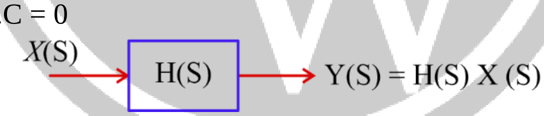
(a) T.F can be calculated by putting initial condition = 0

(b) Output can not be calculated .

- Regenerate initial condition to calculate output

Block Diagram Representation

- Used to represent a system .
- T.F can be calculated by forcing I.C = 0



$r(t)\longrightarrow R(S)$ input

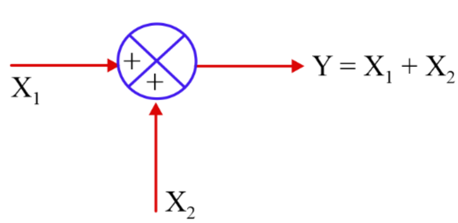
$c(t)\longrightarrow c(S)$ output

$g(t)\longrightarrow C(S)$ Transfet function

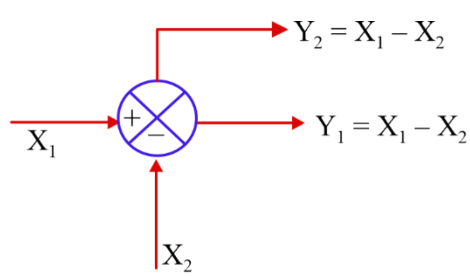
Important Concepts

(1) Summer – It should have 2 or more then 2 inputs.

Symbol

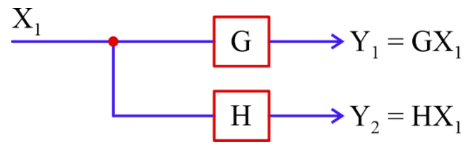


Multi Input Single Output



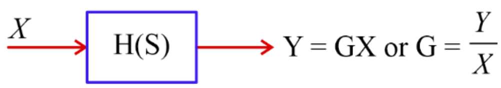
Multi Input Multi Output

(2) Take of points - single input and Multi output



Used for input distributions

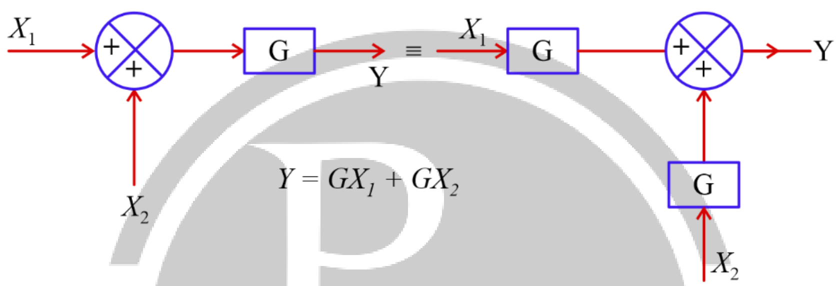
(3) Forward Gain - Direction always from input to output



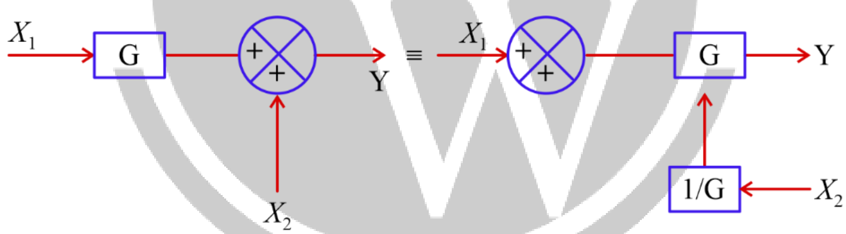
Rules

Case 1 : Summer and forward Gain

Case A

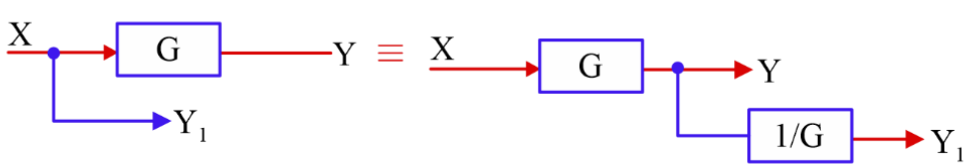


Case B

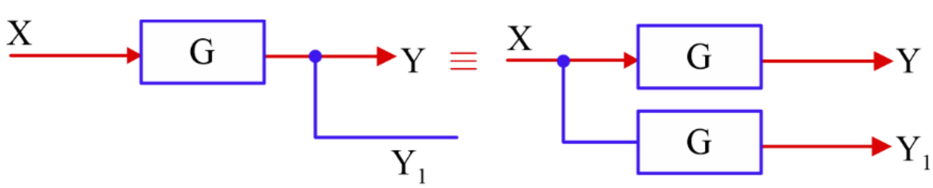


Case 2 : Take off points and forward gain

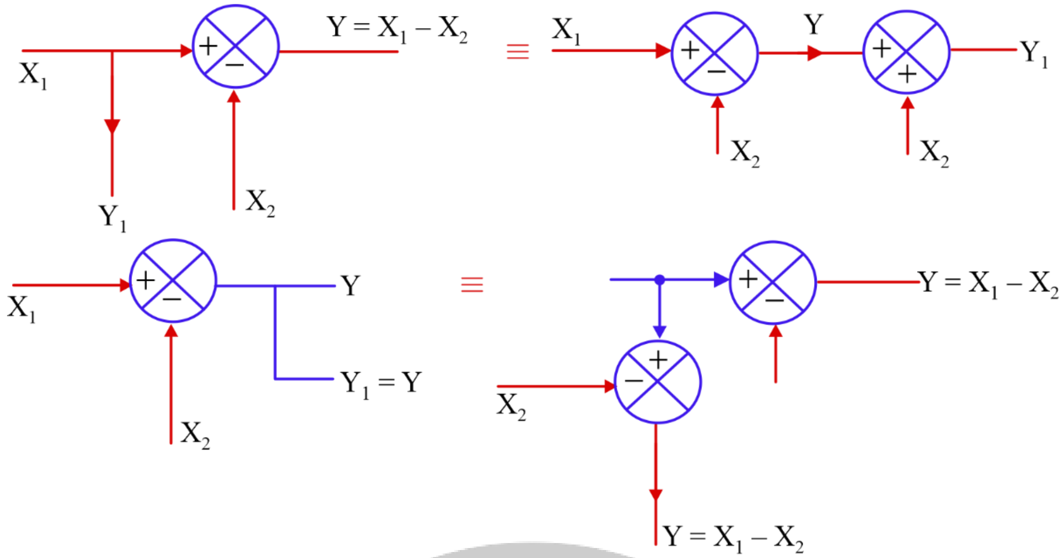
Case A



Case B



Case 3 : Take off point and summer

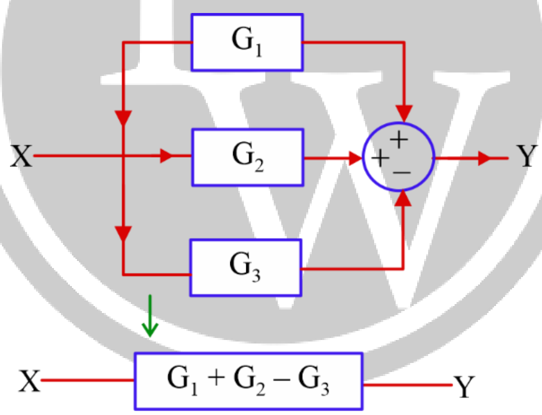


Gain Connected in Parallel

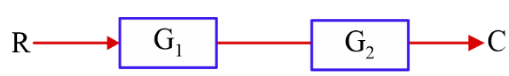
(1) Direction of flow should be from input to output from all the gains .

(2) Summing block should be common.

(3) Input should be common.

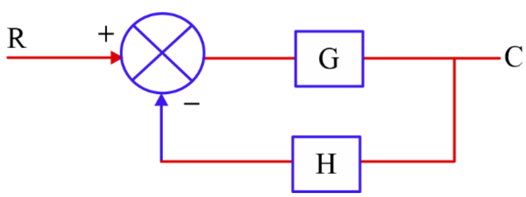


Gain Connected in Cascade



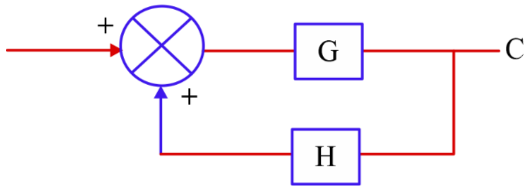
$$\frac{C}{R} = G_1G_2$$

Feedback –



$$\frac{C}{R} = \frac{G}{1+GH}$$

Negative Feedback



$$\frac{C}{R} = \frac{G}{1 - GH}$$

Problem solving Techniques -

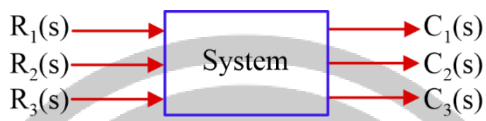
(1) Try to eliminate common node by using parallel paths .

(2) Convert 3 input summer to Two , 2 input Summer

(3) Try to bring two summers side by side by changing their inputs if required.

1.2. MIMO

(1) T.F can not be calculated



(2) $\left.\frac{C_1(S)}{R_1(S)}\right|_{\substack{R_2(S)=0\\R_3(S)=0}}$ Ratio parameter

(3) $\left.\frac{C_2(S)}{R_3(S)}\right|_{\substack{R_1(S)=0\\R_2(S)=0}}$ Ratio parameter

(iii) Output can be calculated by super position .

$$C_1(S) = ? \quad \left.\frac{C_1(S)}{R_1(S)}\right|_{\substack{R_2=0\\R_3=0}} \quad \left.\frac{C_1(S)}{R_2(S)}\right|_{\substack{R_1=0\\R_3=0}} \quad \left.\frac{C_1(S)}{R_3(S)}\right|_{\substack{R_1=0\\R_2=0}} = H_3(S)$$

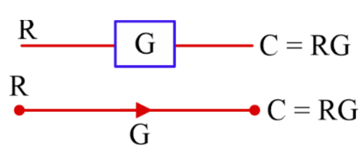
$$\downarrow \qquad\qquad \downarrow$$

$$= H_1(S) \qquad = H_2(S)$$

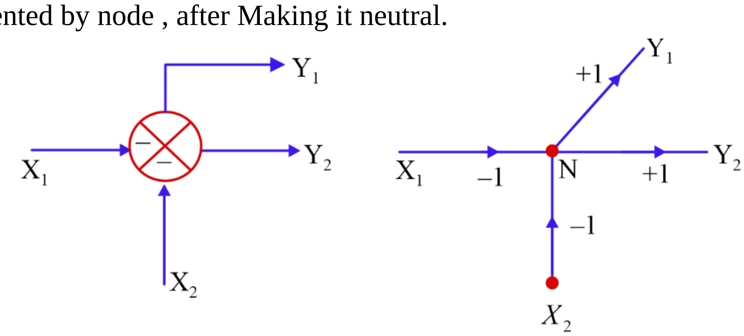
$$\boxed{C_1(S) = R_1(S)H_1(S) + R_2(S)H_2(S) + R_3(S)H_3(S)}$$

Signal flow Graph - Alternative Representation of a system .

(1) Gain Block Representation

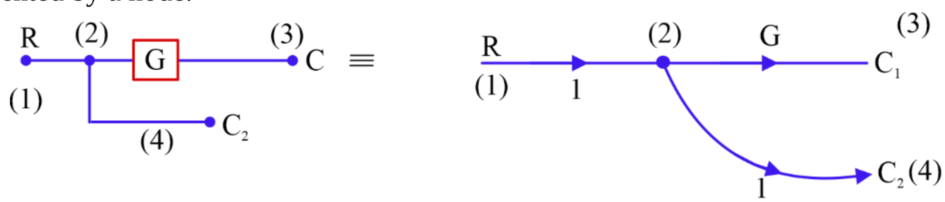


(2) Summer Block - Represented by node , after Making it neutral.



$\boxed{N = -X_1 - X_2}$ $\boxed{Y_1 = N}$ $\boxed{Y_2 = N}$

(3) Take off – Represented by a node.



Note : If take off point comes other summer both of them represented by same node, but if it comes before summer then two nodes .
Input Node - Only outgoing branches
Initial Node - may have incoming and outgoing branches

1.

R: input node

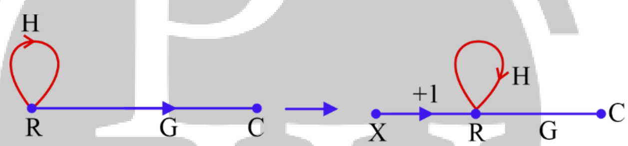
$\frac{C}{R} = G(M.G)$

$R = X$

R: Initial Node

X: Input Node

2.



$\frac{C}{R} \rightarrow$ MG not allowed R=HR+X

R: initial node

$\frac{C}{X} \rightarrow$ MG allowed X: input node

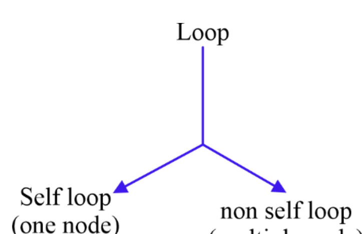
$\frac{R}{X} \rightarrow$ MG allowed

Output node – Having only incoming branches, when this condition not present then forward drugging Fill above condition.

Forward Path - Path connecting input to output.

- Direction from input to output.
- Any node should not traversed twice.
- Not in loop from.

Loop – A path which originate and terminate at same node



Macon Gain formula – used to calculate for

Limitations

(1) $\dfrac{\text{Output Node}}{\text{Input Node}}$ (2) $\dfrac{\text{Intermidiate Node}}{\text{Input Node}}$

$$\frac{Y}{X} = \frac{\sum_{k=1}^{n} p_k \Delta_k}{\Delta}$$

n = no of forward path from X to Y

Δ = 1-(sum of all loop gains) + (sum of product of two non touching loop gain) (sum of product of 3 non touching loop gains)

Δ_K = It is dependent on forward path

Δ_K = 1- [sum of all loop gains not touching K^{th} forward path] + (sum of product of 2 non touching loop gains not touching K^{th} Forword path) + --------------

Steps :

(1) $\frac{C}{R} \longrightarrow M.G$ Applicable

(2) Calculate total no of forward paths

(3) Calculate all types of loops .

(4) Calculate Δ and Δ_k

Note :

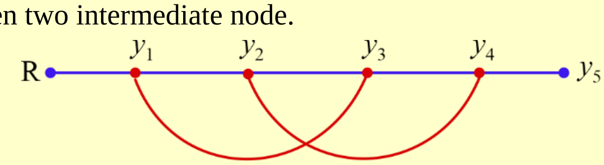
(1) Self loop at initial (first) node of unity /non unity can always be ignored .

(2) Self loop at intermediate nodes having unity gain –

(i) Result in inconsistent nodal equation $\frac{C}{R} \to \infty$,

(ii) To obtain finite $\frac{C}{R}$ such loops can be ignored .

(3) MGF can not be applied between two intermediate node.



$$\frac{y_5}{y_2} = ? = \left(\frac{\frac{y_5}{R}}{y_2 / R}\right) \longrightarrow \text{calculate by MGF}$$

Mapping of Block diagram to SFG

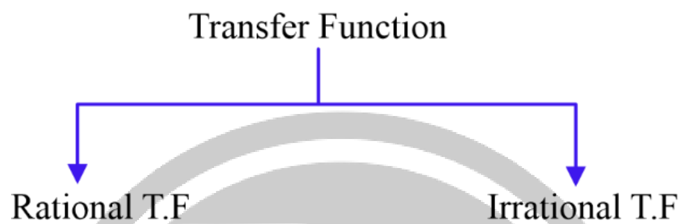
Summer ⟶ Node

Take off ⟶ Node

Gain ⟶ Line

2 TIME RESPONSE ANALYSIS

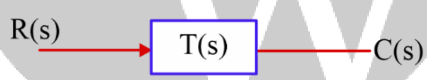
2.1. Introduction



$$T(S) = \frac{N(S)}{D(S)} = \frac{S+1}{S+2} \qquad T(S) = \frac{N(S)}{D(S)} = \frac{e^{S}}{(S+1)}$$

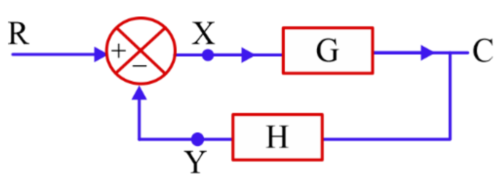
$$\text{Pole – Zero format } = \frac{k(S-z_1)(S+z_2)}{S^N(S-p_1)(S+p_2)} \rightarrow \text{Root locus Nyquist}$$

$$\text{Time constant from } T(S) = \frac{K(1+ST_1)(1+ST_2)}{S^N(1+ST_a)(1+ST_b)} \rightarrow \text{Bode , Nyquist}$$



Open loop system

Open loop Transfer function $T(S) = \dfrac{C(S)}{R(S)}$



Close loop system

$$\text{C.L.T.F} = T(S) = \frac{C(S)}{R(S)} = \frac{G}{1+GH}$$

C.L.S May hare C.L.T.F or O.L.T.F

$$\text{O.L.T.F of C.L.S} = \boxed{\frac{Y}{X} = GH}$$

2.1. Degree of T.F

⇒ Highest order of D(S) after pole zero cancellation.

$$T(S)=\frac{(S+1)}{S^4(S+2)(S+1)(S+3)^3}\ \text{, degree = 8}$$

Type of a system

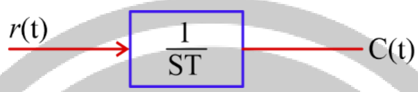
(1) Defined for C.L.S only .

(2) To calculate Type of C.L.S , O.L.T.F or G(S) H(S) of C.L.S is used.

(3) Pole at origin in O.L.T.F of C.L.S – Type

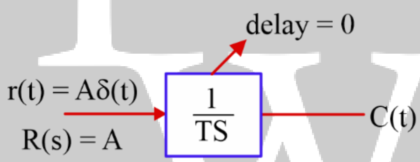
First order system (O.L.S)

$$\frac{C(S)}{R(S)}=\frac{1}{ST}$$



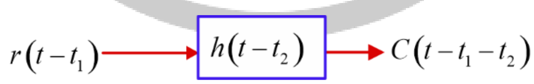
If $r(t)=A\delta(t),\quad C(S)=\frac{A}{TS},\ c(t)=\frac{A}{T}u(t)$

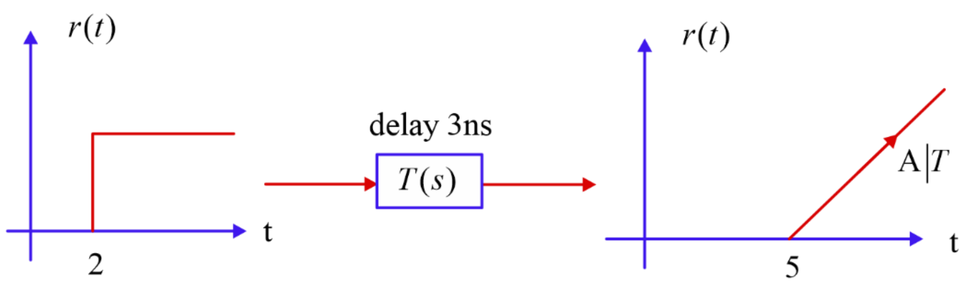
$$r(t)=A\delta(t-t_o)\ \ C(S)=\frac{Ae^{-St_0}}{TS}=c(t)=\frac{A}{T}u(t-t_o)$$



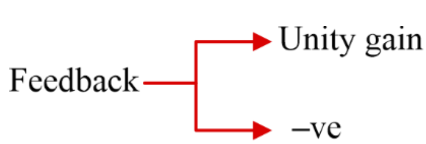
If system delayed by to $h(t-t_o)=H(S)=\frac{e^{-St_o}}{TS}$

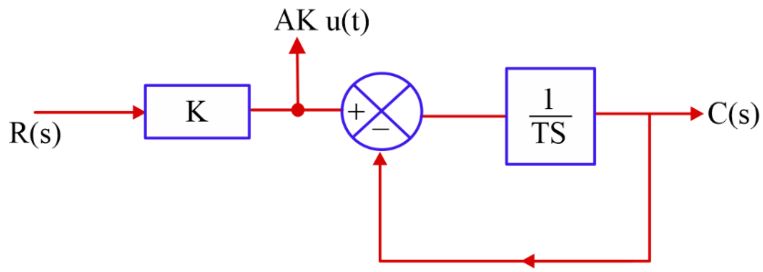
$$C(S)=\frac{Ae^{-st_o}}{TS}\quad , C(t)=\frac{A}{T}u(t-t_o)$$



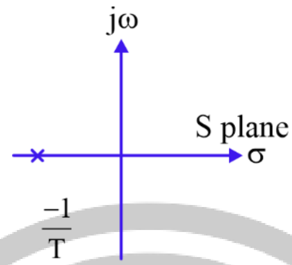


First order Close Loop System

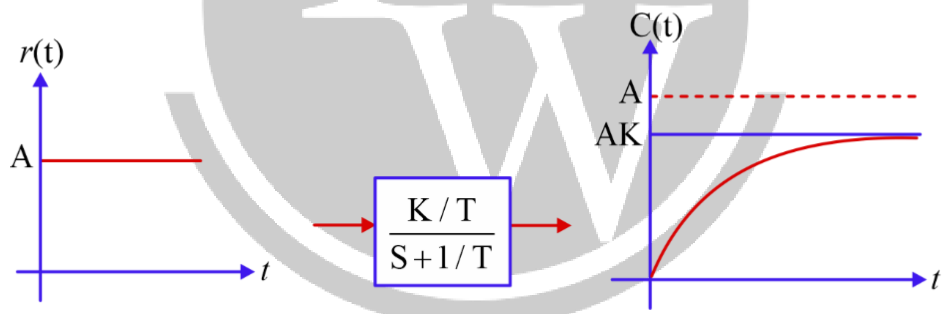




$$T(S)=\frac{C(S)}{R(S)}=\frac{K/T}{S+1/T}\xrightarrow{s=0}K(\text{d.c gain})$$



I/P	C(S)	C(t)
$r(t)=A\delta(t)$	$C(S)=\frac{AK/T}{S+1/T}$	$C(t)=\frac{AK}{T}e^{-t/T}u(t)$
$r(t)=AU(t)$	$C(S)=AK\left[\frac{1}{S}-\frac{1}{s+1}\right]$	$C(t)=A\left(1-e^{-\frac{t}{T}}\right)u(t)$



1. Initial slope at t = 0 can be calculate from graph and can be equated to $\left.\frac{dc(t)}{dt}\right|_{t=0}=\frac{AK}{T}$

2. Net input at summer = output approaches or settled .

Transient and Steady state Response –

$$C(t)=(1-e^{-t/T})u(t)=1-e^{-t/T}$$

$C_1(t)=1\rightarrow$ Steady state Response (constant)

$C_2(t)=-e^{-t/T}\rightarrow$ Transient Response [exponential]

$\lim_{t\to\infty} c_1(t)=1$, $\lim_{t\to\infty} c_2(t)=0$

Transient Response

It is part of total step Response which tend to 0, When large time frame is Considered 0.

- Tends to 0 as $t \to \infty$
- To calculate transient response from D.E $IC \neq 0,\ i/p = 0$
- Zero input response

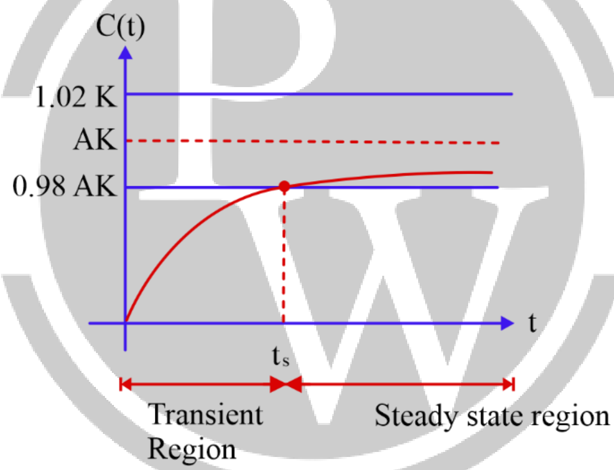
Steady State Response

Part of total step response which remains after transient dies out .

- To calculate the steady state response from D.E = I.C = 0 and input $\neq 0$
- zero state response

$$c(t) = c_{tr}(t) + c_{ss}(t)$$
$$\lim_{t \to \infty} c(t) = c_{ss}(t)$$

Setting Time



(1) t_s = Time required to settle in a predefined "Error Band".

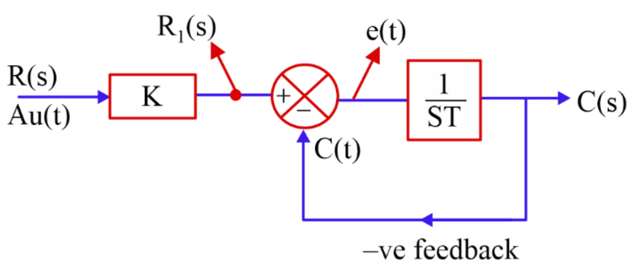
(2) Error Band $= \pm$ m% of input + d.c gain

(3) t_s = f (error band)

(4) $t_s \downarrow$ as error band $\uparrow$

(5) $t_s \to \infty$ for 0% error band.

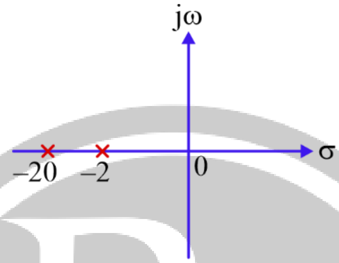
Error Signal



$$e(t) = Kr(t) - c(t)$$
$$e(t) = r_1(t) - c(t)$$
$$e(t) = \text{Error signal (unity f/b)}$$
$$e(t) = Ake^{-t/T}u(t)$$

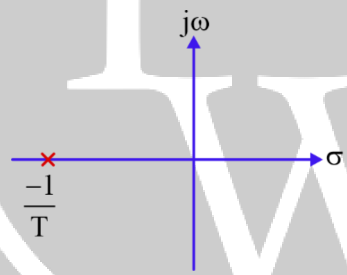
- Steady state error = final value of error signal = $\lim_{t \to \infty} e(t) = 0$
- Time constant = $\dfrac{1}{|\text{Real part of dominant pole}|}$

$$T = \frac{1}{2}\text{sec}$$



- 3 dB band width of RC circuit $= \dfrac{1}{RC}\text{rad/sec} = \dfrac{1}{\text{Time constant}}$

$$T(s) = \frac{K/T}{S + \frac{1}{T}} \longrightarrow$$



Time constant =T

3dB BW=$1/T$ rad/sec

Rise Time

Time taken by step response of first order CLS to reach from 10% to 90% of its final value.

$t_s = 4T$ 2% error band

$t_r = 2.2T$

$t_d = 0.693T$

Second order system :

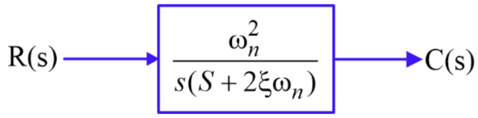
- For first order system = one parameter = Time constant
- For 2nd order $\rightarrow$ ξ : damping ratio

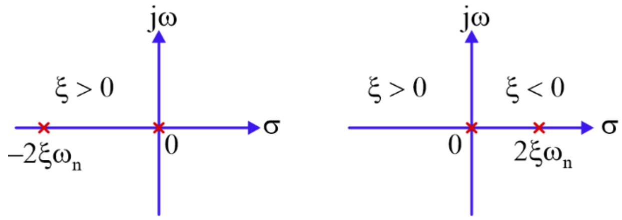
  ω_n : undamp natural frequency

$\xi\omega_n$ = damping tactor $\omega_n\sqrt{1-\xi^2}$ = Damped Natural frequency.

ω_d

2nd order O.L.S :

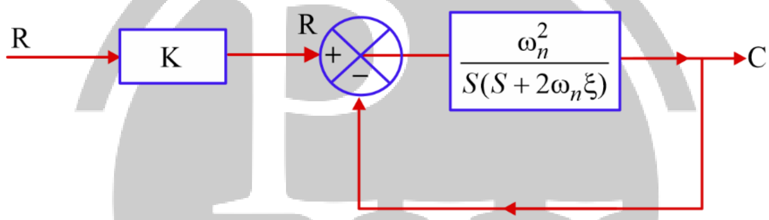




$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S(S+2\xi\omega_n)} \quad 0<\omega_n<\infty \qquad -\infty<\xi<\infty$$

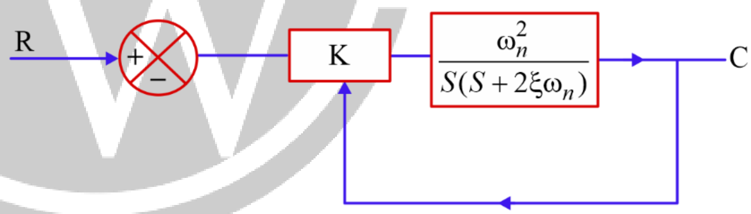
2nd Order C.L.S

(1) $\boxed{\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n s+\omega_n^2}}$ $S=0\rightarrow 1$ d.c gain



(2) $\frac{C(S)}{R(S)} = \frac{K\omega_n^2}{S^2+2\xi\omega_n s+\omega_n^2}$

d.c gain = 0



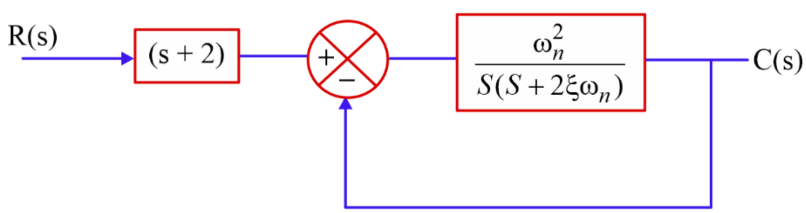
(3) $\frac{C(S)}{R(S)} = \frac{K\omega_n^2}{S^2+2\xi\omega_n S+K\omega_n^2}$

d.c gain = 1

$$T(S) = \frac{\omega'^2_n}{S^2+2\xi'\omega'_n S+\omega'^2_n} \quad \omega'_n = \sqrt{k}\ \omega_n$$

$$\xi' = \frac{\xi}{\sqrt{K}}$$

Non Standard second order s/s



$\boxed{T(S) = \frac{C(S)}{R(S)} = \frac{(S+2)\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2}}$ Non Standard T.F

For std. 2nd order C.L.T.F $\Rightarrow$ $$C.L.T.F = \frac{OLTF}{1+OLTF}$$

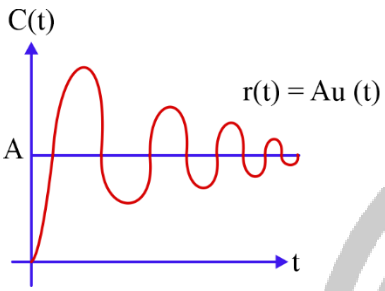
Important Points :

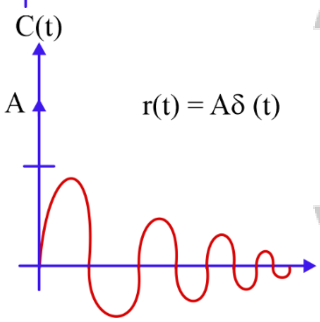
(1) Damping ratio (ξ), dimensionless, represent decay of oscillation in output response .

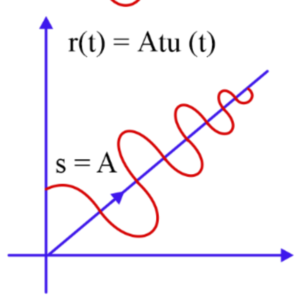
(2) Undamped natural frequency $(\omega_n) \rightarrow$ rad/sec, frequency of oscillation of output response, in absence of damping force, $\omega_n > 0$.

(3) Damped natural frequency $\omega_d = \omega_n\sqrt{1-\xi^2}$ rad/sec frequency of output response oscillation, when Damping force is present, $\omega_d > 0$.

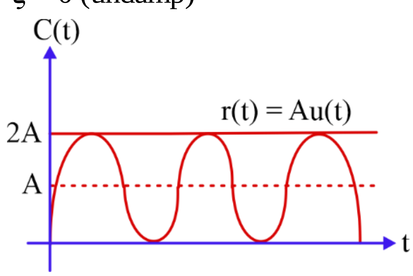
Case 1: $0 < \xi < 1$ (under damp) $c(t)$

(1) 

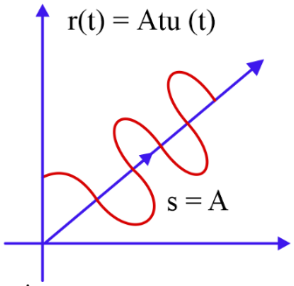
(2) 

(3) 

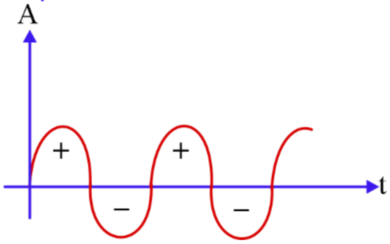
Case 2 : $\xi = 0$ (undamp)

(1) 

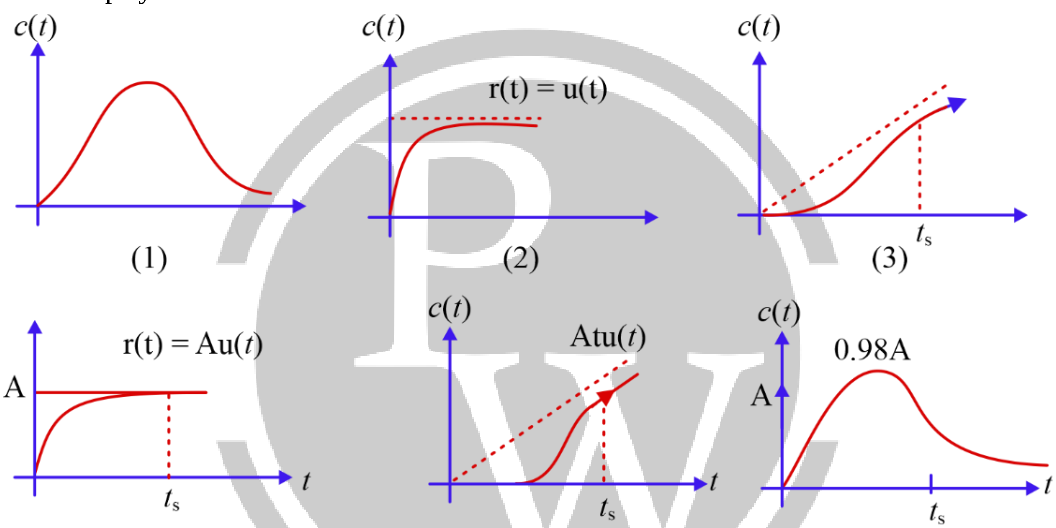
(2)



(3)



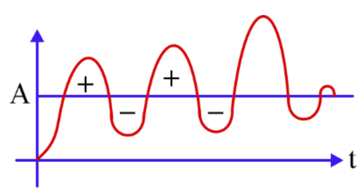
Case 3 : $\xi = 1$ critical damp system



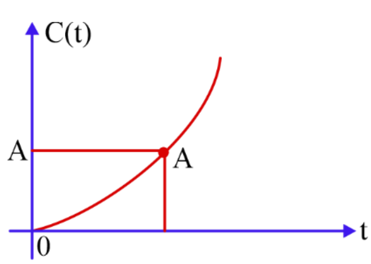
Case 4 : $1 < \xi < 0$ overdamp

- Output response does not oscillate and approaches constant parameter of input not in shortest possible time.

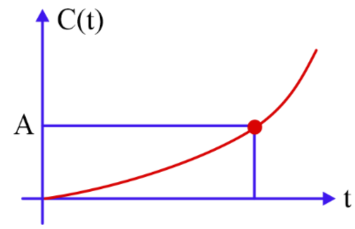
Case 5 : $-1 < \xi < 0$



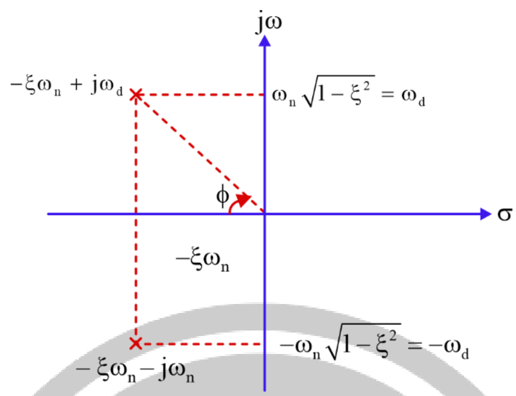
Case 6 : $\xi = -1$



Case 7 : $\xi < -1, \qquad -\infty < \xi - 1$



Under Damped System

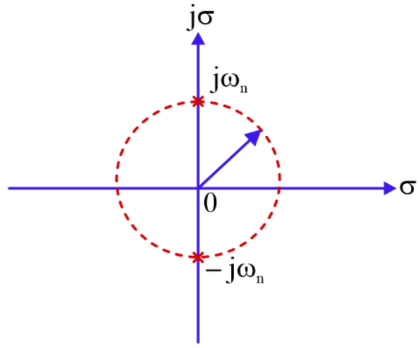


- $0 < \xi < 1$ $\quad T(S) = \dfrac{\omega_n^2}{S^2 + 2\xi\omega_n^2 S + \omega_n^2} = \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{N(s)}{D(s)}$
- Characteristic equation $= \boxed{1+G(S)H(S)=0}$

  $S^2 + 2\xi\omega_n S + \omega_n^2 = 0$
- Poles $S = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$
- Complex poles, left half of S-plane.
- Time constant $= T = \dfrac{1}{\xi\omega_n}$
- $t_s = 4T$ 2% Error Band

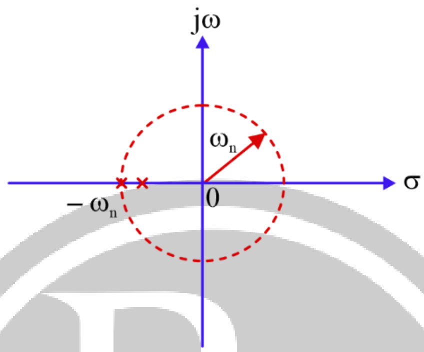
  $= 3T$ 5% Error Band
- $\cos\phi = \xi$
- Locus of poles = semi circle of radius ω_n
- Always stable.

Undamp System :



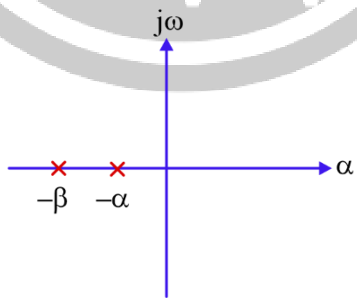
- $\xi = 0$
- $D(s) = 1 + G(S)H(S) = S^2 + \omega_n^2 = 0$
- Poles , $S = \pm j\omega_n$ (purely imaginary)
- $T = \infty$
- $t_s = \infty$
- marginally stable (Non repeated poles on imaginary axis)
- Locus $\rightarrow$ circle of radius ω_n

Critical Damp System :



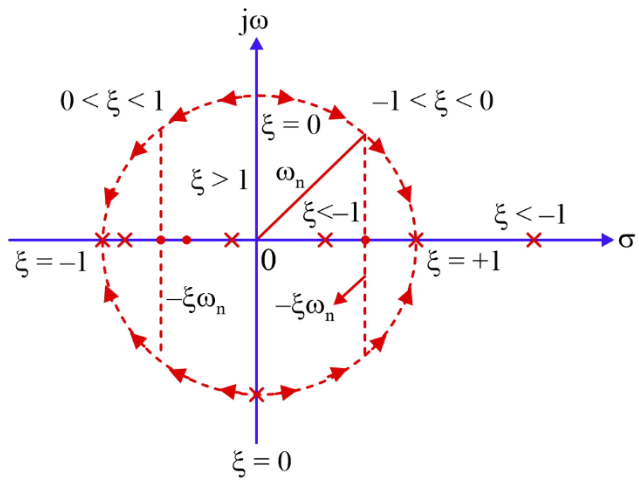
- $\xi = 1$
- $D(S) = S^2 + 2\omega_n S + \omega_n^2 = 0 \qquad (S + \omega_n)^2 = 0$
- Poles $S = -\omega_n,\ -\omega_n$
- $T = \dfrac{1}{\omega_n}$
- $t_s = 6T$ for 2%
- Always Stable
- Poles lies on circle of radius ω_n

Over damp System :

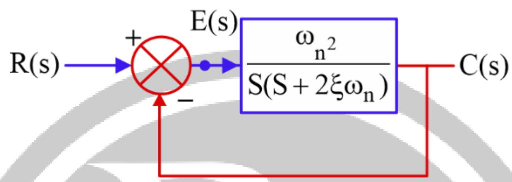


- $1 < \xi < \infty$
- $D(S) = S^2 + 2\xi\omega_n S + \omega_n^2 = 0 = (S + \alpha)(S + \beta)$
- Poles = $-\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$ real poles
- $T = \dfrac{1}{\alpha}$

  $t_s = 4T$ 2%
- Always stable



Step Response of under damp system



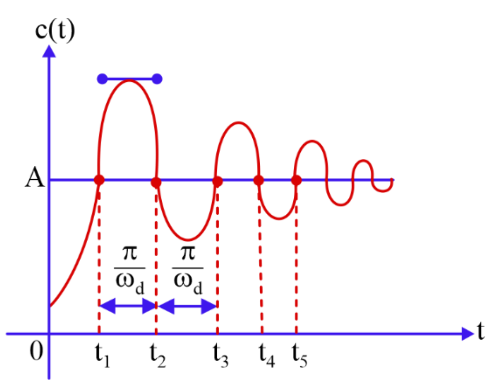
$$r(t) = Au(t)$$

- Error signal E(S) = R(S) - C(S)
- $$C(S) = \frac{A\omega_n^2}{S(S^2 + 2\xi\omega_n S + \omega_n^2)}$$ $$C(t) = A\left\{1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right\} \to u(t)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\cos\theta = \xi, \quad \tan\phi = \frac{\sqrt{1-\xi^2}}{\xi}$$

- $$e(t) = A\left[\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)\right]u(t)$$



(1) t_k [Time Constant When $C(t) = A$] $$t = \frac{n\pi - \phi}{\omega_d}$$

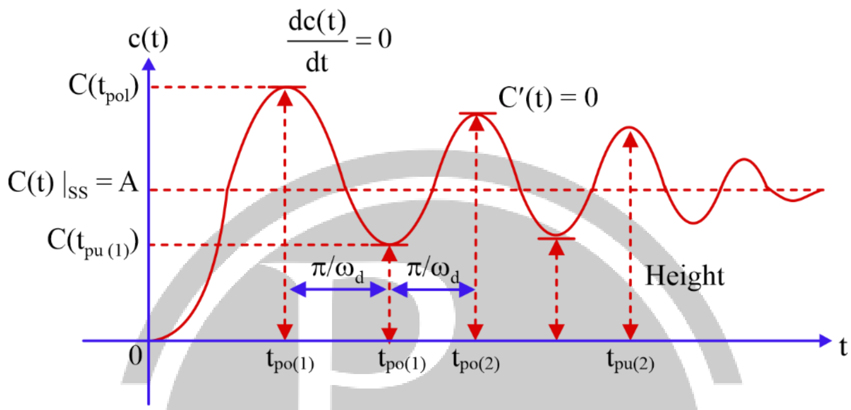
(2) $n=1, \boxed{t_1 = \frac{\pi-\phi}{\omega_d}}$ Rise time $\boxed{\Delta t = \frac{\pi}{\omega_d}}$

$n=2$ $\boxed{t_2 = \frac{2\pi-\phi}{\varepsilon_d}}$ Time instant of A

$\Rightarrow$ 2nd order rise time $\Rightarrow$ Time taken to reach from 0 to 100% of final value.

$t_r = \frac{\pi - \cos^{-1}\xi}{\omega_n\sqrt{1-\xi^2}}$ $\pi = 3.14$, $\phi = \text{radians}$

(ii) Peak Overshoot and Peak undershoot time :



Peak Time $\boxed{t = \frac{n\pi}{\omega_d}}$ $n = 1, 2, 3$

(1) $t_1 = \frac{\pi}{\omega_d} = t_{po(1)} \rightarrow$ First peak overshoot time

(2) $t_2 = \frac{2\pi}{\omega_d} = t_{pu(1)} \rightarrow$ first peak undershoot time

(3) $t_3 = \frac{3\pi}{\omega_d} = t_{po(2)} \rightarrow 2^{nd}$ peak overshoot time

(4) $\boxed{t_{po(K)} = \frac{(2K-1)\pi}{\omega_d}}$, K = 1,2

K^{th} peak overshoot time

(5) $\boxed{t_{pu(K)} = \frac{2K\pi}{\omega_d}}$ K = 1, 2

K^{th} peak undershoot time

(6) At peak overshoot time $\frac{dc(t)}{dt} = 0$, same for peak undershoot also

(7) At 1^{st} peak overshoot time value, output is maximum $c\left(t = t_{po(1)}\right) = [c(t)]_{max}$

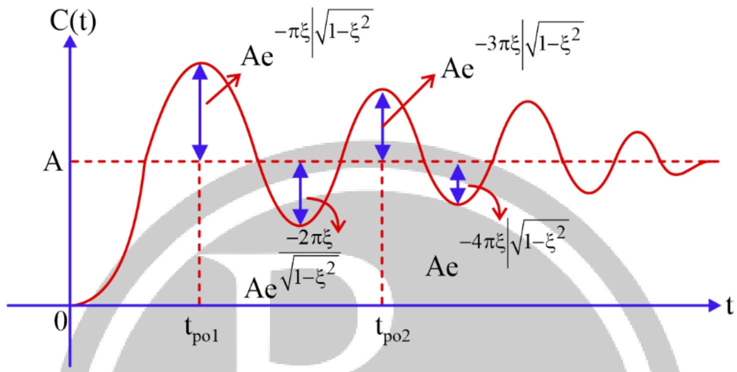
(8) Time gap b/w two successive $\rightarrow p.o|p.u$ is $2\pi/\omega d$

(9) First peak undershoot time = time period of damped oscillation $t_{pu(1)} = T_D = \frac{2\pi}{\omega_d}$

(10) $C(t)\big|_{SS} = C(t)\big|_{t=\infty} = A$

(11) Output maxima $\Rightarrow$ $\boxed{C\left(t_{po(K)}\right) = A\left(1 + e^{\frac{-\pi\xi(2K-1)}{\sqrt{1-\xi^2}}}\right)}$ $K = 1, 2$

(12) Output Minima $\Rightarrow$ $\boxed{C\left(t_{pu(K)}\right) = A\left(1 - e^{\frac{-\pi\xi(2K)}{\sqrt{1-\xi^2}}}\right)}$ $K = 1, 2$



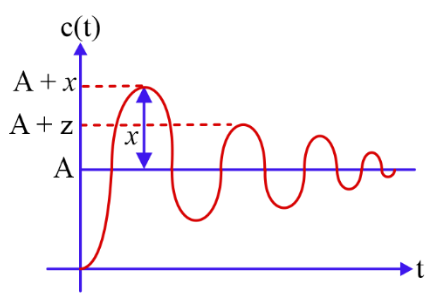
(1) Maxima of $C(t) \Rightarrow C(t)\Big|_{\max} = C(t)\Big|_{\max} = A\left(1 + e^{-\pi\xi/\sqrt{1-\xi^2}}\right)$

(2) Peak overshoot = height of overshoot $\Rightarrow C\left(t_{po(K)}\right) - C(t)\big|_{SS} = Ae^{-\frac{\pi\xi(2K-1)}{\sqrt{1-\xi^2}}}$

(3) Max. peak overshoot $= C(t)\big|_{\max} - C(t)\big|_{SS} = Ae^{-\pi\xi/\sqrt{1-\xi^2}}$

(4) Max. peak percentage overshoot - $\%M_{po} = \frac{C(t)\big|_{\max} - C(t)\big|_{ss}}{C(t)\big|_{ss}} \times 100\% = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100\%$

(5) Graphical Relation



(i) $x = Ae^{-\pi\xi/\sqrt{1-\xi^2}}$

$z = Ae^{-3\pi\xi/\sqrt{1-\xi^2}}$

(ii) $\frac{x}{A} \times 100\% = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100\%$

(iii) $\frac{z}{A}\times 100\% = e^{-3\pi\xi/\sqrt{1-\xi^2}} \times 100\%$

(6) Minima of $C(t) \to C(t)\big|_{\min} = C\left(t_{pu_1}\right) = A\left(1-e^{-2\pi\xi/\sqrt{1-\xi^2}}\right)$

(7) Peak undershoot = height of undershoot $C(t)\big|_{ss} - C\left(t_{pu(K)}\right) = Ae^{-\pi\xi(2K)/\sqrt{1-\xi^2}}$

(8) Maxima peak overshoot $C(t)\big|_{ss} - C\left(t_{pu1}\right) = C(t)\big|_{ss} - C(t)\big|_{\min} = Ae^{-2\pi\xi/\sqrt{1-\xi^2}}$

(9) Maximum peak percentage undershoot

$$\%M_{pu} = \frac{C(t)\big|_{ss} - C(t)\big|_{\min}}{C(t)\big|_{ss}} \times 100\% = e^{-2\pi\xi/\sqrt{1-\xi^2}} \times 100\%$$

(10) Decay ratio $= \frac{M_{po(2)}}{M_{po(1)}} = \frac{M_{Pu(2)}}{M_{pu(1)}} = e^{-2\pi\xi/\sqrt{1-\xi^2}}$

(11) $\%M_{p0} = m\% = \%M_{po(1)}$

$$s.1 \quad p = \frac{m}{100}$$

$$s.2 \quad \xi = \sqrt{\frac{(\ln p)^2}{\pi^2 + (\ln p)^2}}$$

(12) $r(t) = Au(t)$, effective input = A

(13) System dependent parameters

$$t_K = \frac{K\pi - \phi}{\omega_d}, t_r = \frac{\pi - \phi}{\omega_d}, t_{po(K)} = \frac{(2K-1)\pi}{\omega_d}, t_{pu(k)} = \frac{2K\pi}{\omega_d}$$

$$t_s = 4T = \frac{4}{\xi\omega_n}, T = \frac{1}{\xi\omega_n}, \omega_d = \omega_n\sqrt{1-\xi^2}, T_D = \frac{2\pi}{\xi_d}$$

Total no. of cycle before oscillation dies out $A = \frac{t_s}{T_D}$

$\%M_{p0(K)}$, $\%M_{pu(K)}$

(14) Input Independent Parameters

(i) $C(t)$

(ii) $C\left(t_{poK}\right)$

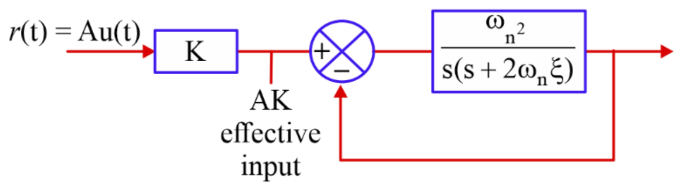
(iii) $C(t)\big|_{\max}$

(iv) Peak overshoot

(v) $C\left(t_{puK}\right)$

(vi) $C(t)_{\min}$

(vii) Peak undershoot



$$\frac{C(s)}{R(s)} = \frac{k\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2}$$

$$C(t)|_{ss} = AK$$

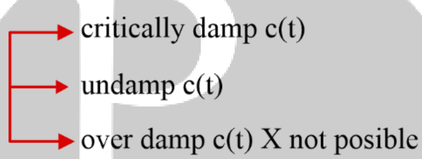
- System dependent parameters $\rightarrow$ No charge
- Input dependent parameters $\boxed{\rightarrow A \Rightarrow AK}$

Note : If $r(t) \rightarrow Au(t - t_0)$ then all time formulas will be replaced by $(t - t_0)$

Step Response of undamped System

(1) $C(s) = \left[\frac{AK\omega_n^2}{s^2 + \omega_n^2}\right]\frac{1}{s} \Rightarrow c(t) = Ak[1 - \cos\omega_n]u(t)$

- From c(t) of underdamp



- $\cos\phi = 0$, $t_s = \frac{4}{\xi\omega_n} = \infty$
- $\omega_d = \omega_n$, $\%M_p = 100\%$
- $t_r = \frac{\pi}{2\omega_n}$,
- $t_{po(K)} = \frac{\pi}{\omega_n}(2K - 1)$
- $t_{pu}(k) = \frac{2k\pi}{\omega_n}$

Step Response of critically damped s/s

$$C(S) = \frac{AK\omega_n^2}{S(S + \omega_n)^2}, \quad C(t) = AK[1 - e^{-\omega_n t}(1 + \omega_n t)]u(t)$$

- Output reaches to steady state without oscillation in short time.

Step response of overdamp s/s-

- $C(S) = \frac{AK\omega_n^2}{S(S^2 + 2\xi\omega_n S + \omega_n^2)} \xrightarrow{\xi > 1} \frac{AK\omega_n^2}{S(S + p_1)(S + p_2)} \longrightarrow C(\infty) = AK$

$$c(t) = [a_0 + a_1 e^{-p_1 t} + a_2 e^{-p_2 t}]u(t)$$

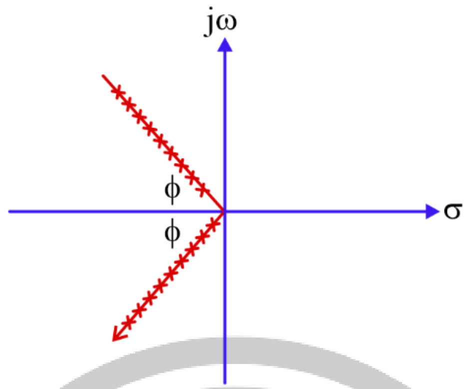
Rise time:

(1) 0 to 100% of s. s value $\rightarrow$ underdamp and undamp system

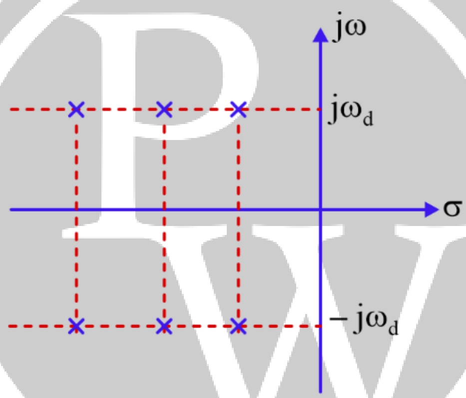
(2) 10 to 90% of s .s value $\rightarrow$ critical and overdamp system

Locus

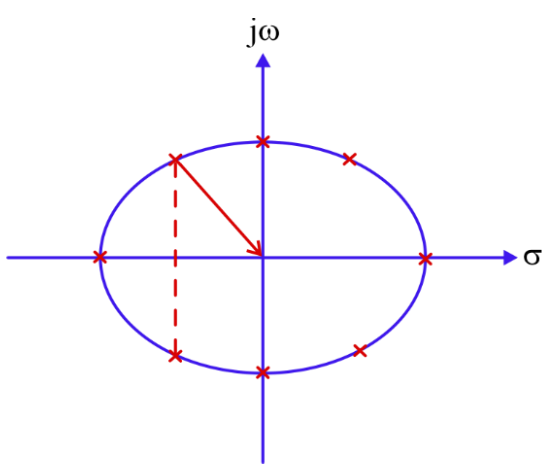
(1) Constant ξ



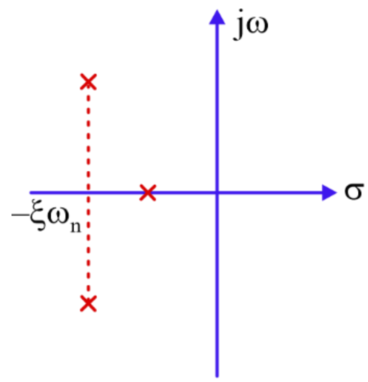
(2) Constant ω_d



(3) Constant ω_n



(4) Constant time constant (setting time) $T = \frac{1}{\xi\omega_n}$



Impulse Response of 2nd order s/s

(1) Under damped system $0<\xi<1$

$$C(t)=\frac{AK\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\left(\omega_d t\right)u(t)$$

$$t_r=\frac{\pi-\phi}{\omega_d}, t_p(\text{peak time})=\frac{n\pi}{\omega_d}, t_s=\frac{4}{\xi\omega_n}, \%M_p e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}\times 100}}$$

(2) Undamp system $\xi=0$

$$C(t)=AK\omega_n\sin\omega_n t$$

(3) Critically damped System $\xi=1$

$$C(t)=AK\omega_n^2 te^{-\omega_n t}u(t)$$

(4) Overdamped System $1<\xi<\infty$

$$C(t)=a_0e^{-p_1t}+a_1e^{-p_2t}$$

Dominant Pole Concept

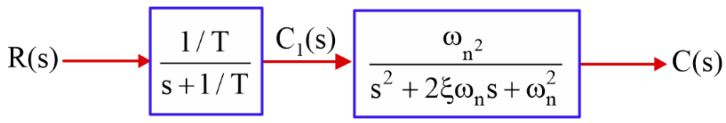
$$\frac{C(S)}{R(S)}=\frac{M}{(S+p_1)(S+p_2)(S+p_3)(S+p_4)}$$

- $p_1, p_2, p_3 p_4 \rightarrow$ Magnitude of real part of pole
- Smallest of $p_{1,} p_2, p_3, p_4 = p_1$
- (i) $\frac{p_2}{p_1}\geq 5 \;\; S+p_2 \xrightarrow{S=0} p_2$

  (ii) $\frac{p_3}{p_1}\geq 5, \;\; (S+p_3)\xrightarrow{S=0} p_3$

  (iii) $\frac{p_4}{p_1}\geq 5 \;\; (S+p_4)\xrightarrow{S=0} p_4$

Cascading of 2nd order under damped System



$$\frac{C_1(s)}{R(s)} = \frac{1/T}{s+1/T}$$

S – IIst order

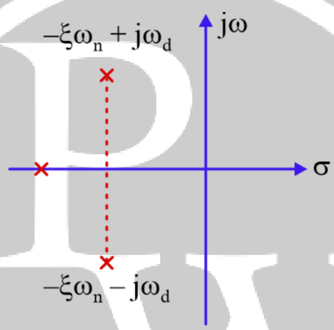
$$\frac{C(s)}{C_1(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$ IInd order

S – II

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{T}\omega_n^2}{\left(s+\frac{1}{T}\right)\left(s^2+2\xi\omega_n s+\omega_n^2\right)}$$ 3rd order

Case 1:

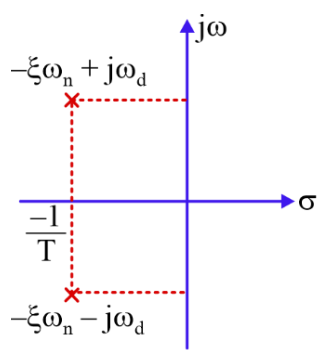
$$\frac{C(s)}{R(s)} = \frac{N(s)}{s^2+2\xi\omega_n s+\omega_n^2}$$



$N(s) = K\omega_n^2$ → $\xi\omega_n$ comparable; → Formula Applicable

N(s) = Polynomial → Comparision of ω, ξ allowed; → Formula Allowed

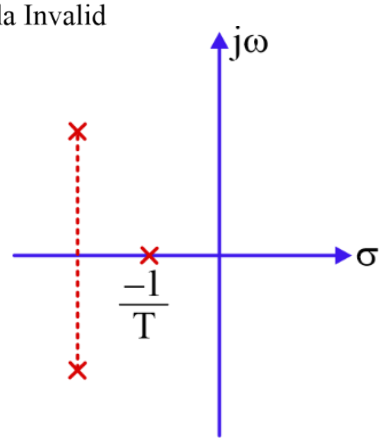
Case 2 :



$N(s) = k\omega_n^2 \rightarrow$ comparision of ξ, ω_n allowed

$$\frac{C(s)}{R(s)} = \frac{N(s)}{s^3+\alpha s^2+\beta s+T}$$

Case 3: N(s) = Polynomial → Comparison allowed; → Formula Invalid

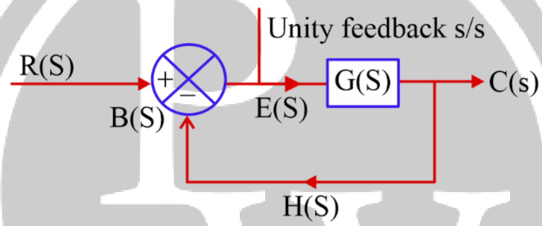


3 STEADY STATE ERROR AND ROUTH STABILITY

3.1. Error

Error Signal
- System Error
- Amplitude Error

Error
- Unity f/b system
- Non unity f/b system



$e(t) = r(t) - b(t)$ amplitude error

System Error

(1) Must tends to 0 as $t \to \infty$

(2) $e_{sys}(t) = r(t) - c(t)$ for $r(t) = Au(t)$

(3) $e_{sys}(t) = \int_{-\infty}^{t} r(t) - \int_{-\infty}^{t} c(t)dt$ for $r(t) = A\delta(t)$

(4) $e_{sys}(t) = \frac{dr(t)}{dt} - \frac{dc(t)}{dt}$ for $r(t) = At\ u(t)$

Amplitude Error

$$e(t) = r(t) - c(t) \to \text{output}$$

Effective input at summer

$r(t) = Au(t)$ $e(t) = Au - c(t)$

r(t) = Atu(t) e(t)= Atu(t) – c(t)

r(t) = $\frac{At^2u(+2)}{2}$ $e(t) = \frac{At^2}{2}u(t) - c(t)$

Calculation of amplitude error

(1) $e(t)$ or $E(s)$ at the output of summer.

(2) Steady state amplitude error $\lim_{t\to\infty} e(t) = \lim_{S\to 0} SE(S) \rightarrow$ will be finite only when all poles of SE(S) will be strictly on LHP.

$$e_{SS} = \lim_{S\to 0} \frac{SR(S)}{1+G(S)}$$

Steps :

(1) Identify feedback (unity)

(2) $ess = \lim_{S\to 0} \frac{SR(S)}{1+G(S)}$

(3) Pole location of $\frac{SR(S)}{1+G(S)}$ is strictly on L.H.P then perform calculation.

Steady state Error for different inputs.

Input	Check	e_{ss}
Au(t)	Poles of $\frac{A}{1+G(S)}$	$e_{SS} = \frac{A}{1+K_p}$
Atu(t)	Poles of $\frac{A}{S(1+G(S))}$	$e_{SS} = \frac{A}{K_V}$
$\frac{At^2}{2}u(t)$	Poles of $\frac{A}{S^2(1+G(s))}$	$ess = \frac{A}{K_a}$

$Kp = \lim_{S\to 0} G(s) =$ Positional Error constant /coefficient

$Kv = \lim_{S\to 0} SG(S) =$ velocity Error constant / coefficient

$Ka = \lim_{S\to 0} S^2G(S)$ Acceleration Error constant / coefficient

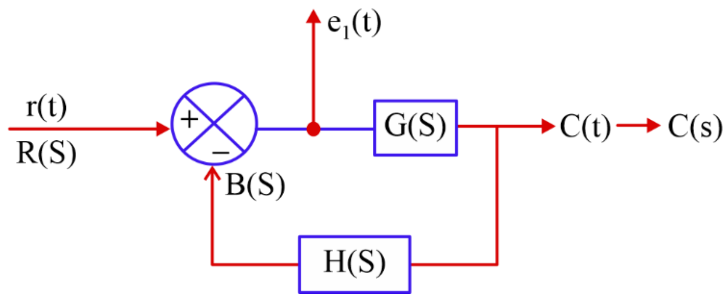
Effect of Type

Input	e_{SS}		
	T-0	T-1	T-2
Au(t)	Finite	0	0
Atu(t)	∞	Finite	0
$\frac{At^2}{2}u(t)$	∞	∞	Finite

(1) Finite steady error is independent of shift in input signal.

(2) Steady state error can be undefined or ∞ even if CLS is stable.

Error Analysis for Non unity f/b –

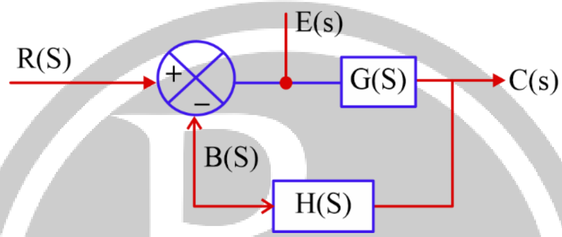


(1) $e_1(t) = r(t) - b(t) \rightarrow$ Can be shown in diagram

(2) $e_2(t) = r(t) - c(t) \rightarrow$ Can not be shown in diagram

(3) $e_3(t) =$ ref signal - $c(t)$

Ref signal = Value of c(t) due to which output of summer is 0.



Case : 1

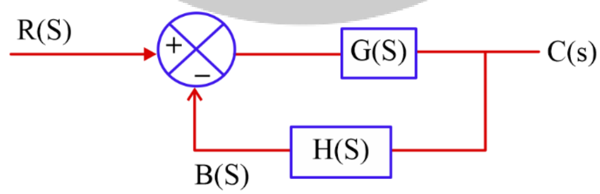
Error Signal $= e(t) = r(t) - b(t)$

$$\boxed{e_{ss} = \underset{s\to 0}{Lim}\, SE(S) = \underset{s\to 0}{Lim} \frac{SR(S)}{1+G(S)H(S)}}$$

Steps: (1) feedback – Non unity $\rightarrow$ error was shown in frequency .

(2) $e_{ss} = \underset{S\to 0}{Lim}\left[\frac{SR(S)}{1+G(S)H(S)}\right]$ all poles must be in L.H.P

Case: 2 $e(t) = r(t) - c(t)$



$$e_{ss} = \underset{S\to 0}{Lim} \frac{SR(S)[1+G(S)H(S)-G(S)]}{1+G(S)H(S)}$$

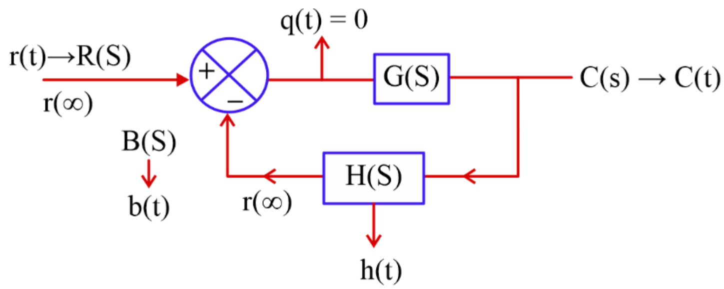
Steps :

(1) f/b $\rightarrow$Non unity $e(t) = r(t) - c(t)$

(2) $ess = \underset{S\to 0}{Lim}\, SE(S) = \underset{S\to 0}{Lim}\, S[R(S) - C(S)]$

(3) All poles of SE(S) must be in L.H.P

Case: 3



$$r(\infty) = H(0)C(\infty)$$

$$H(0) = K_H$$

$$\boxed{e_{ss} = r_f(\infty) - c(\infty)}$$

$r_f(t) =$ Value of C(t) due to which $q(t) = 0$

$r_f(\infty) =$ Value of $C(\infty)$ due to which $q(\phi) = 0$

$$e_{ss} = \lim_{S\to 0} \frac{SR(S)}{K_H}[1 - K_H\ T(S)] \qquad K_H = H(S)|_{S=0}$$

Steps: (1) Non unity f/b, $e(t) =$ ref signal $- c(t)$

(2) $e_{ss} = \lim_{s\to 0}\left\{\underbrace{\frac{SR(S)}{K_H}[1 - K_H T(S)]}\right\} = \frac{r(\infty)}{K_H} - c(\infty)$

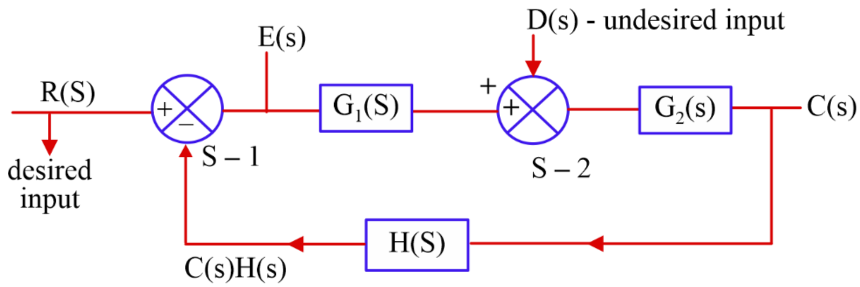
all poles must be in L.H.P

Note : $H(S) = S^N F(S)$

$$K_H = \lim_{S\to 0} F(S)$$

$$e_{ss} = \lim_{s\to 0} \frac{SR(S)}{K_H S^N}(1 - K_H T(S) S^N)$$

Concept of Disturbance or Noise signal :



$$\boxed{C = \left[\frac{G_1G_2}{1 + G_1G_2H}\right]R + \left[\frac{G_2}{1 + G_1G_2H}\right]D}$$

Error signal at output of S_1, $E = R - CH$ S_1, $E = R - CH = \left[\frac{R}{1 + G_1G_2H}\right] + \left[\frac{-D}{1 + G_1G_2H}\right]G_2H$

Stead state error, $e_{ss} = \underset{s\to 0}{Lim}\, SE(S)$

for addition at s-2 $\overset{(+)}{}$

$\uparrow$

$$e_{ss} = \underset{s\to 0}{Lim}\left\{\frac{SR(S)}{1+G_1(S)G_2(S)H(S)}\right\} + \underset{s\to 0}{Lim}\left\{\frac{-SD(S)}{1+G_1(S)G_2(S)H(S)}\right\}G_2(s)H(s)$$

$\downarrow$ due to input r(t)[desired input] at s-1

$\downarrow$ due to undesired input d(t) or Noise or diturbance at S-2

➢ We can reduce e_{ss} if G_1 should be as high as passible and G_2 as low as possible

Sensitivity of a parameter :

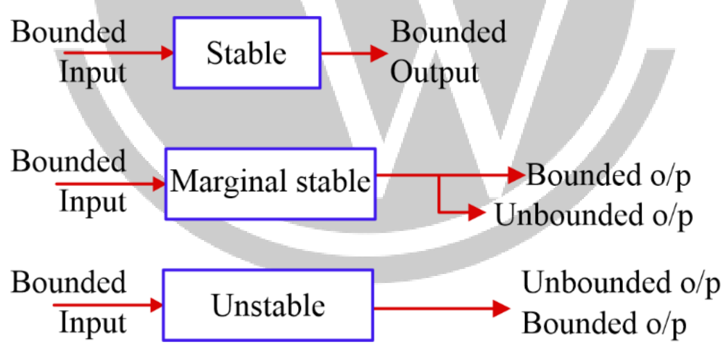
S_T^K = sensitivity of K w.r.t T

$$S_T^K = \frac{(\partial K / K)}{(\partial T / T)}$$

Stability of an LTI System :

(1) For an LTI S/S to be stable must follow BIBO criteria .

(2) Bounded (in amplitude) signal ; $|X(t)| \le M < \infty, M$:finite No. X(t) is bounded signal



(3) LTI system $\rightarrow h(t)$

Unit impulse response must be absolutely integrable

$$\int_{-\infty}^{\infty} |h(t)|\, dt \rightarrow h(t)$$

(4) For an LTI s/s to be stable ROC of H(s) must include $j\omega$ axis [ROC never include poles]

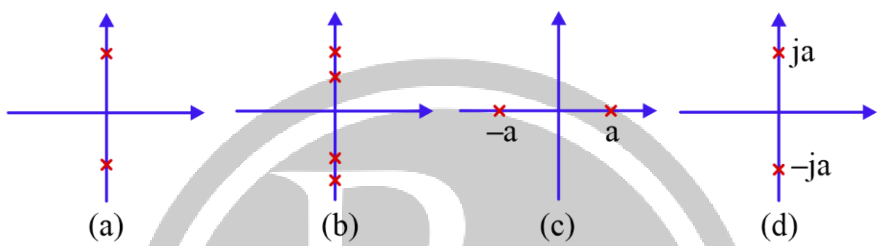
Conclusion :

Causal LTI system can be stable, it all poles of H(s) must be strictly on L. H. P

Location of poles in H(s)	Stability
(1) Repeated or Non – repeated poles on L.H.P	Stable

(2) Single pole at origin	Marginal stable
(3) Non repeated poles on $j\omega$ axis	Marginal stable
(4) Multiple Non repeated poles on $j\omega$ axis	Marginal stable
(5) Multiple poles on $j\omega$ axis	Unstable
(6) Repeated poles on $j\omega$ axis	Unstable
(7) Poles on R.H.P	Unstable
(8) No poles	Stable

Imaginary vs image Location :



Imaginary location (a), (b) image location (c), (d)

1st Order Polynomial

If all coefficient are same then roots will be in L.H. P

D(S) = S +2 , D(S) $= -S-3$

L. H. P L. H. P

2nd Order

All coefficient having same sign and No coefficient is zero, then all roots will be in LHP

D(S) = S^2+2S+2 D(S) $= -S^2-S-1$

L.H.P L. H. P

3rd Order

No missing power and all coefficient has same sign then –

(A) All real roots in L. H. P

(B) No comment on complex roots

R – H Table

$$T(s)=H(S)=\frac{N(s)}{D(s)}$$ Root of D(s) = Poles of T(s)

$$D(s)=a_0s^6+a_1a^5+a_2s^4+a_3s^3+a_4s^2+a_5s+a_6$$

s^6	a_0	a_2	a_4	a_6
s^5	a_1	a_3	a_5	0

$$b_2=\frac{a_1a_4-a_0a_5}{a_1}$$

s^4	$b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$	b_2	0	0	$c_1 = \frac{b_1a_3 - b_2a_1}{b_1}$
s^3	c_1	c_2	0	0	$c_1 = \frac{b_1a_3 + b_2a_1}{b_1}$
s^2	d_1	a_6	0	0	$c_2 = b_1a_5 - a_6a_1 \vert c_1$
s^1	c_1	0	0	0	$d_1 = c_1b_2 - b_1c_2 \vert c_1$
s^0	a_6	0	0	0	$e_1 = \frac{d_1c_2 - c_1a_6}{d_1}$

Key Point:

(1) If any now of RH table multiply or divided by + constant result remains same

(2) 1^{st} column elements have same sign $\rightarrow$ No roots in

(3) Any row becomes zero then roots will be at image location

(4) If all elements in 1^{st} column $\rightarrow$ same sign + No row becomes zero

Then all roots in LHP

(5) The no of sign changes in 1^{st} column = No of roots lying in RHP

(6) If any power of *s* is missing then

(i) 1 or more than 1 root may exist in RHP

$D(s) = s^2 - 1$

$\text{Roots} = s = \pm 1$

(ii) Non repeated roots on $j\omega$ axis may exist

$$D(s) = s^2 + 1 \Rightarrow s = \pm j$$

(iii) Repeated roots may exist on $j\omega$ axis

(iv) There may be complex roots

(v) If only even power of *s* exist then root will be at image location.

(vi) If only odd power of *s* exist then few roots will be at origin and remaining roots at image location

(vii) In RH table $\rightarrow$ odd ROZ $\rightarrow$ some roots will be at image location

$\downarrow$

Sign of elements in 1^{st} Column is same $\rightarrow$ Image location will be imaginary axis

(viii) In RH table $\rightarrow$ No odd ROZ $\rightarrow$ No root at image location

$\downarrow$

No root at imaginary location

3.2. Root Calculation when Odd Row is never zero

For nth order polynomial

(1) Form RH Table

(2) Observe the first column

(3) 1st column elements → Same sign → all roots in LHP
→ K sign change → K roots in RHP (n - K) roots in LHP

Note :

$$a_3s^3 + a_2s^2 + a_1s + a_0 = 0$$

I. P

E. P

I.P – Inner product

E. P – External Product

All coefficient should have same sign

(1) IP > EP → all roots in LHP

(2) IP = EP → 1 root LHP
→ 2 root on image location

(3) $IP = EP$ ↗ 2 root RHP
↘ 1 root LHP

Special Cases

(1) 1st element of row = 0 other elements non zero

Ex. $3s^4 + s^3 + 3s^2 + s + 2$

s^4	3	3	2
s^3	1	1	0
s^2	$0 \to d$	2	0
s^1	$\frac{d-2}{d}$	0	0
s^0	0		

$d = +ve$ quantity

$$\frac{d-2}{d} = 1 - \frac{2}{d} = 1 - \frac{2}{0} = -ve$$

No odd row is zero, two sign changes

$2 \to R.H.P$

$2 \to L.H.P$

(2) If all element of odd row are zero

(i) Few roots at image location

(ii) Form auxiliary characteristic equation has been formed from the row just above the odd row

$A(s) = 0$

(3) Then

$$\frac{d}{ds}A(s) = B(s)$$

(4) Replace odd row of zeros with B(s) coefficients

(5) Roots of A(s) = Roots of D(s), A(s) roots will be at image location A(s) is always a factor of D(s)

$$\frac{D(s)}{A(s)} = P(s) \rightarrow \text{Remaining roots}$$

(6) Both location and exact value of roots can be calculated

If odd row becomes zero once:

(1) Roots of A(s) will be to image location and non-repeated in nature

(2) If A(s) is of 2nd order and roots of A(s) is on $j\omega$ axis then roots will represent, undamp natural frequency of 2nd order system

(3) $\frac{D(s)}{A(s)} = P(s) = 0 \rightarrow$ Remaining roots

3.2.1. Odd Row becomes Zero Twice

(1) 2 Auxillary equation

A_1 (s) : Higher order AE A_2 (s) : Lower order AE

(2) Roots of $A_2(s)$ will automatically be covered by $A_1(s)$

(3) Roots of $A_1(s)$ will be image location

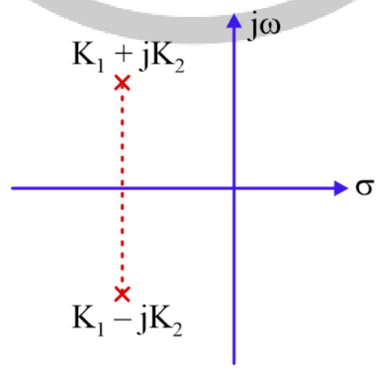
(4) $\frac{D(s)}{A_1(s)} = P(s) \rightarrow$ Remaining roots

$$\begin{matrix}\text{No. of roots of } D(s) \text{ on} \\ \text{Imaginary axis}\end{matrix} = \begin{pmatrix}\text{Highest} \\ \text{order AE}\end{pmatrix} - 2\times\begin{pmatrix}\text{No. of sign changes} \\ \text{below highest order AE}\end{pmatrix} - C$$

No. of roots of D(s) is RHP = No of sign charge in 1st column ..... (2)

No. of roots of D(s) in LHP = order of D(s) – (i) – (ii)

Conditionally Stable :



$K_1 \rightarrow \text{variable}, K_2 \rightarrow \text{Constent}$

- Stability depends on K_1 or conditionally stable
- Use wavy curve

Marginally Stable :

S – 1 form RH table

S – 2 All sign should be same

S – 3 Odd row become zero once. Non repeated roots on imaginary axis and system becomes marginally stables

Oscillating system with undamp natural frequency

S – 1 form RH table

S -2 All elements should be +ve

S – 3 Odd row zero once Auxiliary C.E. is of 2nd order

Note :

- If polynomial has only even power the roots are symmetrical about origin or image location.
- Random power of s missing then at least 1 root in RHP

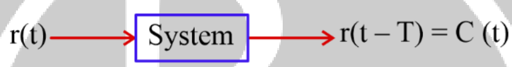
Limitation of Routh:

(1) Applicable to finite order polynomial only

(2) $D(s) = e^s, \tan s, \cos s \rightarrow$ RH invalid

(3) Coefficient of polynomial showed be constant

3.3. Transportation Lag System



T : delay time or log time

Common Mistake

$$T(s) = e^{-sT}$$

$$e^{-x} = 1 - x \text{(When } x \text{ is very small)}$$

$$D(s) = 1 + Ke^{-ST} = 0$$

Routh invalid, use basic root calculation approach

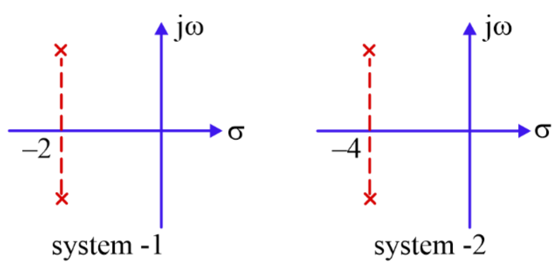
- Both polynomial and exponential present thin R.H. applicable

$$s^2 + s + Ke^{-sT} = 0$$

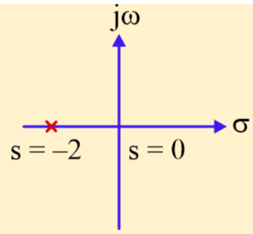
$$s^2 + s + K(1 - sT) = 0$$

Shifting Of origin

- System 2 is more stable



Note : D(s) = s + 2



shifting the origin S = 0 to S = - 1

$$S \begin{cases} \nearrow Z+1 \xrightarrow[Z=0]{} s=1 \\ \searrow Z-1 \xrightarrow[Z=0]{} s=-1 \end{cases}$$

Put S = Z – 1

D(s) = Z + 1

Note : $D(s) = s^2 + s + 1$

(1) How many roots are more negative than $\sigma = 0$ $\Rightarrow$ Roots in LHP R – H criteria in D(s)

(2) How many roots are more +ve than $\sigma = 0 \Rightarrow$ Roots in RHP R – H criteria in D(s)

(3) How many roots have $\sigma = 0 \Rightarrow$ Roots on $j\omega$ axis R – H criteria on D(s)

(4) How many roots are more negative than $\sigma = -1$

Put S = Z – 1

D(z) $\rightarrow$ R. H criteria

No of roots in RHP in z plane $\Rightarrow$ No of roots having $\sigma > -1$ in s – plane

No of roots in LHP in z plane $\Rightarrow$No of roots having $\sigma < -1$ in s – plane

4 ROOT LOCUS

4.1. Root Locus

Locus of roots of characteristic equation or Locus of zeros of characteristic equation or Locus of poles of closed loop system .

D.R.L $\Rightarrow$ Direct root locus

C.R.L $\Rightarrow$ Complementary root Locus

4.1.1. Angle and Magnitude Criteria

Case 1: For D.R.L , C.E , KF(S) = 1 , K= +ve constant

$$|KF(S)| = 1$$

$$< KF(S) = (2n+1)\pi$$

Case 2: For C.R.L KF(S) =1

$$|KF(S)| = 1$$

$$\angle KF(S) = 2n\pi$$

G(S)H(S)	K	Feedback	Locus
KF(S)	$0<K<\infty$	-Ve	D.R.L
-KF(S)	$0<K<\infty$	-Ve	C.R.L
KF(S)	$0<K<\infty$	+Ve	C.R.L
-KF(S)	$0<K<\infty$	+Ve	D.R.L

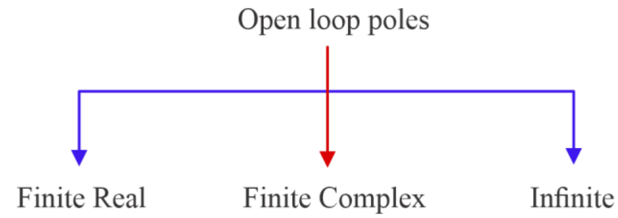
Rules to plot D.R.L

Rule 1 : To plot D.R.L all the coefficient of S should be + ve.

Rule 2 : Origination of D.R.L

(1) DRL originate from open loop poles

(2)



$G(S)H(S)=\frac{K}{S+2}$ $\quad G(S)H(S)=\frac{K}{S^2+S+1}$ $\quad G(S)H(S)=KS^2$

(3) Open loop pole : finite Real – DRL branch will originate from the open loop pole and move on the real axis in the section D.R.L present .

(4) Open loop pole : finite complex – D.R.L branch will originate form open loop pole in the directions of angle of departure

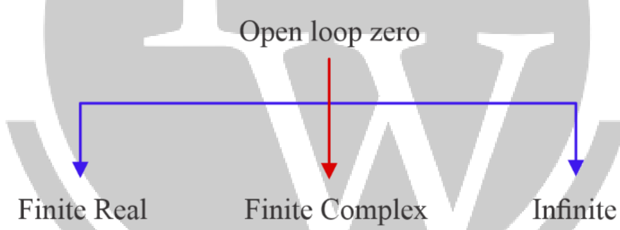
(5) open loop pole present at ∞ - D.R.L branch will originate from ∞ by gazing the asymptotic lines given by angle of asymptotes.

(6) at open loop pole value of K = 0

Rule 3 : Termination of D.R.L

(1) Terminate at open loop zeros .

(2)



(3) open loop zero : finite Real – Terminate at open loop zero by moving on the real axis in the section D.R.L exist .

(4) open loop zero finite complex – DRL branch will terminate at open loop zero in the direction given by angle of arrival.

(5) open loop zero at ∞ - DRL branch will terminate at open loop zero by gazing angle of asymptotes.

(6) Value of K at open loop zero is

$0<K<\infty \rightarrow K=\infty$

$-\infty<K<0 \rightarrow K=-\infty$

Rule 4 : Existence of D.R.L on Real axis.

- Segment of real axis where D.R.L exist
- Segment where DRL exist must hare " ODD no of open loop poles zeros towards its night.

Rule 5 : Identification of a point $S=S_o$ is

(i) Part of root locus
(ii) Poles of C.L.S
(iii) Roots of C.E
(iv) zeros of C.E

Case 1: $S = S_o$ is real

Method 1 check if $S = S_o$ is part of D.R.L

Method 2 Angle sub started by all open loop poles and zeros towards desired point must be odd Multiple of π.

Method 3 (i) put $S = S_0$ in CE , calculate K then if K is real and $+Ve \Rightarrow S = S_0$ is part of D.R.L

K is real and +ve $\Rightarrow S = S_0$ is part of D.R.L

K otherwise $\Rightarrow S = S_0$ is not part of D.R.L.

Method 4 verify magnitude and angle criteria at $S = S_0$

D.R.L $\angle KF(S) = (2n+1)\pi$

$|KF(S)| = 1$

Case 2 : $S = S_0$ Complex.

Method 1 fails

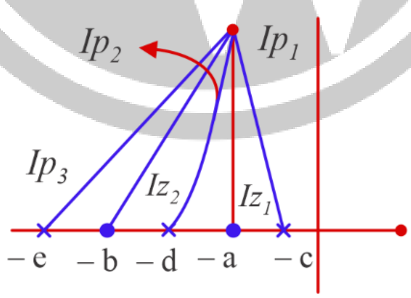
Method 2,3,4 are applicable

Rule 6 : Calculation of K at $S = S_0$ (if $S = S_0$ is part of RL)

$$G(S)H(S) = \frac{K(S+a)(S+b)}{(S+c)(S+d)(S+e)} \qquad 0 < K < \infty$$

$-ve \ f/b$

$$\boxed{K = \frac{lp_1 lp_2 lp_3}{lz_1, lz_2}}$$



Rule 7: Angle of asymptotes

Only for those branches which ether originate or terminate at ∞.

Formula :

(1) $P - Z \neq 0,\ n = 0,1,2,3,--P-Z-1$ P = finite open loop poles

(2)

If $P > Z$	$Z > P$
$\theta_n = \dfrac{(2n+1)180^0}{(p-z)}$	$\theta_n = \dfrac{(2n+1)180^0}{(z-p)}$

Z = finite open loop zeros

Rules 8 : **Centroid**

(i) Calculated When $P \pm Z$

(ii) Needed only when A.O.A are calculated.

(iii) The A.O.A drown from a point on real axis known as centroid, (originating point of asymptotes)

(iv) All the asymptotic lines meets at common point on real axis know as centroid.

(v) Always present on real axis .

(vi) May or may not be part of R.C

(vii) Value of centroid $\rightarrow \sigma = 0, +ve, -ve$

(viii) formula $\sigma = \frac{\sum p - \sum z}{p - z}$ σ :Real numbers

Rule 9: Break point

(1) Where 2 or more then 2 poles of C.L.S coincides simultaneously . If is part of R.L

Types :

(1) Break away point

(1) 2 or more then 2 poles of C.L.S coincides .

(2) After B.A.P R.L Breaks into some parts and it can not remains on real axis. It moves into different parts in complex conjugate location

(3) BAP means shifting of R.L from real axis into complex conjugate Location.

(4) At BAP k achieves max value for which root remains on real axis if $K \uparrow$ then R.L moves on complex conjugate location

$0 \le K \le K_{BAP}$: Root locus is on real axis .

$K > K_{BAP}$ Root locus on complex conjugate location.

(2) Break In point

(1) 2 or more then 2 poles of C.L.S coincides .

(2) After the BIP R.L Breaks into some parts and if can not remain on complex conjugate location . If move on into different parts on real axis .

(3) BIP means shifting of root locus from complex conjugate location to real axis .

(4) At BIP K achieves min value for which root locus is on real axis . If $K \uparrow$ if remains on real axis $0 < K \le K_{BIP} \rightarrow$ complex conjugate location

$K \ge K_{BIP} \rightarrow$ Real axis

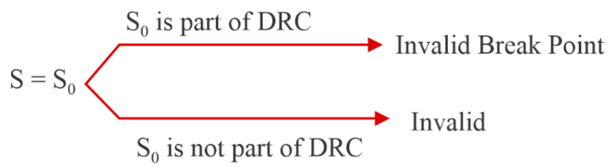
S.1 from the CE $1 \pm G(S)H(S) = 0$

$1 \pm KF(S) = 0$

$$K = \pm \frac{1}{F(S)} = Q(S)$$

$$\frac{dk}{ds} = 0, \text{ Possible Break point}$$

S.2 By $\frac{dk}{ds} = 0,$



S.3 If so is valid Break point .

$$S_0 : \text{Real} \quad \left(\frac{d^2K}{ds^2}\right)_{S=S_0} < 0$$

$$K = Q(S) \rightarrow \text{maxima at } S = S_0 \rightarrow B.A.P$$

$$\left(\frac{d^2k}{ds^2}\right)_{S=S_0} > 0$$

K = Q(S) $\rightarrow$Minima at $S = S_0 \rightarrow B.I.N$

4.2. Properties of Break points

(1) At Break point ,RL branches from an angle of $\pm \frac{180^0}{n}$ with real axis where n is number of closed loop poles arriving or departing from signal breakpoint on the real axis .

(2) If 2 adjacent open loop poles on real axis and segment between them is part of DRL then there will be at least one BAP between then .

(3) For 2 adjacent open zeros of OLS $\rightarrow$At least 1 BIP exist

(4) If 2 or more then 2 poles of OLS coincide at K = 0 this itself represent BAP .

for K = 0 , OLP = CLP

(5) If 2 or more then 2 zeros coincide at $K = \infty$ then this itself becomes Break point for

$K = \infty, OLZ = CLP$

Rule 10 : Angle of departure

➢ For complex OLP , given originating direction to Branch of DRL

$\theta_d = 180^0 - [\phi_p - \phi_z]$

$\phi_p =$ Angle sustained by remaining OLP towards desired pole

$\phi_z =$ Angle sustained by remaining all OLZ towards desired pole

Rule 11 : Angle of arrival

- For complex OLZ , gives terminating direction .

$\theta_a = 180^0 - [\phi_z - \phi_p]$

$\phi_z =$ Angle sustained by remaining OLZ toward desired OLZ

$d_p =$ Angle sustained by remaining all OLP toward desired OLZ .

Rule 12 : Intersection with jω axis

- Identification of CLP on $j\omega$ axis .

S-1 From C.E and R.H table .

S-2 Odd Row ←
- → Not Zero
- → Zero but K = -ve
- → Zero and K = + ve But roots of AE not on $j\omega$ axis

} Root Locus does not intersect $j\omega$ axis

Odd Row → Zero and K = +ve and Root of AE on $j\omega$ axis → RL intersect $j\omega$ axis

Rules to PLOT a C.R.L

- Rule 1 – Same as DRL
- Rule 2 – Same
- Rule 3 – Same
- Rule 4 – Replace odd with Even
- Rule 5 – Identification of $S = S_0$ on CRL

Case 1 : $S = S_0$ is real

- Method-1 So is part of CRL
- Method -2 Angle by all OLP and zero are even multiple of $\pi = 2n\pi$
- Method -3 $1 \pm G(S)H(S) = 0 \xrightarrow{S=S_o} K = \text{Real and} + ve$
- Method -4 $|KF(S)|_{S=S_0} = 1$

$\angle KF(S)|_{S=S_0} = 2\pi$

Case 2 : $S = S_0$ Complex

M-1 Fall

M-2 Angle by all OLP and OLZ should be $2n\pi$

M-3 $C \cdot E \xrightarrow{S=S_0} K$ real and $+ve$

M-4 $|KF(S)| = 1,\ \angle KF(S)|_{S=S_0} = 2n\pi$

Rule 6 : **Angle of asymptotes**

$P > Z \qquad P < Z$

$$\theta_n = \frac{2n\pi}{P-Z} \qquad \theta_n = \frac{2n\pi}{P-Z}$$

Rule 7 : Same

Rule 8 : Calculate K at $S = S_0$, if $S = S_0$, is part of C.R.L

Rule 9 : Breakpoint

S.1 – Same as DRL

S.2 – Validate $S = S_0$ by following C.R.L criteria

Rule 10: Angle of departure

D.R.L	C.R.L
$\theta_d = 180^0 - (\phi_p - \phi_z)$	$\theta_d = 0^0 - (\phi_p - \phi_z)$

Rule 11: Angle of arrival.

$\theta_a = 0^0 - (\phi_z - \phi_p)$

Rule 12: Same

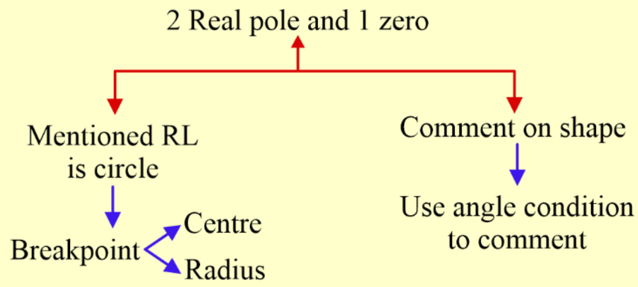
Note – RL always symmetrical about real axis

Few Important Result

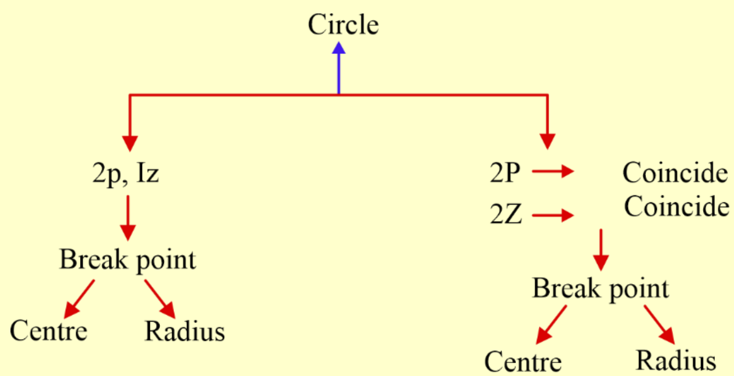
(1) $G(S)H(S) = \dfrac{K(s+b)}{(s+a)}$ Breakpoint $= -b \pm \sqrt{b^2 - ab}$ Radius of circle = $\sqrt{b^2 - ab}$ Centre = (-b, 0)	(2) $G(S)H(S) = \dfrac{-K(s-b)}{S(s+a)}$ Breakpoint $\Rightarrow s = b \pm \sqrt{b^2 + ab}$ Centre = (b, 0) Radius $= \sqrt{b^2 + ab}$
(3) $G(S)H(S) = \dfrac{KS}{(S-a)(S-b)}$ Breakpoint $\Rightarrow s = \pm\sqrt{ab}$ Centre = (0 , 0) Radius $= \sqrt{ab}$	(4) (4) $G(S)H(S) = \dfrac{-KS}{(S+a)(S+b)}$ Breakpoint $\Rightarrow s = \pm\sqrt{ab}$ Centre = (0, 0) Radius = $\sqrt{ab}$
(5) $G(S)H(S) = \dfrac{K(S+a^2)}{(S+b)^2}$ Centre $= \left[-\left(\dfrac{a+b}{2}\right), 0\right]$ Radius $= \dfrac{1}{2}\lvert a-b \rvert$ Breakpoint , $s = -a, -b$	

Note :

(1)



(2)



Min phase System : All polls and zeros must be in L.H.P

Non Minimum phase system : Which are not minimum phase

5 FREQUENCY RESPONSE ANALYSIS

5.1. Introduction

Test inputs : Sinusoidal input

$x(t) = A\cos\omega_o t \rightarrow$ const frequency $= \omega_o$ $\quad S = j\omega$

$$A\cos\omega_o t \longrightarrow \boxed{H(S)} \longrightarrow y(t) = A|H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_o))$$

$$A\sin\omega_o t \longrightarrow \boxed{H(S)} \longrightarrow y(t) = A|H(j\omega_o)|\sin(\omega_o t + \angle H(j\omega_o))$$

- Only $j\omega$ axis of S domain needed .
- Steady state output when a sinusoidal signal is applied –

  $\{y(t)\}_{SS} = \lim_{t\to\infty}[a_0 e^{-j\omega_0 t} + a_1 e^{j\omega_o t}]$

Few observations

$$A\cos(\omega_o t + \phi) \longrightarrow \boxed{H(S)} \longrightarrow y(t)$$

$$\boxed{y(t) = A|H(j\omega_o)|\cos(\omega_0 t + \phi + \angle H(j\omega_o))} \rightarrow [y(t)]_{SS}$$

$\downarrow$

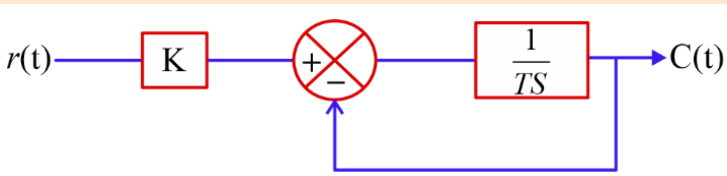
Replace With sin for sin i/p .

$H(j\omega) \rightarrow$ Frequency response of an LTI S/S

$|H(j\omega)| \rightarrow$ Magnitude

$\angle H(j\omega) \rightarrow$ Phase Response

5.1.1. Frequency domain analysis of 1st order



$$T(S) = \frac{K/T}{S + 1/T}$$

$$T(j\omega) = \frac{K/T}{j\omega + 1/T}$$

$$|T(j\omega)| = \frac{K/T}{\sqrt{\omega^2 + 1/T^2}}, \angle T(j\omega) = -\tan^{-1}\omega T$$

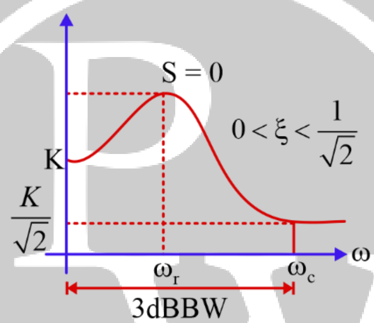
$$3.dB\ \ BW - \omega_c = \frac{|T(j_0)|}{\sqrt{2}} \text{rad/sec}$$

5.1.2. Frequency domain analysis of 2nd Order System

$$T(S) = \frac{K\omega_n^2}{s2 + 2\xi\omega_n s + \omega_n^2} \quad 0 \le \xi \le 1 \text{ stability can be decided.}$$

$$|T(j\omega)| = \frac{K}{\sqrt{\left(i - \left(\frac{\omega^2}{\omega_n^2}\right)^2 + 4\xi^2\frac{\omega^2}{\omega_n^2}\right)}}, \angle T(j\omega) = -\tan^{-1}\left\{\frac{2\xi\omega/\omega_n}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)}\right\}$$

Plot 1



For Resonant frequency

$$\frac{d}{d\omega}|T(j\omega)| = 0$$

$\omega_r = \omega_n\sqrt{1 - 2\varepsilon^2}$

$\omega_n \rightarrow$ undamped Natural frequency

$\omega_r \rightarrow$ Resonant frequency

$\omega_d \rightarrow$ Damped frequency

Resonant Peak

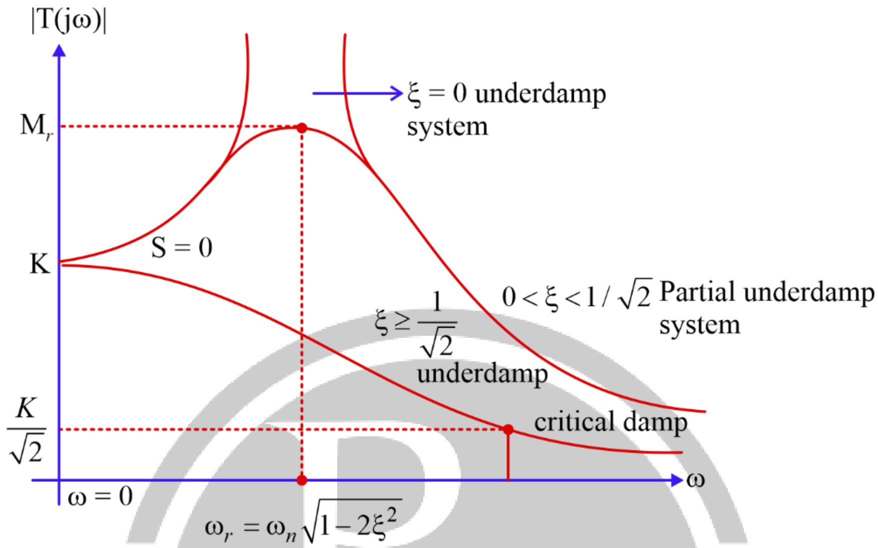
$$\boxed{M_r = \frac{K}{2\xi\sqrt{1 - \xi^2}}}$$

ω_r is real only when $\xi < 1/\sqrt{2}$

for $\xi \ge \frac{1}{2}, \omega_r$ does not exist

$$\omega_r = \begin{cases} \omega_n : \xi = 0 \\ \omega_n\sqrt{1 - 2\xi^2} : 0 < \xi < 1/\sqrt{2} \\ 0 \qquad : \xi \ge 1/\sqrt{2} \end{cases}$$

$$M_r = \begin{cases} \infty & : \quad \xi = 0 \\ \dfrac{K}{2\xi\sqrt{1-\xi^2}} & : \quad 0 < \xi < 1/\sqrt{2} \\ K & : \quad \xi \geq \dfrac{1}{\sqrt{2}} \end{cases}$$



$\rightarrow \omega_r \downarrow$ as $\xi \uparrow$ if ω_n constant

$\rightarrow \omega_r \alpha\, \omega_n$ if ξ is constant

$\rightarrow M_r \downarrow as\ \xi \uparrow (0 < \xi < 1/\sqrt{2})$

$$\omega_c = \omega_n \sqrt{(1-2\xi^2) + \sqrt{(1-2\xi^2)^2 + 1}}$$

$$\boxed{\omega_c = \omega_n \sqrt{(1-2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}}}$$

$\rightarrow \omega_c \downarrow$ as $\xi \uparrow$, ω_n constant

$\rightarrow \omega_c\ \alpha\ \omega_n$ When ξ *is* constant

Bode Plot :

$\dfrac{\text{Exact frequency Analysis of a system}}{T(S)}$:

S-1 Put $S = j\omega \quad 0^+ < \omega < +\infty$

S-2 $T(j\omega) = |T(j\omega)| e^{j\angle T(j\omega)}$

S-3 Plot $|T(j\omega)|\ VS\ \omega \rightarrow$ Exact Magnitude plot

$\angle T(j\omega)\ VS\ \omega \rightarrow$ Exact phase plot

- Exact plots are non linear in shape ; drawn on normal graphs .
  Stability of T(S) $\rightarrow$ can be defined by plotting Bode plot of OLTF G(S)H(S).

Bode Plot:

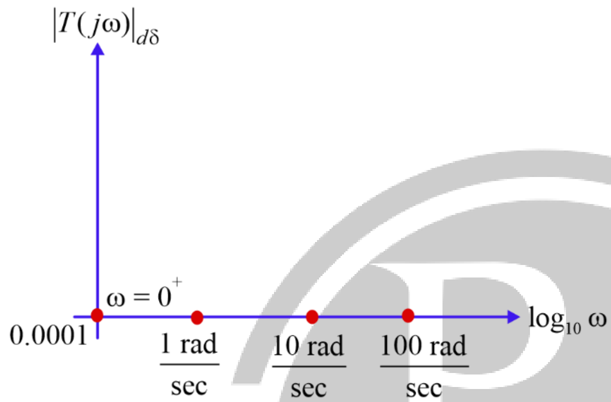
Let OLTF is G(S)H(S)

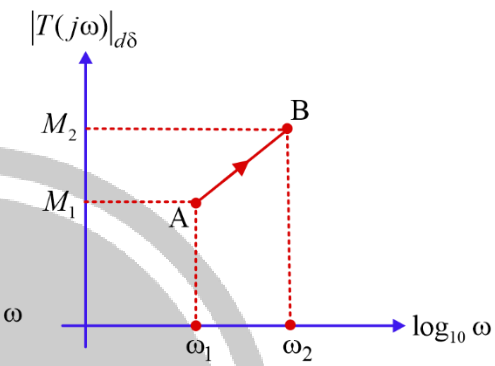
1. $S = j\omega \;\; G(j\omega)H(j\omega) = T(j\omega)$
2. $|G(j\omega)H(j\omega)| = |T(j\omega)|$

   $20\log_{10}|G(j\omega)H(j\omega)| = 20\log_{10}|T(j\omega)| = |G(j\omega)H(j\omega|_{dB}$

   Plot $|G(j\omega)H(j\omega)|$ vs $\log_{10}\omega \rightarrow$ Should be linear
3. $\angle G(j\omega)H(j\omega) = \angle T(j\omega)^0$

   Plot : $\angle G(j\omega)H(j\omega)$ vs $\log_{10}\omega \rightarrow$ This need not to be linear .
4.

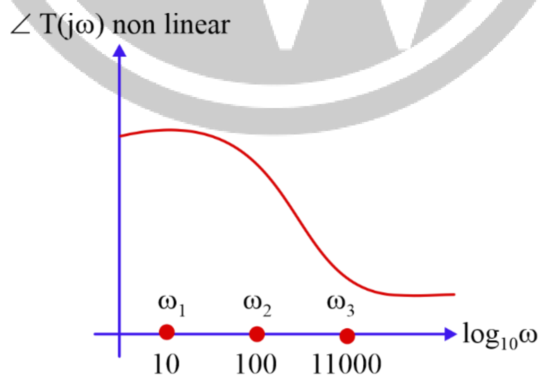




$A(\log_{10}\omega_1, M_1 dB)$ $\quad B(\log_{10}\omega_2, M_2 dB)$

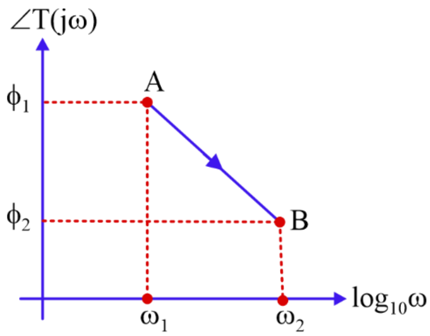
$|T(j\omega)| dB$ VS $\log_{10}\omega \rightarrow$ Make sure it is linear.

$$\text{Slope} = \frac{(M_2 - M_1)}{\left(\log_{10}\frac{\omega_2}{\omega_1}\right)} = \frac{dB}{\text{decade}} = \frac{dB}{\text{Octave}}$$



$\angle T(j\omega)$ ↗ linear ↘ Non linear

(7)



$$S=\frac{\phi_2-\phi_1}{\log_{10}\left(\frac{\omega_2}{\omega_1}\right)}\ \frac{\text{degree}}{\text{decode}}\ or\ \frac{\text{degree}}{\text{octave}}$$

$$+\frac{20dB}{\text{decode}}=\frac{+6dB}{\text{octave}}$$

Exact Plot -

(1) Low frequency Range $\omega << \omega_c : |T(j\omega)|_{dB} = 20\log_{10} K$

Summary Table for $T(S) = KS^p$

$G(S)H(S)$	Initial slope	0 dB axis Int.	Slope at $\omega\to\infty$	PHASE
K	0 dB / decode	×	0 dB/decode	0^0
Ks	+20 dB/dec	$\omega=\frac{1}{K}$	+20 dB/dec	$+90^0$
Ks^2	+40 dB/dec	$\omega=\frac{1}{(K)^{1/2}}$	+40 dB/dec	$+180^0$
:	;	;	;	:
KS^p	+20p dB/dec	$\omega=\frac{1}{(K)^{1/p}}$	+20 dB/dec	$+90^0\ p$

Summary table for $G(S)H(S)=K/S^p$

G(S)H(S)	Initials lobe	0dB axis Int .	Final slope $\omega\to\infty$	Phase
$\frac{K}{S}$	-20dB / dec	ω=K	-20dB / dec	-90^0
$\frac{K}{S^2}$	-40dB / dec	$\omega=(K)^{\frac{1}{2}}$	-40dB / dec	-180^0
$\frac{K}{S^3}$	-60dB/dec	$\omega=(K)^{\frac{1}{3}}$	-60dB/dec	-270^0
:	;	:	:	;
$\frac{K}{S^p}$	-20pd B/dec	$\omega=(K)^{\frac{1}{p}}$	-20pdB/dec	$-90\,p^0$

Important Observation :

Slop of Initial line	Initial Phase	Type
+20 p dB /dec	$+90P^0$	"0"
+0dB/decode	0^0	"0"
-20dB / decade	-90^0	"1"
-40dB/decade	-180^0	"2"
-20p dB/decade	$-90^0\ p$	"p"

Steady state error from Bode plot –

Initial Line slope	Information	e_{ss}
0dB / decade	$Amp = 20\log_{10} K_p$	$\frac{A}{1+K_p}$
-20dB/decade	0dB axis Intersection $= K_v$	A/K_v
-40dB/decade	0 dB axis Intersection $= \sqrt{K_a}$	A/K_a

Exact Plot

(1) Low frequency Range $\omega << \omega_c : |T(j\omega)|_{dB} = 20\log_{10} K$

(2) Mid frequency Range $\omega = \omega_c : |T(j\omega)|_{dB} = 20\log_{10} K + 3dB$

(3) High frequency Range $\omega >> \omega_c : |T(j\omega)|_{dB} = 20\log_{10} \frac{\omega}{\omega_c} + 20\log_{10} K$

G(S)H(S)	Initial Line with slope	Change in slope at $\omega = \omega_c$	error in mag. at $\omega = \omega_c$	Slop at $\omega \to \infty$
$K\left(\frac{S}{\omega_c}+1\right)$	0dB /decade with mag $20\text{Log}_{10}K$	+20 dB/dec	+3dB	+20dB / dec
$K\left(\frac{S}{\omega_c}+1\right)^2$	0dB /decade with mag $20\text{Log}_{10}K$	+40dB/dec	+6dB	+40dB/dec
⋮				
$K\left(\frac{S}{\omega_c}+1\right)^p$	0dB /decade with mag $20\log_{10} K$	+20pdB/dec	+3pdB	+20pdBdecade

G(S)H(S)	Initial Line With slope	Change in slope at ω = ω_c	Error in Mag at ω = ω_c	Slope at ω = ∞
$\dfrac{K}{\left(\dfrac{S}{\omega_c}+1\right)}$	0dB/decade with mag $20\log_{10} K$	–20dB/dec	–3dB	–20dB/dec
$\dfrac{K}{\left(\dfrac{S}{\omega_c}+1\right)^2}$	0dB/decade with mag $20\log_{10} K$	–40dB/dec	–6dB	–40dB/dec
$\dfrac{K}{\left(\dfrac{S}{\omega_c}+1\right)^P}$	0dB/decade with mag $20\log_{10} K$	–20pdB/dec	–3pdB	–20pdB/dec

- If magnitude plot given , then recovered T.F is not unique .
- If Bode magnitude and phase plot is given , then reordered T.F is unique

G(S)H(S)
- → Min phase system $\left(\dfrac{K}{1+\dfrac{S}{\omega_c}}\right), \dfrac{K}{\left(\dfrac{-S}{\omega_c}-1\right)}$
- → Non min phase system $\dfrac{K}{\left(\dfrac{S}{\omega_c}-1\right)}, \dfrac{Ke^{-S}}{\left(\dfrac{S}{\omega_c}+1\right)} \Rightarrow \dfrac{K(1-S)}{\left(\dfrac{S}{\omega_c}+1\right)}$

Time constant from

$$T(S)=\frac{1}{\left(1+\dfrac{S}{T}\right)}$$

↓

Corner frequency

Approximation of T.F.

$$T(S)=\frac{5(S+20)(S+50)}{(S+10)(S+100)}=\frac{5\left[1+\dfrac{5}{20}\right]\left[1+\dfrac{5}{50}\right]}{\left[1+\dfrac{5}{10}\right]\left[1+\dfrac{5}{100}\right]}$$

(i) $0<\omega<10$ $\quad T(S)=\dfrac{5(1)(1)}{(1)(1)}=5$

(ii) $\omega=10$ $\quad T(S)=\dfrac{5(1)(1)}{(1)(1)}=5$

(iii) $10<\omega<20$ $\quad T(S)=\dfrac{5(1)(1)}{\left(\dfrac{5}{10}\right)(1)}=\dfrac{50}{5}$

(iv) $\omega = 20$ $\quad T(S) = \dfrac{5(1)(1)}{\dfrac{S}{10}.1} = \dfrac{50}{5}$

(v) $20 < \omega < 50$ $\quad T(S) = \dfrac{5\left(\dfrac{S}{20}\right)1}{\left(\dfrac{S}{10}\right).1} = \dfrac{5}{2}$

(vi) $\omega = 50$ $\quad T(S) = \dfrac{5}{2}$

Similany

- How to calculate T.F from Bode plot – 0 dB / decode $\rightarrow$ K

Step - 1 Identify initial slope $20p\text{dB/decode} \rightarrow KS^p$

$-20p\text{dB/decode} \rightarrow K/S^p$

Step - 2 Identify corner frequency (where slope changes change in slop = (final – Initial) slope

$$\Delta S = +20p\left(1+\frac{s}{\omega_c}\right)^p, \Delta S = -20p\frac{1}{\left(1+\dfrac{S}{\omega_c}\right)^p}$$

Step -3 For calculation of K

M-1 Approximation

M-2 (i) If initial line is 0dB/de code = $20\log_{10} K = M$

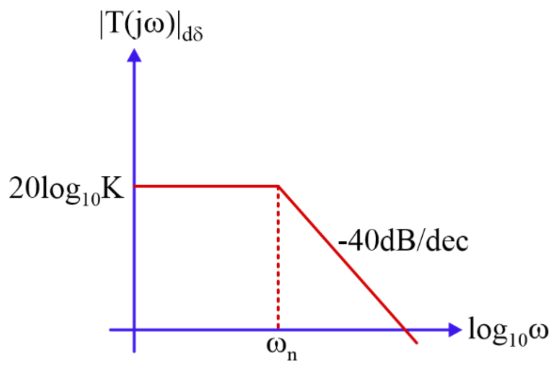
(ii) If slope is there

Magnitude at $\omega = 1 \longrightarrow N$ (magnitude of initial line) $20\log_{10} K = N$

Bode plot of $G(S)H(S) = \dfrac{K\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2} = ?$

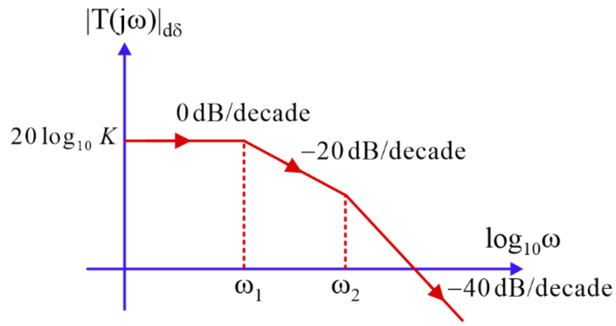
Case - 1 Critical damping $(\xi = 1)$

$$G(S)H(S) = \frac{K}{\left(\dfrac{S}{\omega_n}+1\right)^2}$$



Case – 2 Overdamped $(\xi > 1)$

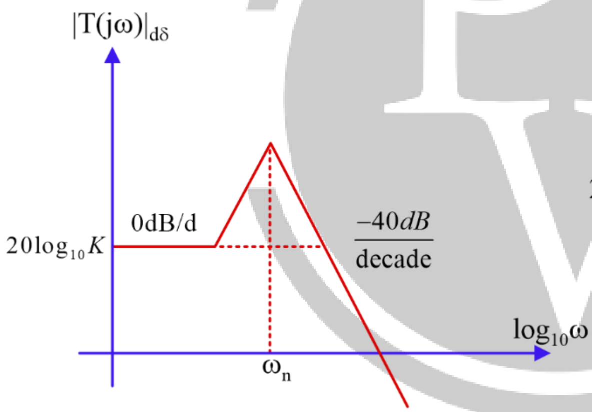
$$G(S)H(S) = \frac{K}{\left(\frac{S}{\omega_1}+1\right)\left(\frac{S}{\omega_2}+1\right)}$$

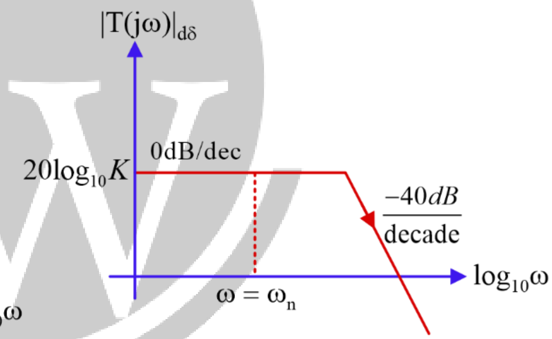


Case -3 Underdamp $(0 < \varepsilon < 1)$

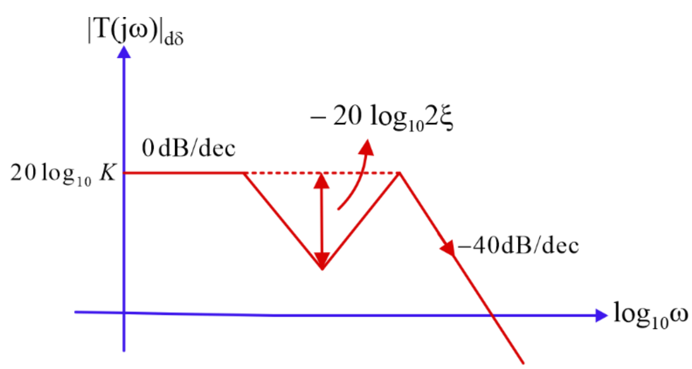
(1) $0 < \xi < \frac{1}{2}$

(2) $\xi = \frac{1}{2}$

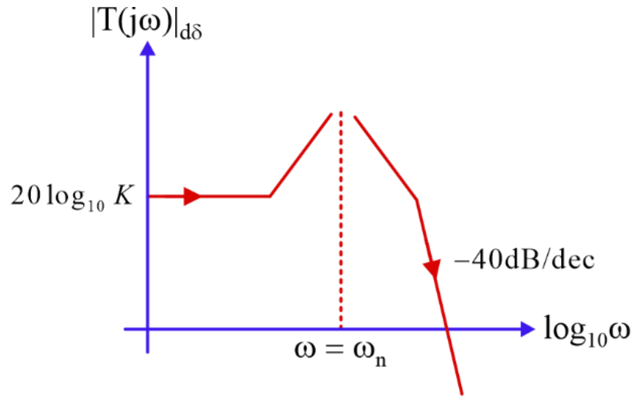




(3) $\frac{1}{2} < \xi < 1$



Case-4: $\xi = 0$



Important Points :

(1) Calculation of unknown frequency

change in magnitude between 2 frequencies = ΔM

$$\frac{\Delta M}{20p} = r$$

factor affecting frequency = 10^r

$$\frac{\Delta M}{6p} = r$$

factor affecting frequency = 2^r

5.2. Nyquist Stability and Plot

- Contour (closed curve in S – plane) or (specified region in s plane) encircles

  encircles (contains) m poles of Q(S) strictly inside if .

  ↓

  Q(S) plot in Q(S) plane encircles origin (0,0) m times.

- Contour in splane passes through one or more pole of $\theta(S)$

  Q(S) plot in Q(S) plane remain open curve. Hence poles do not contribute in encirclement of origin.

- Contour in s-plane encircle m zeros of Q(S) strictly inside it

  ↓Same durection

  Q(S) plot in Q(S) plane encircles the origin m times.

- Contour in s plane has m zero on the boundary of the contour

  ↓ same sense

  (i) Q(S) plot in Q(S) plane is closed.

  (ii) Q(S) plot crosses origin m times.

  (iii) Such zeros do not contribute in encirclement of origin

5.2.1. Rules of mapping from S plane to Q(S) Plane

$$Q(S)=\frac{N(S)}{D(S)},\quad \text{C contour in S plane}$$

P_C = No of poles of Q(S) present strictly inside C,

Z_C = No of Zeros.

N = No . of encirclement of origin by Q(S) plot.

$N = P_C - Z_C$
- N = +ve C and Q(C) are in opposite direction
- N = –ve C and Q(C) are in same direction

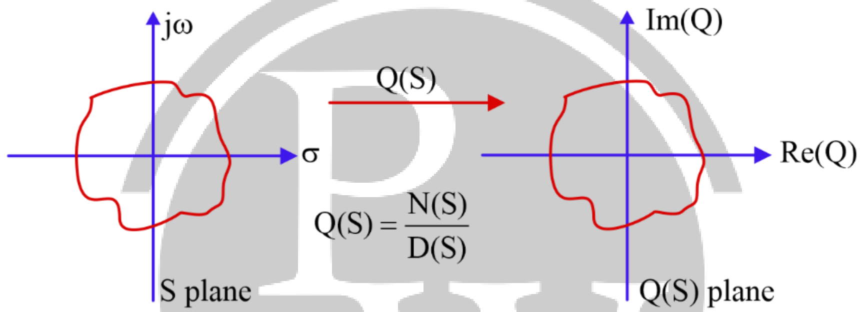
Limitation

(a) If pole of Q(S) lies on boundary of C

(b) If zero of Q(S) lies on boundary of C

Important Points :

(1) Principle of argument



➢ If P_C = Pole inside Contour, Z_C = zero inside contour , Q(S) plot in Q(S) plane

(i) $P_C > Z_C \rightarrow Q(S)$ plot in Q(S) plane encircles the origin (0,0) $(P_C - P_Z)$ time in direction opposite to the contour in S plane.

(ii) $P_c < Z_c \rightarrow$ Encircle origin $(Z_C - P_C)$ time in same direction .

(iii) $P_C = Z_C \rightarrow$ Does not encircle origin .

(2) Rules of Mapping

valid for T.F $Q(S)=\frac{N(S)}{D(S)}$

$$A+BQ(S)=A+\frac{N(S)}{D(S)}B=\frac{AD(S)+BH(S)}{D(S)}$$

(i) Let T.F $= Q(S)=\frac{N(S)}{D(S)}$

P_C = No of poles of Q(S) inside contour .

Z_C = No of Zeros of Q(S) inside contour

N = No of encirclement of (0,0) by Q(S) plot.

N = +ve , if C and Q(S) has opposite direction

N = - ve , if C and Q(S) has same direction

(ii) Let T.F is A + BQ(S)

N = $P_C - P_Z$

$P_C \rightarrow$ No . of poles of T(S) inside contour

$Z_C \rightarrow$ No. of zero .

$N \rightarrow$ No. of encirclement of (0,0) by A+BQ(S) plot .

(iii) If A+BQ(S) plot encircle origin then Q(S) plot will encircle $\left(-\frac{A}{B},0\right)$.

Or

If Q(S) plot encircles (-A/B,0) then A+BQ(S) plot will encircle (0,0)

Nyquist

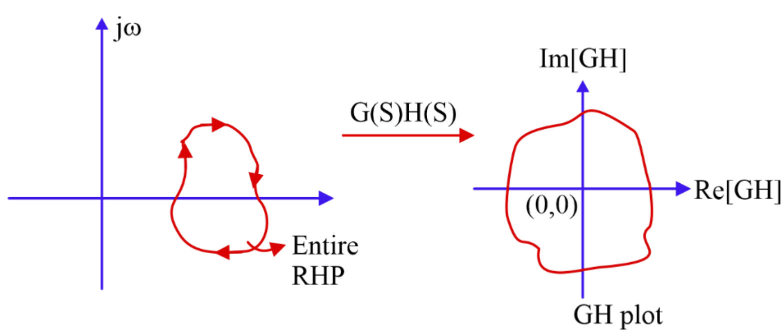
If $G(S)H(S)=\frac{N(S)}{D(S)}$

$$1+G(S)H(S)=\frac{N(S)+D(S)}{D(S)}$$

$$A+BG(S)H(S)=\frac{AD(S)+BH(S)}{D(S)}$$

- Poles of T.F $A+BG(S)H(S)$ will be same as T.F G(S)H(S).
- Zeros of $1+G(S)H(S)=$ Root of $[1+G(S)H(S)]$ = Poles of CLS
- Poles of $1+G(S)H(S)=$ Poles of G(S)H(S)

Case 1



Fixed : Clockwise

Rule of Mapping : let TF is $G(S)H(S) \Rightarrow N = P_C - Z_C$

N = No of encirclement of (0,0) by G(S)H(S) plot in G(S)H(S) plane.

N → $+ve$ $G(S)H(S)$ plot in ACW direction
N → $-ve$ $G(S)H(S)$ plot in CW direction

P_C = No of poles of G(S)H(S) lying inside contour in S plane .

OR

No of poles of G(S)H(S) lying in right side plane $P_C = P_t$

Z_C = No of zero of G(S)H(S) lying inside contour in S plane .

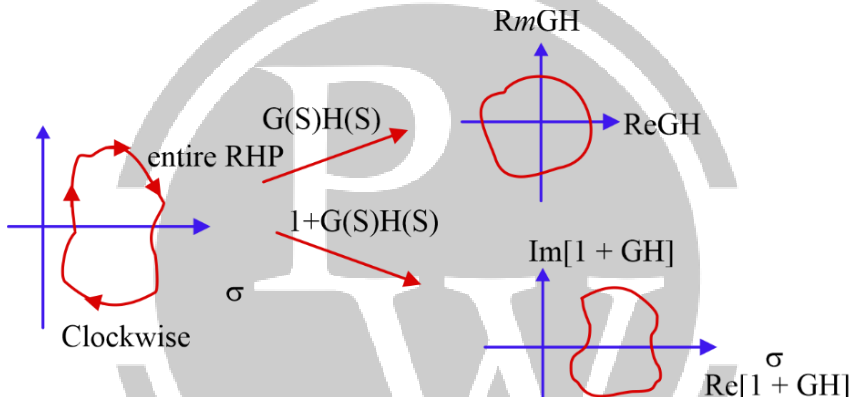
OR

No of zeros of G(S)H(S) lying in RHP $Z_C = Z_t$

$$\boxed{N = P_+ - Z_+}$$

For OLS to be stable $P_C = 0$

Case 2



Rule of Mapping : Let T.F is $1+G(S)H(S) = \dfrac{N(S)+D(S)}{D(S)}$

$$N = P_C - Z_C$$

N = No of encirclement of (0,0) by 1+GH plot in 1+GH plane .

OR

No of encirclement of (-1,0) by GH plot in GH plane .

P_C = no of poles of $[1+G(S)H(S)]$ lying in R.H.P

OR

No of poles of G(S)H(S) lying in $P_C = P_t$

Z_C = no of zeros of [1+G(S)H(S)] lying in RHP.

OR

No of poles of closed loop system lying in R.H.P. $Z_C \rightarrow Z_t$

Note: After Nyquist modified the mapping Rule .

(i) S plane contour : Entire R.H.P.

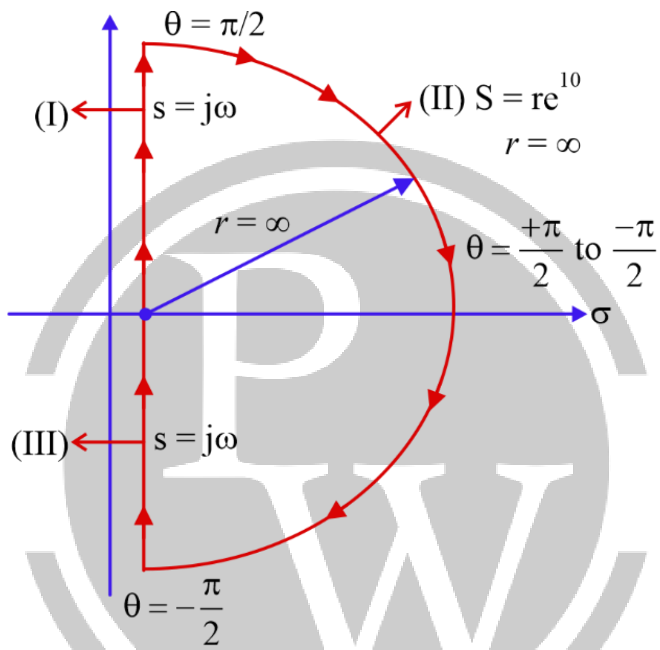
(ii) Plot of G(S)H(S) is needed only

- Stability of all the T.F. of type A+BG(S)H(S) can be determined.
- Also stability of C.L.S can be determined by applying N.S.C. in $1 \pm G(S)H(S)$.

  S plane contour $\rightarrow$ Entire R.H.P. : "NYQUIST CONTOUR "

Nyquist Contour : "Contour containing entire R.H.P."

5.2.2. "Types Of Nyquist Contour"



$$G(S)H(S) = \frac{N(S)}{D(S)}$$

$\downarrow$

Should not have any pole or zero on $j\omega$ axis

Nyquist Contour

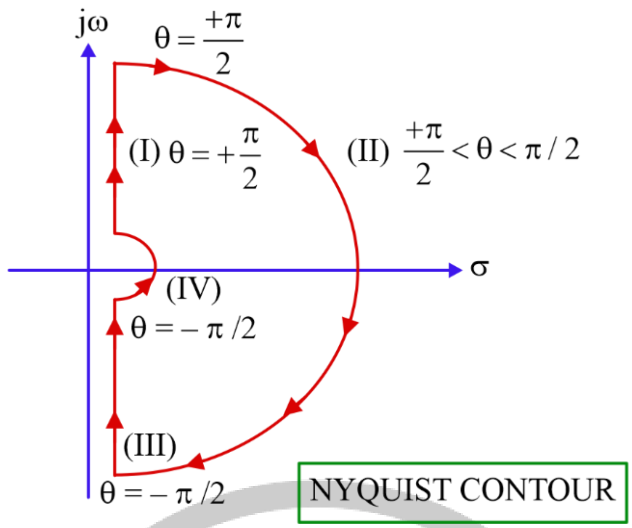
[I]: $S = j\omega \quad 0 < \omega < \infty$

[II]: $S = re^{j\theta} \quad r = \infty$

$$\frac{-\pi}{2} < \theta < \frac{-\pi}{2}$$

[III] — $S = j\omega \quad -\infty < \omega < 0$
— $S = -j\omega \quad 0 < \omega < \infty$

2. $G(S)H(S) = \dfrac{N(S)}{D(S)} \rightarrow$ It has poles or zeros at origin



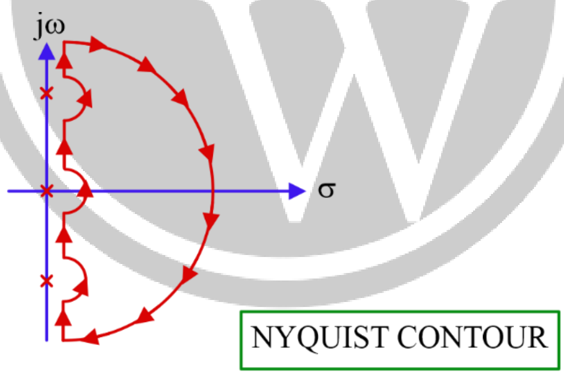
(I) $S = j\omega \qquad 0 < \omega < \infty$

(II) $s = re^{10} \qquad r = \infty$

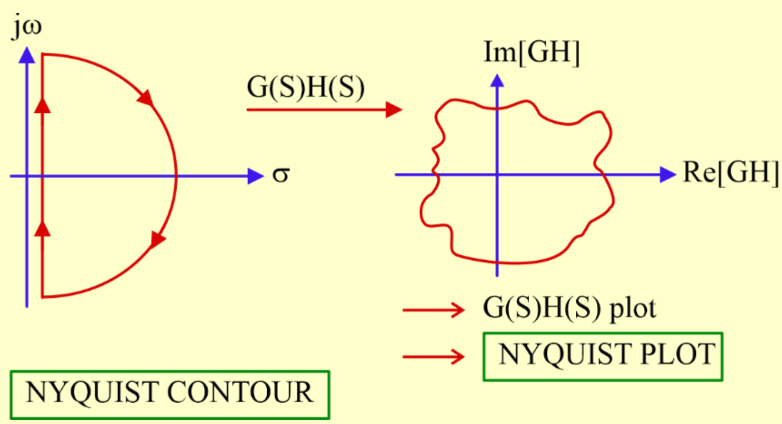
(III) $S = -j\omega \qquad o < \omega < \infty$

(IV) $S = re^{10} \qquad r \rightarrow 0 \qquad -\pi/2 < \theta < +\pi/2$

3. $G(S)H(S) = \dfrac{N(S)}{D(S)} \longrightarrow$ It has poles and zeros on $j\omega$ axis.



Note:



N.S.C. for various T.F. Let Nyquist contour is clockwise:

Case 1 : N.S.C. for T.F. G(S)H(S)

$$N = P_+ - Z_+$$

$N =$ number of encirclement of (0,0) by GH plot in GH plane

N → +ve if GH is ACW
N → –ve if GH is C.W.

P_+ = Number of poles of GH lying in RHP

Z_+ = Number of zeros of GH lying in RHP

For TF GH to be stable : $P_+ = 0$

Case 2 : N. S. C. for T.F. 1 + G(s)H(s)

$N = P_+ - Z_+$

N = Number of encirclement of (0, 0) by 1 + GH plot in 1 + GH plane

Or

No. of encirclement of (–1, 0) by GH plot in GH plane

P_+ = Number of poles of 1 + GH lying in R.H.P.

Or

Number of poles of GH lying in R.H.P

Z_+ = Number of zeros of 1 + GH lying in RHP

Or

Number of poles of $\frac{G}{1+GH}$ (C.L.S) lying in RHP

(i) For TF 1 + GH to be stable $\longrightarrow P_+ = 0$

(ii) For closed loop system $\left(\frac{G}{1+GH}\right)$ to be stable $\longrightarrow Z_+ = 0$

Case 3: N.S.C. for T.F. 1 – G(s)H(s)

$N = P_+ - Z_+$

N = Number of encirclement of (0,0) by 1 - GH plot in 1- GH plane

Or

Number of encirclement of (1,0) by GH plot in GH plane

P_+ = Number of poles of 1- GH lying in RHP

Or

Number of poles of GH lying in RHP

Z_+ = Number of zeros of 1 – GH lying in RHP

Or

Number of poles of $\frac{G}{1+GH}$ (C.L.S.) lying in R.H.P.

(i) For TF 1 - GH to be stable $\longrightarrow P_+ = 0$

(ii) For C.L.S. $\left(\frac{G}{1-GH}\right)$ to be stable $\longrightarrow Z_+ = 0$

Case 4 : N.S.C. for T.F. 4 + 3 G(s)H(s)

$N = P_+ - Z_+$

N = Number of encirclement of (0,0) by 4 + 3GH plot in 4 + 3 GH plane

Or

Number of encirclement of $\left(\frac{-4}{3}, 0\right)$ by GH plot in GH plane

P_+ = Number of poles of $4+3GH$ lying in RHP

Or

No. of poles of GH lying in RHP

Z_+ = Number of zeros of $4+3GH$ lying in RHP

Or

Number of poles of 0 system $\frac{G}{4+3GH}$ lying in R.H.P.

Case 5 : N.S.C. for T.F. $A+BG(s)H(s)$

$N = P_+ - Z_+$

N = Number of encirclement of (0,0) by $A+BGH$ plot in $A+BGH$ plane

Or

Number of encirclement of $\left(\frac{-A}{B}, 0\right)$ by GH plot in GH plane

P_+ = Number of poles of $A+BGH$ lying in RHP

Or

Number of poles of GH lying in RHP

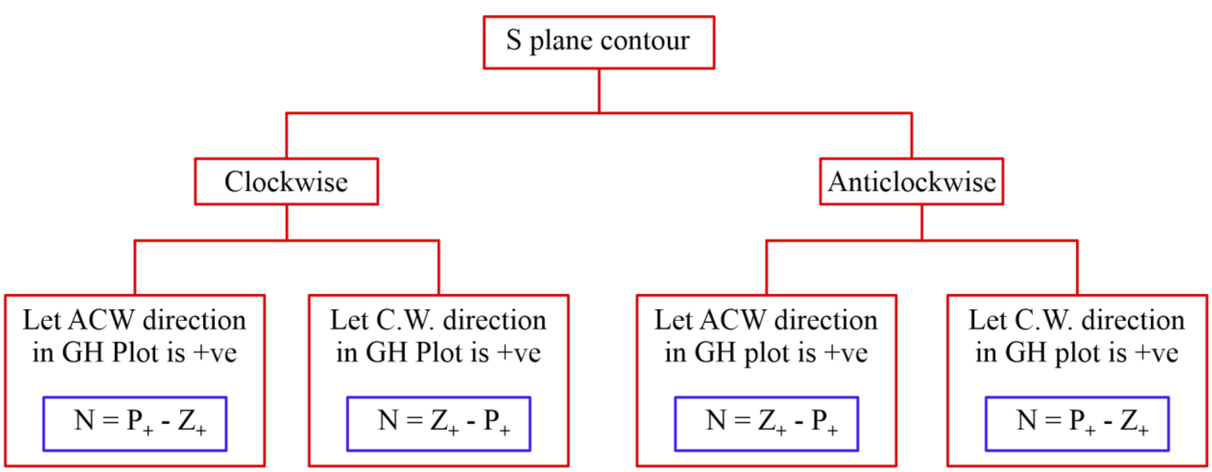
Z_+ = Number of zeros of $A+BGH$ lying in RHP

Or

Number of poles of 0 system $\frac{G}{A+BGH}$ lying in R.H.P.

5.3. Problem Solving Approach

1. Flow Chart



Plotting Nyquist Plot

Assumptions

(1) Nyquist contour is clockwise.

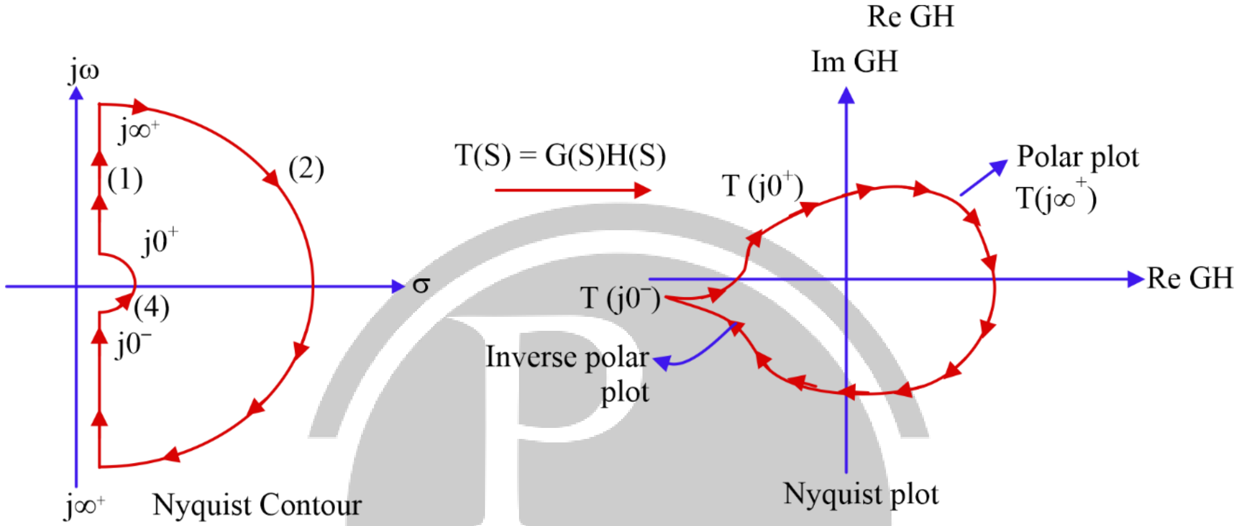
(2) Does not contain pole or zero on jω axis.

(3) Mapping on G(s) H(s) plane

Prerequisite

(1) Rang of $\omega \rightarrow -\infty < \omega < +\infty$

(2) Consider generalized Nyquist contour



Closing of Nyquist Plot

(i) $s = j\omega \rightarrow 0^+ < \omega < \infty^+ \longrightarrow T(s) : T(j0^+)$ to $T(j\infty^+)$

(ii) $s = \text{Re } j\theta \rightarrow \infty^+ < \omega < \infty^- \longrightarrow T(s) : T(j\infty^+)$ to $T(j\infty^-)$

(iii) $s = j\omega \rightarrow \infty^- < \omega < 0^- \longrightarrow T(s) : T(j\infty^-)$ to $T(j0^-)$

(iv) $s = \text{Re } j\theta \rightarrow 0^- < \omega < 0^+ \longrightarrow T(s) : T(j0^-)$ to $T(j0^+)$

Nyquist Contour Nyquist plat

(3)

(i) $s = j\omega \;\; 0^+ < \omega < \infty^+$

(ii) $s = \text{Re}^{j\theta} \;\; R \rightarrow \infty \;\; \theta \rightarrow +\frac{\pi}{2}$ to $\frac{-\pi}{2}$ (C W)

(iii) $s = j\omega \;\; \infty^+ < \omega < 0^-$

(iv) $s = \text{Re}^{j\theta} \;\; R \rightarrow 0 \;\; \theta \rightarrow -\frac{\pi}{2}$ to $\frac{\pi}{2}$ A.C.W

(4)

$\theta \rightarrow \frac{+\pi}{2}$ to $\frac{-\pi}{2}$:CW

$-\theta \rightarrow \frac{-\pi}{2}$ to $\frac{+\pi}{2}$:ACW

$$T(s)=\frac{1}{1+s}$$

(5) segment $-1 s = j\omega$

$$T(j\omega)=\frac{1}{1+j\omega}$$

Case 1: $T(j\omega)=\frac{1}{1+j\omega}$

$$|T(j\omega)|=\frac{1}{\sqrt{1+\omega^2}}, \angle T(j\omega)=-\tan^{-1}\omega$$

$\omega=0^+$ $\quad |T(j0^+)|=1$ $\quad \omega=\infty^+$ $\quad |T(j\omega)|=0$

$\angle T(j0^+)=0^0$ $\quad \angle T(j\omega)=-90^0$

Case 2: $T(j\omega)=\frac{1}{1+j\omega}$

$$\omega=0^+ \quad T(j0^+)=\frac{1}{1+j0^+}=1=1\angle 0^0$$

$$|T(j0^+)|=1, \ \angle T(j0^+)=0^0$$

$$\omega=\infty \quad T(j\infty^+)=\frac{1}{1+j\infty^+}=\frac{1}{j\infty^+}=0\angle -90^0$$

- Mapping of segment I on $G(s)H(s)$ plane is polar plot.
- Mapping of segment III on $G(s)H(s)$ Inverse polar plot

  Inverse polar plot = error image of polar plot w.r.t. horizontal axis keeping the same flow direction

Step to Draw Polar Plot

(1) Put $S=j\omega$ $T(j\omega)$ $0^+<\omega<\infty t$

(2) $T(j0^+)=M_1\angle\theta_1$

$T(j\infty^+)=M_2\angle\theta_2$

(3) Rationalise $T(j\omega)=\text{Re}\{T\}+j\,\text{Im}\{T\}$

Nyquist Plot

S-1 Draw pole – zero diagram on S plane and select proper contour.

S-2 Map segment I of Contour and draw polar plot .

S-3 Map segment II of contour and draw the respective mapping (generally circle).

S-4 Map segment III of contour and draw inverse polar plot .

S-5 Map segment IV of contour and draw respective mapping (generally circle) .

All Pass System/filter

Poles and zeros are at mirror image w.r.t. to $j\omega$ axis . $T(S)=\frac{(1-S)}{1+S}K$

- Nyquist plot of all pass filter is always circle, with radius K and center (0,0).

5.3.1. Closing of Nyquist Plot from Polar Plot

Case 1 If OLTF contains n no. of poles at origin .

- $G(\infty+)$ *and* $G(\infty-)$ will be connected by O^+ radius circle (short circle)
- $G(O^+)$ and $G(O^-) \to O.C$
- To close this $n\pi$ clockwise encirclement is performed from $G(O^-)$ to $G(O^+)$

Case 2 If Type of OLS = 0, and order of zero is greater then of pole .

- $G(0^-)$ and $G(0^+) \div$ short circuit
- $G(\infty^+)$ and $G(\infty^-) \div O.C$
- $(m-n)\pi$ clockwise encirclement from $G(\infty^+)$ *to* $G(\infty^-)$.

Case 3 Type of OLS = 0, order of zero $\leq$ order of pole

- $G(0^-)$ and $G(0^+) \div S.C$
- $G(\infty^+)$ and $G(\infty^-) \div S.C$

Gain Margin – Phase Margin

Minimum Phase System : All poles and zero must be on L.H.P

- Poles and zeros at origin or $j\omega$ axis are allowed

Non Minimum Phase System : Which are not minimum .

- All poles in L.H.P , few zeros in RHP $\to$ Type A
- All zero in LHP few poles are in RHP $\to$ Type B

Gain Margin

(1) Can determine stability

(2) Amount of gain K_1 that is needed to multiplied in OLTF such that corresponding C.L.S becomes stable .

(3) Amount of gain (in dB) that need to be added in OLTF such that corresponding C.L.S becomes marginally stable .

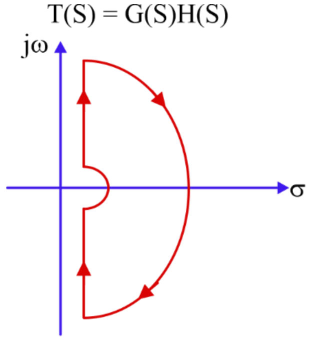
Phase Margin

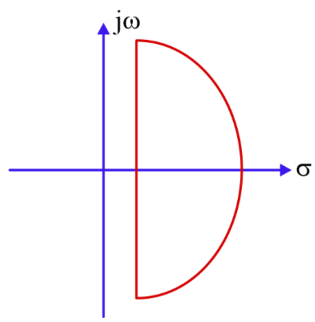
(1) Can determine stability

(2) Amount of phase angle that is needed to added in of G(S)H(S) such that C.L.S become Marginally stable

5.3.2. Mathematical calculation of G.M

T(S) = G(S)H(S)





S-2 Put $S = j\omega$ and determine range of ω

Contour 1: $0 < \omega < \infty$ $T(S)|_{S=j\omega} = T(j\omega)$

Contour 2: $0 \le \omega < \infty$

S-3 $\angle T(j\omega) = -180^0$

Solve and calculate ω, possible phase crossover frequency .

S-4 Validation of ω

(i) ω is real and +ve (i) n (ii) $\omega = \omega_{pc}$

(ii) $\angle T(j\omega) = -180^0$

S-5 At $\omega = \omega_{PC}$ $|T(j\omega)_{PC}| = M$

Gain Margin = $\frac{1}{M}$

Note: (1) If No valid ω_{PC} then G.M will be either $+\infty$ dB or $-\infty$ dB, depending on nature of OLS and absolute stability of CLS .

(2) $GM = +\infty$ dB or $-\infty$ dB represent absolute stable / unstable nature .

Method 2

S-1 Put $S = j\omega$ and find range of ω

S-2 $T(s = j\omega) = TR(j\omega) + jT_I(j\omega)$:Rationalize

S-3 $T_I(j\omega) = 0$ Possible ω_{pc}

S-4 validity $\omega_{pc} = (\text{Real and +ve}) \cap (TR(j\omega_{pc}) = -ve)$

S-5 $|T(j\omega_{PC})| = M$, $G.M = \frac{1}{M}$

Note : If ω_{pc} is invalid $\rightarrow$same procedure as Method 1.

Note: G.M cannot be 0 or ∞ in ratio

C.L.T.F	O.L.T.F	GM(dB)	PM in degree
Stable unstable	Min phase system	+ve (dB) -ve(dB)	+ve in degree -ve in degree
Stable unstable	Non Minimum Type –A	+ve(dB) -ve(dB)	+ve in dgree -ve in degree
Stable unstable	Non Minimum Type – B	-ve (dB) +ve(dB)	-ve in degree +ve in degree

Mathematical calculation of phase Margin–

Given T(S)

S-1 $S = j\omega$ and range of ω

S - 2 $|T(j\omega)| = 1$ passible : gain crossover frequency $(\omega_g c)$

S-3 Validity : $\omega \rightarrow$Real and +ve

S-4 P.M = $\angle T(j\omega_{gc}) + 180^0$

Note :

(1) Changes in ω_{pc}

(2) Change gain margin ← Introduction of Transportation log

↓

(1) ω_{gc} remain same

(2) PM changes

(3) PM ↓, so stability ↓

Shortcut for G.M

$T(S) = G(S)H(S)$

- Let K Multiplied in $G(S)H(S) \rightarrow K, G(S)H(S),$ So that roots of $1 + KG(S)H(S)$ represents Marginal Stability
- S-1 from Routh table

S-2

ODD Row		LAST Row		G.M
Invalid	→	Invalid	→	$+\infty / -\infty$
Invalid	→	valid	→	finite
Valid	→	Invalid	→	finite
Valid	→	valid	→	Absured case

S-3 **Valid Odd Rows :**

Odd row = 0 , K= +ve , A.E roots are non repeated on $j\omega$ axis then $K_1 = GM$ in ratio and roots of $AE \rightarrow \omega_{PC}$

S-4 **Valid last Row :**

Last Row = 0

$K = +ve \rightarrow 0 < K < \infty$

G.M and P.M from Nyquist

G.M (1) $\omega_{pc} = ?$ $\angle T(j\omega) = -180^0$ [Nyq plot must cross –ve real axis $\rightarrow \omega_{pc}$ exist]

(2) $G.M = \dfrac{1}{|T(j\omega_{PC})|} = \dfrac{1}{|\text{length on negative real axis tiee } \omega_{PC}|}$

Phase Margin –

(i) $\omega_{gc} \rightarrow |T(j\omega_{gc})| = 1$

Nyquist plot intersects unity radius circles then ω_{gc} exist .

(ii) $PM = \angle T(j\omega_{ge}) + 180^0$

$\angle T(j\omega_{gc}) \rightarrow$ from+ve real axis
- CW = - ve
- ACW = +ve

For OLS : Min phase

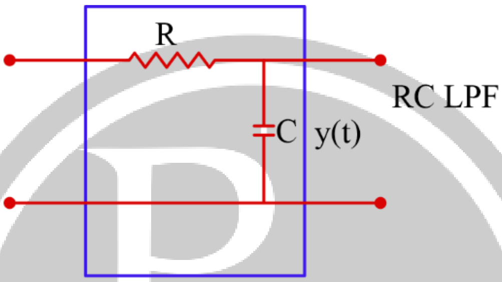
		G.M (in dB)	PM in degree	C.L.S
$K = K_1$	$\omega_{pc} > \omega_{gc}$	+ve (dB)	$+ve^0$	Stable
$K = K_2$	$\omega_{pc} > \omega_{gc}$	0 (dB)	0^0	M.S
$K = K_3$	$\omega_{pc} > \omega_{gc}$	- ve (dB)	$-ve^0$	Unstable

❑❑❑

6 STATE SPACE ANALYSIS

6.1. Introduction

Single I/P single output Input $\rightarrow u(t)$



$$H(s)=\frac{1/RC}{s+1/RC}, I.C=0$$

$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{RC}u(t)\rightarrow \text{ initial condition not zero}$$

$$y(t)=x_1(t)\rightarrow \text{ stable variable (A parameter across memory element)}$$

$$\frac{dy(t)}{dt}=\dot{x}_1(t)$$

$$\boxed{\begin{aligned}\dot{x}_1(t)&=\frac{-1}{RC}x_1(t)+\frac{1}{RC}u(t)\\ y(t)&=x_1(t)\end{aligned}}\rightarrow \text{State equation}$$

$\downarrow$

Output equation

$$\boxed{\begin{aligned}[\dot{x}_1(t)]_{|X|}&=\left[\frac{-1}{RC}\right]_{|X|}[x(t)]_{|X|}+\left[\frac{1}{RC}\right]_{|X|}u(t)\\ [y(t)]_{|X|}&=[1][x_1(t)]_{|X|}+[0]_{|X|}[u(t)]\end{aligned}}$$

State Model of above system

$$\left.\begin{aligned}[x(t)]&=[A][x(t)]+[B][u(t)]\\ [y(t)]&=[C][x(t)]+[D][u(t)]\end{aligned}\right]\text{Mathematical Representating of a physical system}$$

For MIMO System

State Equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}_{n\times 1} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix}_{n\times n} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} + \begin{bmatrix} \\ B \\ \\ \end{bmatrix}_{n\times l} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_l(t) \end{bmatrix}_{l\times 1}$$

Output Equation

$$\begin{bmatrix} y_1(t) \\ y_0(t) \\ \vdots \\ y_n(t) \end{bmatrix}_{m\times 1} = \begin{bmatrix} \\ C \\ \\ \end{bmatrix}_{n\times n} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} + \begin{bmatrix} \\ D \\ \\ \end{bmatrix}_{m\times l} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_l(t) \end{bmatrix}_{l\times 1}$$

$$[\dot{x}(t)]_{n\times 1} = [A]_{n\times n}[x(t)]_{n\times 1} + [B]_{n\times l}[u(t)]_{l\times 1}$$
$$[y(t)]_{m\times 1} = [C]_{m\times n}[x(t)]_{n\times 1} + [D]_{m\times l}[u(t)]_{l\times 1}$$

State Model Representation from DE

$$\frac{d^3y(t)}{dt^3} + \frac{3d^2y(t)}{dt^2} + \frac{6dy(t)}{dt} + 7y(t) = 6u(t)$$

$y \rightarrow$ output, $\mu \rightarrow$ input

Let $y(t) = x_1(t)$, $\quad dy(t) = x_2(t) = xi(t), \dfrac{d^2y(t)}{dt^2} = x_3(t) = \dot{x}_2(t)$

$$\frac{d^3y(t)}{dt^3} = \dot{x}_3(t)$$

Then solve question

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = [A]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [B]u(t) \qquad \text{and } y(t) = [C]\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + Du(t)$$

From Transfer function

Let T.F is T(S) $= \dfrac{b(c_3d^3 + c_2s^2 + c_1s + c_0)}{s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}$

Case 1 : Controllable canonical from

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}(u)$$

$$[y] = [C_0 \; C_1 \; C_2 \; C_3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + [0][u]$$

Note :

1. No of state variable = Highest order of D^r
2. Coefficient of highest order of D^r should be 1.

Case 2 : Observable canonical from

X = AX + BU

Y = CX + DU

$$[A]_{OCF} = [A]^T_{CCF}, [B]_{OCF} = [C]^T_{CCF}, [C]_{OCF} = [B]^T_{CCF}$$

Case 3 : Diagonal canonical form

$$T(S) = \frac{b_1}{(s+p_1)} + \frac{b_2}{(s+p_2)} + \frac{b_3}{(s+p_3)} + \frac{b_4}{(s+p_4)}$$

$$A = \begin{bmatrix} -p_1 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 \\ 0 & 0 & -p_3 & 0 \\ 0 & 0 & 0 & -p_4 \end{bmatrix} [B]\begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix}$$

$[C] = [m_1, m_2, m_3, m_4]$

$b_1 = K_1 m_1$

$b_2 = K_2 m_2$

$b_3 = K_3 m_3$

$b_4 = K_4 m_4$

Case 4 : Jordan Canonical form

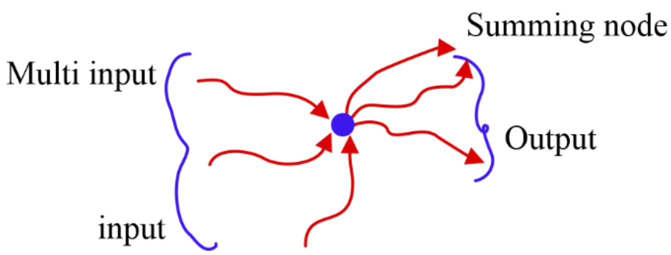
Extension of D.C.F when poles are repeated.

$$T(S) = \frac{b_1}{(s+p_1)} + \frac{b_2}{(s+p_1)^2} + \frac{b_3}{(s+p_1)^3} + \frac{b_4}{(s+p_2)^4}$$

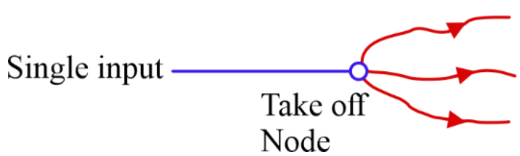
$$[A] = \begin{bmatrix} -p_1 & 0 & 0 & 0 \\ 0 & -p_1 & 0 & 0 \\ 0 & 0 & -p_1 & 0 \\ 0 & 0 & 0 & p_2 \end{bmatrix}$$ Jordan Block

From S.F.G

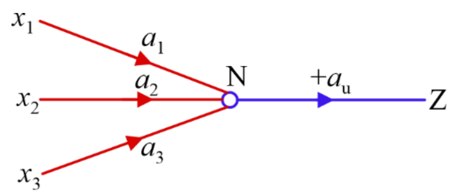
1. Summing Node



2. Take off Node

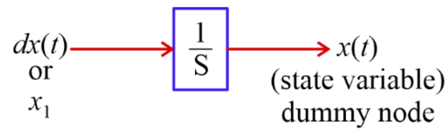


3. Potential of a node

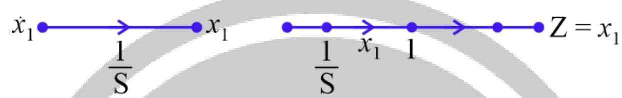


$$N = a_1x_1 + a_2x_2 + a_3x_3$$
$$Z = a_4N$$

4. Integrator Block



5. Integrator SFG



No of integrator = No of state variable

Important :

$$[X(S)] = \underbrace{[SI - A]^{-1}[X(0^-)]}_{\text{Solution of state variable due to non zero I.C}} + \underbrace{[SI - A]^{-1}[B][U(S)]}_{\text{Solution of state Variable due to input.}}$$

State Transition Matrix – (S.T.M)

(a) S.T.M in S-domain = $[SI - A]^{-1}$ $n \times n$

(b) S.T.M in time domain = $[\phi(t)]$ or $[e^{AT}]_{nxn}$

$$[\phi(t)]_{n\times n} \xleftrightarrow{\text{L.T}} [ST - A]^{-1}$$

6.1.2. Properties of STM

$$[e^{At}] = [\phi(t)]$$

(1) $[e^{Ao}] = [\phi(0)] = [I]_{nxn}$

(2) $\left[\left(\frac{de^{At}}{dt}\right)_{t=0}\right] = (A)_{nxn}$

(3) $[\phi(-t)] = \left[\phi^{-1}(t)\right]$

(4) $[\phi(t_1 + t_2)] = [\phi_1(t)\phi_2(t)]$

(5) $\left[\phi^K(t)\right] = [\phi(Kt)]$

(a) Homogeneous State equation $\left[\dot{X}(t)\right]=[A][x(t)]$

Sol. $[X(s)]=[SI-A]^{-1}\left[X(0^{-})\right]$

$$x(t)=\left[e^{AT}\right]\left[x(0^{-1})\right] \text{ or } [x(t)]=[\phi(t)][x(0^{-})]$$

$$\phi(t)=ILT\left\{[SI-A]^{-1}\right\}$$

(b) Non Homogeneous state equation $\left[\dot{X}(t)\right]=[A][x(t)]+[B][u(t)]$

$\dot{X}=AX+BU$

Sol. $X(S)=[SI-A]^{-1}\left[x(0^{-})\right]+[\phi(t)]*[B][u(t)]$

Solution of y(t)

$[y(t)]=[C][X(t)]+[D][u(t)]$

$[Y(S)]=[C][X(S)]+[D][U(S)]$

$$[Y(S)]=[C]\left\{[SI-A]^{-1}[x(0^{-})]+[SI-A]^{-1}[B][U(S)]\right\}+[D][U(S)]$$

$$[Y(S)]=\underbrace{[C][SI-A]^{-1}[x(0^{-})]}_{\text{Zero Input Response}}+\underbrace{\left\{[C][SI-A]^{-1}[B]+[D]\right\}[U(S)]}_{\text{Zero State Response}}$$

Total Response

For 2 Input 2 Output

$$\begin{bmatrix}Y_1\\Y_2\end{bmatrix} \text{ and } \begin{bmatrix}U_1\\U_2\end{bmatrix}$$

For 1 Input and 1 Output

$$\frac{Y(S)}{U(S)}=\left[[C][SI-A]^{-1}[B]+[D]\right]$$

$$\left[x(0^{-})\right]=0$$

$[SI-A]=$ Poles of the system $=$ eigen values of matrix A $=D(S)$

$$\boxed{\frac{Y(S)}{U(S)}=\frac{[C]adj[SI-A][B]+[D]|SI-A|}{|SI-A|}}$$

Controllability and observability

$$\dot{X}=AX+BU$$
$$Y_2=CX+DU$$

$A\rightarrow$ Square matrix

$\rightarrow|A|$

$\rightarrow$ Rank of matrix $A=\rho(A)$

$\rightarrow n\times n$

Method 1

Kalman Test

- **Controllability**

(1) $[Q_C] = [B : AB : A^2B : ----- A^{n-1}B]$ ↗ Square ↘ Rectangular

(2) Q_C : Rectangular, $\rho(Q_C) = \rho(A) \to$ Controllable

$\rho(Q_C) < \rho(A) \to$ Uncontrollable

- **Observability**

(1) $[Q_0] = \left[C^T : A^T C^T : (AT)^2 C^T .... \left(A^T\right)^{n-1} C^T\right]$

(2) Q_0 : Square

$|Q_0| = 0$ Non observable

$|Q_0| \neq 0$ observable

(3) Q_C : Rectangular, $\rho(Q_0) = \rho(A) \to$ Observable

$\rho(Q_C) < \rho(A) \to$ Not observable

Method 2

If A is diagonal Matrix with distinct diagonal

$$\dot{X} = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}_{n\times n} X + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ : & : \\ b_{n1} & b_{n2} \end{bmatrix} \longrightarrow \text{They should not be all zero (Controllable)}$$

$$Y = \begin{bmatrix} C_{11} & C_{12} & ---- & C_{1n} \\ C_{21} & C_{22} & ---- & C_{2n} \end{bmatrix} X + [\;]U$$

They should not be all zero (observable)

Method 3

Gilbert Test

Upper Triangular Matrix

- UTM having Jordan block
- Jordan block is used when E.V are repeated

$$\begin{bmatrix} d_1 & a_1 & a_2 \\ 0 & d_2 & a_3 \\ 0 & 0 & d_3 \end{bmatrix} \longrightarrow \text{They all Should not be zero}$$

Lower Triangular Matrix

$$\begin{bmatrix} d_1 & 0 & 0 \\ a_1 & d_2 & 0 \\ a_1 & a_3 & d_3 \end{bmatrix} \longrightarrow \text{Should not be all zero}$$

Method 4

- Controllable Canonical form

$$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix}=B$$

- O.C.F

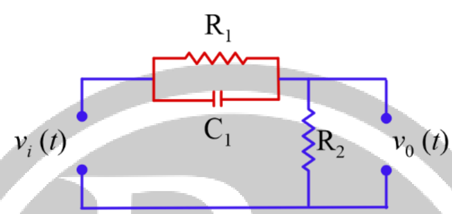
$$[A]=\begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix}[C]=[C, \quad 0, \quad 0]$$

❑❑❑

7 CONTROLLER AND COMPENSATOR

7.1. Introduction

(1) Phase lead compensator



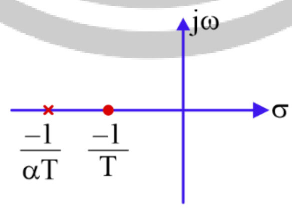
$$T = R_1C_1$$

$$T(S) = \frac{R_2(1+SR_1C_1)}{R_1+R_2\left\{1+\dfrac{SR_1C_1}{R_1+R_2}\right\}}$$

$$\boxed{T(S) = \frac{\alpha(1+ST)}{1+S\alpha T}} \qquad \frac{1}{\alpha} > 1 \qquad \boxed{\alpha = \frac{R_2}{R_1+R_2} < 1}$$

Pole, $S = -\frac{1}{\alpha T}$

Zero, $S = -\frac{1}{T}$



7.1.1. Zero Dominant Compensator

Phase $\phi = \tan^{-1}(\omega T) - \tan^{-1}(\alpha\omega T)$

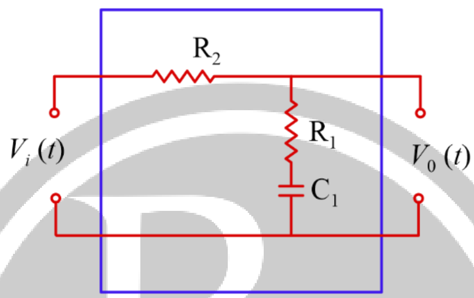
Max. value of phase, $\frac{d}{d\omega}\phi(\omega) = 0$

Max $\leftarrow$ $\boxed{\omega = \frac{1}{T\sqrt{\alpha}}}$ $\qquad$ $\boxed{\tan\phi_{max} = \frac{1-\alpha}{2\sqrt{\alpha}}}$

$$\boxed{\sin\phi_{max} = \frac{1-\alpha}{1+\alpha}}$$

- Behave as HPF
- Decrease gain of system
- Increase steady state error
- Increase $\omega_{gc}, BW \uparrow$
- Increase P.M, improve relative stability
- Increase ξ ↗ Mr decreases, ↘ % Mp decreases
- ω_n increases, t_s decreases
- Improve or reduces the transient region
- Increase the speed of system

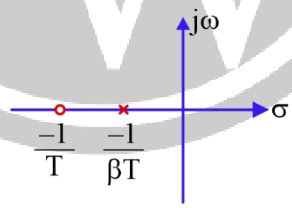
Phase log Compensator



$$T(S) = \frac{1 + SR_1C_1}{1 + SC_1(R_1 + R_2)}$$

$$T(S) = \left(\frac{1 + ST}{1 + \beta ST}\right)$$

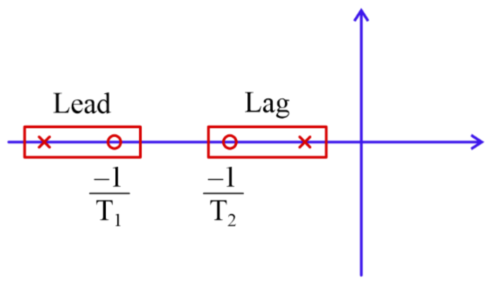
$$S = \frac{-1}{\beta T}, \frac{-1}{T}, \beta = \frac{R_1 + R_2}{R_1} > 1, T = R_1C_1$$



- Pole dominant
- $\phi = \tan^{-1}\omega T - \tan^{-1}\beta\omega T$
- $\omega_m = \frac{1}{T\sqrt{\beta}}$
- $\sin\phi_m = \left(\frac{1-\beta}{1+\beta}\right) \beta > 1$
- LPF
- Gain remains constant
- β → $= 1$ e_{ss} same; → ≥ 1 e_{ss} reduces
- Reduces $\omega_{gc} \rightarrow$ B.W reduced

- Reduces P.M $\rightarrow$ Relative stability decrease
- $\xi\downarrow\rightarrow M_p\uparrow$ and $M_r\uparrow$
- $\xi\downarrow,\omega_n\downarrow\rightarrow t_s\uparrow$
- Increases transient region, speed of operation decreases.

Lead – Log Compensator



$$T(S)=\frac{\alpha(1+ST_1)}{(1+\alpha ST_2)}\cdot\frac{(1+ST_1)}{(1+\beta ST_2)}$$

$$T_1=R_1C_2 \qquad T_2=R_1'C_1'$$

$$\alpha=\frac{R_2}{R_1+R_2} \qquad \beta=\frac{R_1'+R_2'}{R_1'}$$

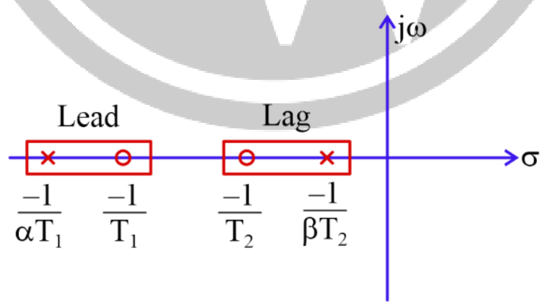
$\boxed{T_1>T_2}$

- ➢ B.P.F

LAG – LEAD Compensator

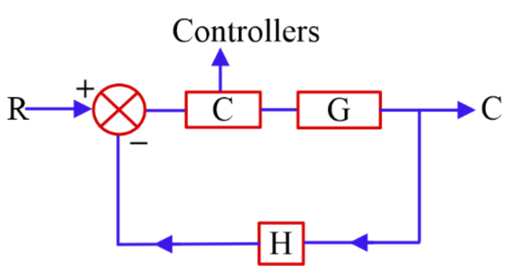
$$T(S)=\left\{\frac{(1+ST)}{(1+\beta ST)}\right\}\times\left\{\frac{\alpha(1+ST)}{(1+\alpha ST)}\right\} \qquad \frac{1}{T_1}>\frac{1}{T_2}$$

$\boxed{T_2>T_1}_{\log}$

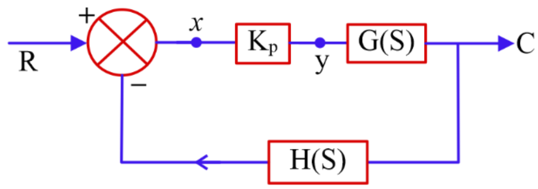


- Band Reject filter

Controllers



7.2. Proportional Controller



T.F of controller: $\frac{Y(S)}{X(S)} = K_P$

$H(S) = 1$

$$G(S) = \frac{\omega_n^2}{S(S+2\xi\omega_n)}$$

$$\omega_n' = \omega_n\sqrt{K_p}, \xi' = \xi/\sqrt{K_p}$$

$$e_{ss} = \frac{A}{K_p}\left(\frac{2\xi}{\omega_n}\right)$$

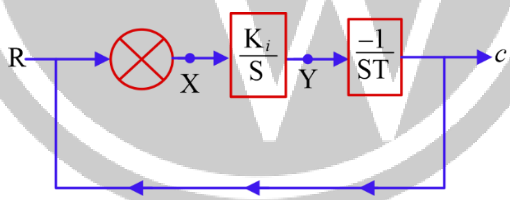
Effects

(1) e_{ss} reduces if $K_P > 1$

(2) $\xi\omega_n =$ constont, $t_s =$ constant, stability same

(3) $\xi\downarrow, \%M_P\uparrow, \omega_d\uparrow, t_r\downarrow$

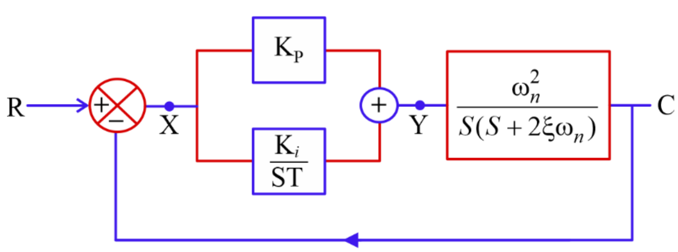
7.2.1. Integral Controller 1st order



$$\text{T.F} = \frac{Y(S)}{X(S)} = \frac{K_i}{S}$$

- Increases type of system by 1
- e_{ss} for same input becomes 0.
- It makes → ↗ 1^{st} order CLS to M.S ↘ 2^{nd} order CLS to unstable

Proportional Integral Controller



$$\frac{Y(S)}{X(S)} = K_P + \frac{K_i}{S}$$

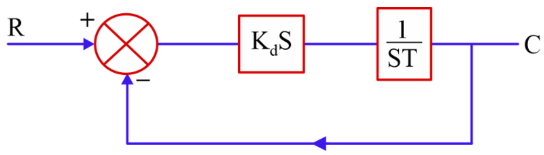
If $K_P = 1$ $$\frac{Y(S)}{X(S)} = 1 + \frac{K_i}{S}$$

$\rightarrow \xi\omega_n \downarrow \rightarrow t_s \uparrow \rightarrow$ No. of oscillations $\uparrow$

$\downarrow$

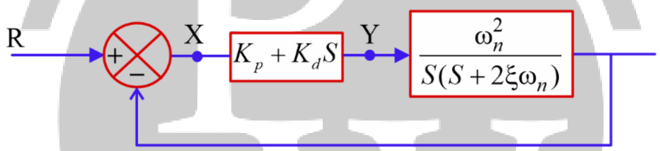
Sluggish $\leftarrow$ Transiend region becomes pronounced

Derivative Controller



- Derivative Controller reduces type of system by 1
- $e_{ss} \uparrow$ for same input
- Transient region reduced

Proportional Derivative Controller (P –D)



$$\frac{Y}{X} = K_P + K_dS \xrightarrow{K_P=1} 1 + K_dS$$

- It reduces $\%M_P, t_s, t_r$
- It improves: relative stability and transient region

PID Controller

Transfer function $= K_P + \frac{K_i}{S} + SK_D$

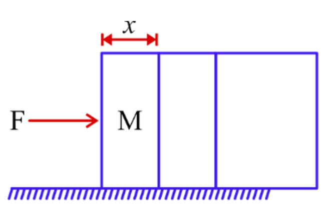
- Improves stability and decreases e_{ss}
- Increases type and decreases e_{ss}

Mathematical Modelling

Mechanical system $\longleftrightarrow$ Electrical system

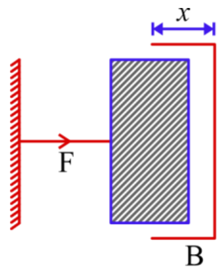
Translational System (Mass Damper System)

(1) Mass

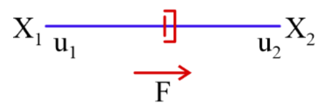


$$F_M = M\frac{d^2x}{dt^2}$$

(2) Damper

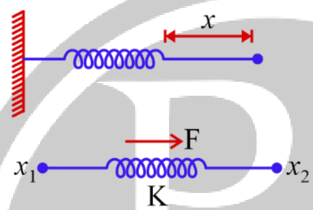


$$F_D = \frac{Bdx}{dt}$$



$$F = B\frac{d}{dt}(x_2 - x_1) = B(v_2 - v_1)$$

(3) Spring

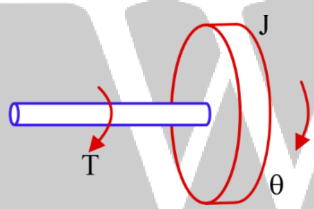


$$F = Kx$$

$$F = K(x_2 - x_1)$$

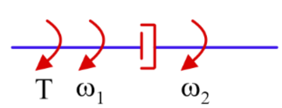
Rotational System

(1) Inertia



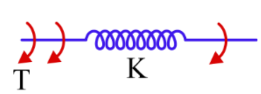
$$\boxed{T = J\frac{d^2\theta}{dt^2}}$$ J : Moment of Inertia

(2) Damper



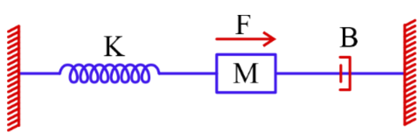
$$\boxed{T = B(\omega_2 - \omega_1) = B\left(\frac{d\theta_2}{dt} - \frac{d\theta_1}{dt}\right)}$$

(3) Spring

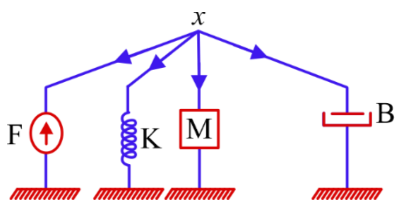


Force voltage – force Current

Given



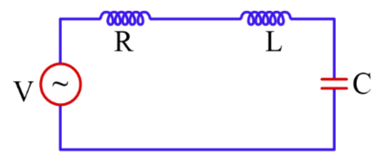
Network



$$F = Kx + M\frac{d^2x}{dt^2} + B\frac{dx}{dt}$$

Force voltage Analogy

$$I = \frac{dQ}{dt}$$

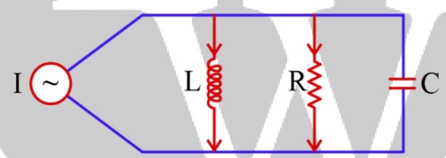


$$V = \frac{Ld^2Q}{dt^2} + \frac{RdQ}{dt} + \frac{1}{C}Q$$

$$F = \frac{Md^2x}{dt^2} + \frac{Bdx}{dt} + K_x$$

$$\begin{vmatrix} F \to V \\ M \to L \\ B \to R \end{vmatrix} \quad \begin{matrix} x \to Q \\ K \to \frac{1}{C} \end{matrix} \quad v \longleftrightarrow I$$

Force current Analogy



$$V = \frac{dQ}{dt}$$

$$\boxed{I = C\frac{d^2Q}{dt^2} + \frac{1}{R}\frac{dQ}{st} + \frac{1}{L}Q}$$

$F \to I$

$M \to C$

$B \to 1/R$ Voltage $\to$ Velocity

$K \to 1/L$

$x \to Q$