Engineering Mathematics

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1 BASIC CALCULUS

1.1 Introduction

1.1.1 Limits, Continuity and Differentiability

a. As x tends to a ($x \to a$) $\Rightarrow$ x is moving towards a

A value l is said to be limit of a function $f(x)$ at $x \to a$ if $f(x) \to l$ as $x \to a$.

It is mathematically defined as

$$\lim_{x\to a} f(x) = l = \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

A function f(x) is said to be continuous at x = a if

$$\lim_{x\to a} f(x) = l = f(a) = f(x)|_{x=a}$$

Note:

For $\lim_{x\to a} f(x)$ to exist, the function need not be continuous at $x = a$.

But for $f(x)$ to be continuous at x = a, $\lim_{x\to a} f(a)$ should exist.

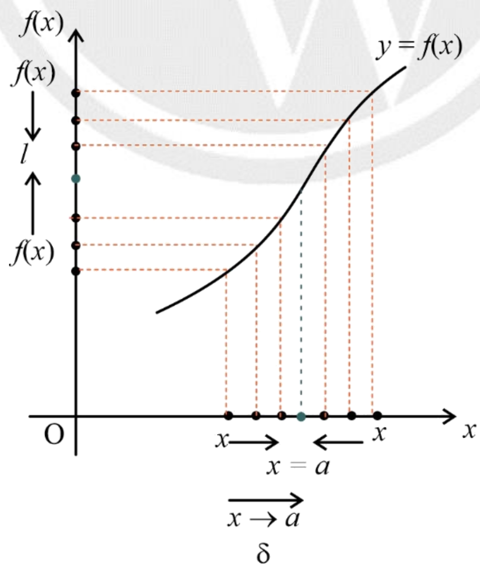

Fig. 1.1

b. Concept of differentiability

A continuous function $f(x)$ is said to be differentiable at $x = a$ if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists.

$$f'(x)|_{x=a} = f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$f'(a) = \tan\theta$ where θ is the angle made by the tangent to the curve at x=a with x – axis.

c. Some Standard Derivatives

(i) $\frac{d}{dx}(x^n) = n.x^{n-1}$

(ii) $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\frac{d}{dx}(\cos x) = -\sin x$

(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$

(v) $\frac{d}{dx}(\cot x) = -\cos e\, c^2 x$

(vi) $\frac{d}{dx}(\sec x) = \sec x . \tan x$

(vii) $\frac{d}{dx}(\cos e\, cx) = -\cos e\, cx \cot x$

(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; -1 < x < 1$

(ix) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

(x) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xi) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

(xiii) $\frac{d}{dx}(\cos e\, c^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

(xiv) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

(xv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(xvi) $\frac{d}{dx}(a^x) = a^x . \log_e a$

(xvii) $\frac{d}{dx}(e^x) = e^x$

(xviii) $\frac{d}{dx}(|x|) = \frac{|x|}{x} (x \neq 0)$

(xix) $\frac{d}{dx}(x^x) = x^x(1 + \log_e x)$

(xx) $\frac{d}{dx}(\sinh x) = \cos h\, x$

d. Product rule of differentiation

$$\frac{d}{dx}\big(f(x).g(x)\big) = f(x).g'(x) + f'(x).g(x)$$

$$d(uvw) = uvw' + uv'w + u'vw$$

e. Quotient rule of differentiation

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).f'(x) - f(x).g'(x)}{(g(x))^2}, (g(x) \neq 0)$$

f. Greatest Integer function / step function / integer part function

$f(x) = [x] = n, \forall\, n \leq x < n+1 \text{ where } n \in Z$

$\lim\limits_{x\to a}[x] = \nexists$ if a is an integer

L.H.L. = $\lim\limits_{x\to a^-}[x] = a - 1$

R.H.L. = $\lim\limits_{x\to a^+}[x] = a$

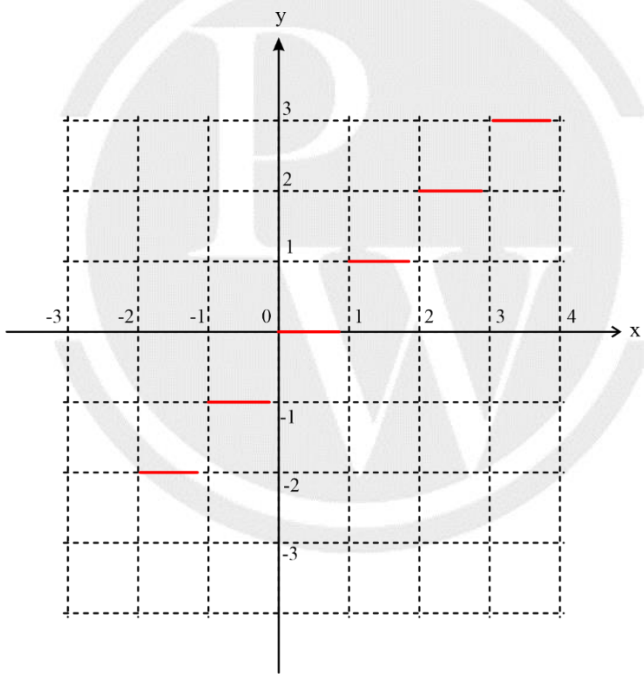

Fig.1. 2 Greatest Integer

g. Properties of Limits

(i) $\lim\limits_{x\to a}(f(x) \pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

(ii) $\lim\limits_{x\to a}\big(f(x).g(x)\big) = \lim\limits_{x\to a} f(x).\lim\limits_{x\to a} g(x)$

(iii) $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \left(\lim\limits_{x\to a} g(x) \neq 0\right)$

(iv) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x) = \nexists$, then $\lim_{x\to a} f(x).g(x)$ MAY exist

Ex: let $f(x) = \sin x$, $g(x) = \frac{1}{x}$, $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0}\frac{1}{x} = \nexists$

But $\lim_{x\to 0} \sin x . \frac{1}{x} = 1$

(v) If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \neq \left(\frac{0}{0}\right)$

If $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$ and so on

(vi) If $\lim_{x\to a}\big(f(x).g(x)\big) = 0 \times \infty \Rightarrow \lim_{x\to a}\frac{f(x)}{\left(\frac{1}{g(x)}\right)} = \frac{0}{0}$ (Apply L- Hospital Rule again)

h. Some Standard Limits

(i) $\lim_{x\to a}\frac{\sin x}{x} = 1$

(ii) $\lim_{x\to a}\frac{\tan x}{x} = 1$

(iii) $\lim_{x\to 0}\frac{1-\cos ax}{x^2} = \frac{a^2}{2}$

(iv) $\lim_{x\to \infty}\frac{\sin x}{x} = 0$

(v) $\lim_{x\to \infty}\frac{\cos x}{x} = 0$

(vi) $\lim_{x\to 0}(1+ax)^{b/x} = e^{ab}$

(vii) $\lim_{x\to \infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$

(viii) $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x} = \sqrt{ab}$

(ix) $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+....+n^x}{n}\right)^{1/x} = \sqrt[n]{n!}$

(x) $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a$; $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

(xi) $\lim_{x\to 0} x.\sin\left(\frac{1}{x}\right) = 0$

1.2 Mean Value Theorems

1.2.1 Lagrange's Mean Value Theorem (LMVT):

If $f(x)$ is continuous in [a, b] and it is differentiable in (a, b) then ∃ at least one point 'C' such that C ∈ (a, b) and

$$f'(C) = \frac{f(b)-f(a)}{b-a}$$

Here $f'(C)$ slope of tangent to $f(x)$ at $x = C$.

Tangent at x = c is parallel to the line connecting the points A and B

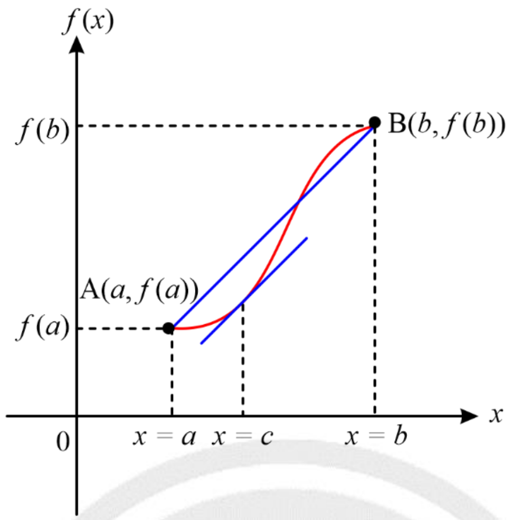

Fig.1.3 LMVT

1.2.2 Rolle's Mean Value Theorem

If $f(x)$ is continuous is $[a, b]$ and differentiable in (a, b) and $f(a) = f(b)$ then ∃ at least one-point C∈ (a, b) such that $f'(C) = 0$.

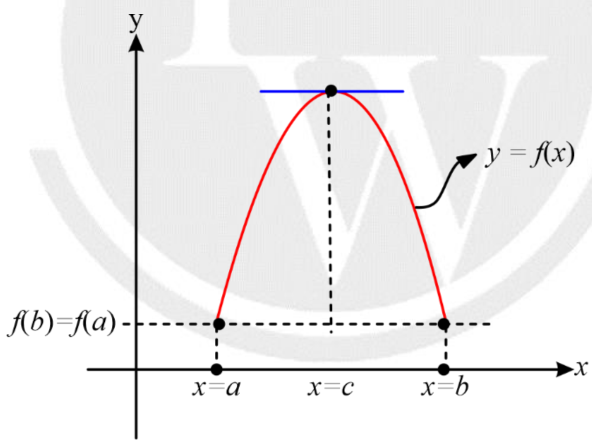

Fig.1.4 Rolle's mean value

1.2.3 Cauchy's Mean Value Theorem

If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b) then ∃ at least one value of 'C' such that C ∈ (a, b) and $\frac{g'(C)}{f'(C)} = \frac{g(b)-g(a)}{f(b)-f(a)}$

1.3 Increasing and Decreasing Functions

1.3.1 Increasing Functions

A function $f(x)$ is said to be increasing if $f(x_1) < f(x_2)\ \forall\ x_1 < x_2$

Or

A function $f(x)$ is said to be increasing if $f(x)$ increases as x increases.

For a function $f(x)$ to be increasing at the point x=a, $f'(a) > 0$.

Example:

e^x, $\log_e x$ → Monotonically increasing functions

sin x in (0, π/2) → non-monotonic functions

1.3.2 Decreasing Functions

A function $f(x)$ is said to be a decreasing function if $f(x_1) > f(x_2) \forall x_1 < x_2$

A function $f(x)$ is said to be decreasing function if $f(x)$ decreases as x increases.

Example: e^{-x} →Monotonically decreasing function $sin\, x$ in $\left(\frac{\pi}{2}, \pi\right)$

1.4 Concept of Maxima and Minima

Let $f(x)$ be a differentiable function, then to find the maximum (or) minimum of $f(x)$.

(1) Find stationary points from the equation $f'(x) = 0$. Let 'x_0' be the stationary point.

(2) Find the value of $f''(x_0)$

Case (i): If $f''(x_0) < 0$, then the function $f(x)$ has maximum at $x = x_0$ and the maximum value of the function is $f(x_0)$.

Case (ii): If $f''(x_0) > 0$, then the function $f(x)$ has minimum value at $x = x_0$ and the minimum value is $f(x_0)$.

Case (iii): If $f''(x_0) = 0$, then the we cannot comment on the existence of maximum (or) minimum of $f(x)$ at $x = x_0$.

Such points are called points of inflection (or) Critical points.

Example: $x = 0$ is a critical point of $f(x) = x^3$

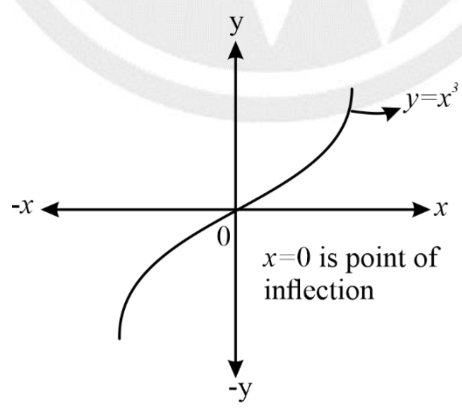

Fig. 1.5 Graph of x^3

$$f(x) = x^3$$

⇒ $$f'(x) = 3x^2 = 0 \quad \Rightarrow \quad x = 0$$

$$f''(x) = 6x \Rightarrow f''(0) = 6(0) = 0$$

1.5 Taylor Series

If $f(x)$ is continuously differentiable $(f'(x), f''(x), f'''(x), ...$ exists) then the Taylor series expansion of $f(x)$ about the point $x = a$ is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.+...\infty$$

If $a = 0$, then $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!} + ...\infty$

The coefficient of $(x - a)^n$ in the Taylor series expansion of $f(x)$ is $\frac{f^n(a)}{n!}$.

The general expansion of Taylor series is given by $f(x+h) = f(x) + h.\frac{f'(x)}{1!} + h^2.\frac{f''(x)}{2!} + h^3.\frac{f'''(x)}{3!} + ...\infty$

- Finding the expansion of e^x about $x = 0$

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1; f''(0) = f'''(0) = f''''(0) = ... = 1$$

$$f(x) = e^x = 1 + (x-0)\frac{1}{1!} + (x-0)^2.\frac{1}{2!} + (x-0)^3.\frac{1}{3!} + ...$$

$$\Rightarrow \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

1.6 Integral Calculus

If $F(x)$ is anti-derivative of $f(x)$ that is continuous and differentiable in (a, b), then we write $\int_{x=a}^{x=b} f(x)\,dx = F(b) - F(a)$. Here $f(x)$ is integrand

If $f(x) > 0\ \forall a \le x \le b, the\ \int_a^b f(x)\,dx$ represents the shaded area in the given figure.

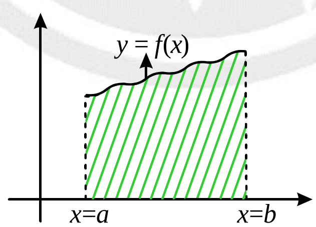

Fig.1.6 Integration of continuous function

1.6.1 Mean Value Theorem of Integration

If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then '∃' atleast one-point $C \in (a, b)$ such that

$$f'(C) = \frac{\int_a^b f(x)dx}{(b-a)}$$

-y

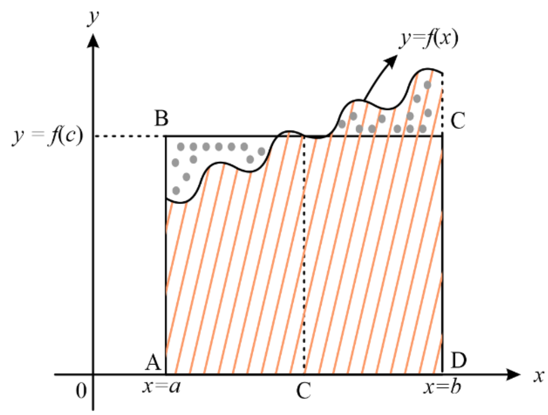

Fig.1.7 Mean value of integration

1.7 Newton-Leibnitz Rule

If $f(x)$ is continuously differentiable and $\phi(x)$, $\Psi(x)$ are two functions for which the 1st derivative exists, then

$$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(x)dx\right) = f\big(\psi(x)\big).\psi'(x) - f\big(\phi(x)\big).\phi'(x)$$

Ex. $\frac{d}{dx}\left(\int_{x}^{x^2} \sin x\, dx\right) = \sin(x^2).2x - \sin x.1 = 2x\sin(x^2) - \sin x$

1.8 Some Standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C(n \neq -1)$

2. $\int \frac{1}{x} dx = \log_e |x| + C$

3. $\int \sin x\, dx = -\cos x + C$

4. $\int \cos x\, dx = \sin x + C$

5. $\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$

6. $\int \tan x\, dx = -\int -\frac{\sin x}{\cos x} dx = -\log_e|\cos x| + C$

$\Rightarrow \int \tan x\, dx = \log_e|\sec x| + C$

7. $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \log_e|\sin x| + C = -\log_e|\cos e\, cx| + C$

8. $\int \sec x\, dx = \log_e|\sec x + \tan x| + C$

$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \log_e|\sec x + \tan x| + C$

9. $\int \cos e\, cxdx = \log_e|\cos e\, cx - \cot x| + C$

10. $\int a^x dx = \frac{a^x}{\log_e a} + C$

11. $\int \frac{1}{x.\log_e a} dx = \log_a x + C$

12. $\int x^x (1 + \log_e x)\, dx = x^x + C$

13. $\int f(x) . f'(x) dx = \frac{1}{2}(f(x))^2 + C$

14. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2.\sqrt{f(x)} + C$

15. If $f(x)$, $g(x)$ are two functions. that are differentiable, then

$\int f(x) . g(x) dx = f(x). \int g(x) - \int f'(x)\,(g(x)dx)dx + C$

Before integrating the product, the functions $f(x)$ and $g(x)$ are to be arranged according to the ILATE Principle.

Here, ILATE stands for INVERSE LOGARITHMIC ALGEBRAIC TRIGONOMETRIC EXPONENTIAL.

1.9 Properties of Definite Integrals

1. If $f(x)$ is differentiable in interval (a, b), then $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

2. If ∃ a point C ∈ (a, b) such that $f(x)$ is not differentiable, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

3. If $f(x)$ is continuously differentiable function,

$$\int_{-a}^a f(x)\, dx = 2 \times \int_0^a f(x)\, dx; \text{if } f(-x) = f(x)$$

⇒ $f(x)$" is even function")

$$0; \text{ if } f(-x) = -f(x) (\Rightarrow f(x) \text{ is odd function})$$

4. $\int_0^{2a} f(x)\, dx = 2 \times \int_0^a f(x)\, dx \text{ if } f(2a - x) = f(x)$

5. $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

6. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \left(\frac{b-a}{2}\right)$

Ex.

(i) $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} = \frac{\pi}{4}$

(ii) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}} dx = \int_0^{\pi/2} \frac{1}{1+\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)} dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx = \frac{\pi}{4}$

(iii) $\int_2^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}} = \left(\frac{3-2}{2}\right) = \frac{1}{2}$

(iv) $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx = \frac{\pi}{4}$

7. $\int_0^{\pi/2} \sin^m x\, dx = \int_0^{\pi/2} \cos^m x\, dx = \frac{(m-1)\times(m-3)\times(m-5)}{m\times(m-2)\times(m-4)} \times \ldots \left(\frac{1}{2}\right) \text{(or)} \frac{2}{3} \times K$

Where $K = \pi/2$ if m is even

$= 1$ if m is odd.

8. $\int_0^{\pi} \frac{dx}{a^2 \cos^2 x+b^2 \sin^2 x} = \frac{\pi}{ab}$

9. $\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x+b^2 \sin^2 x} = \frac{\pi}{2ab}$

1.10 Length of a Curve

If $y = f(x)$ is a differentiable function in (a, b), then the length L of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

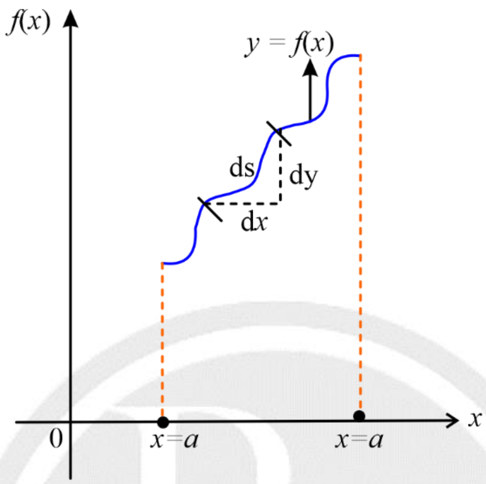

Fig.1.8 Length of the curve

1.11 Surface Area of Solid generated by revolving a curve about a fixed axis.

Elemental Surface Area

$$dA = 2\pi y \times ds = 2\pi y ds$$

$\Rightarrow$ Total surface area = A = $\int_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

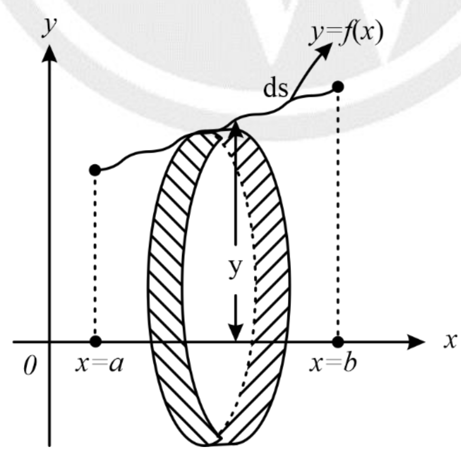

Fig.1.9 Surface area

1.12 Volume of the solid

The volume of the solid obtained by revolving the curve y = $f(x)$ between the lines $x = a$ and $x = b$ is given by

$$V = \int_{x=a}^{x=b} dV = \int_{x=a}^{x=b} \pi y^2 ds = \int_{x=a}^{x=b} \pi y^2 . \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$\Rightarrow$ $$V \approx \int_{x=a}^{x=b} \pi y^2 dx$$

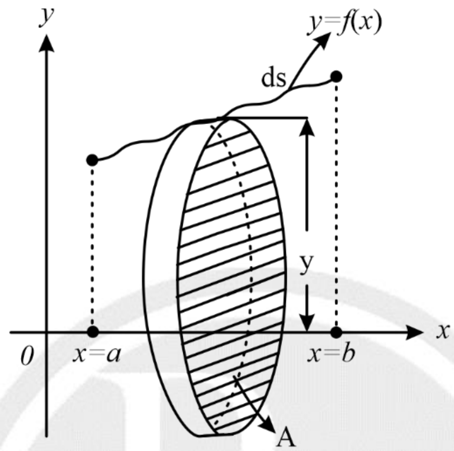

Fig.1.10 Volume of the solid

1.13 Gamma Function

The integral $\int_0^\infty e^{-x}.x^{n-1} dx (n > 0)$ is called Gamma function of n. It is denoted by $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$.

1.13.1 Properties of Gamma Function

(i) $\Gamma n = (n-1)!$

(ii) $\Gamma(n+1) = (n)!$

(iii) $\Gamma(n+1) = n\Gamma n$

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

1.14 Beta Function

The function β $(m, n) = \int_0^1 x^{m-1}.(1-x)^{n-1} dx$ $(m, n > 0)$ is called β function of m and n.

1.14.1 Properties of β function

(i) $\beta(m,n) = \frac{\Gamma m.\Gamma n}{\Gamma(m+n)}$

(ii) $\beta(m,n) = \beta(n,m)$

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$\beta(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

(iv) $sin^p\theta.cos^q\,\theta\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)(p,q > -1)$

1.15 Area under the curves

If the function f(x) > g(x) for all values of x between x=a and x=b then

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$\Rightarrow \quad A = \int_a^b \left(f(x) - g(x)\right) dx$$

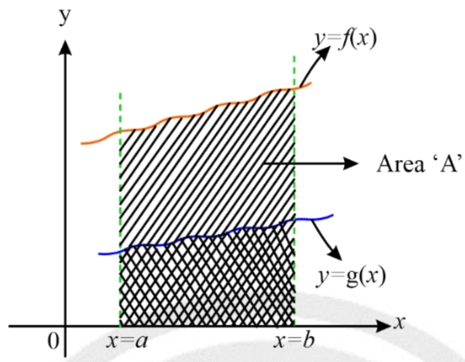

Fig.1.11 Area under curve

1.16 Multi Variable Calculus

a. Continuity of a function

A function *f*(x, y) is said to be continuous at (*a*, *b*) if $\lim_{\substack{x\to a \\ y\to b}} f(x,y) = f(a,b)$

b. Differentiation of a two-variable function

If *f*(*x*, *y*) is a continuous function, then the derivative of *f*(*x*, *y*) with respect to *x* treating *y* as constant is given by p $= \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

The derivative of *f*(*x*, *y*) with respect to *y* treating *x* as constant is given by $q = \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$

c. Homogenous Function

A function *f* (x, y) is said to be homogenous function of degree 'n' if $f(kx, ky) = k^n . f(x,y)$.

Ex. $f(x,y) = x^3 - 3x^2y + 3xy^2 + y^3$

$\Rightarrow \quad f(kx,ky) = (kx)^3 - 3(kx)^2(ky) + 3(kx).(ky)^2 + (ky)^3$

$= \quad k^3(x^3 - 3x^2y + 3xy^2 + y^3)$

$= \quad k^3.f(x,y) \Rightarrow f(x,y)$ is a homogenous function of degree '3'.

d. Euler's Theorem

If *f* (*x*, *y*) is a homogeneous function of degree '*n*' then

(i) $x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nf$

(ii) $x^2.\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

If $f(x, y) = g(x, y) + h(x, y) + \phi(x, y)$ where $g(x, y)$, $h(x, y)$ and $\phi(x, y)$ are homogenous functions of degrees m, n and p respectively, then

$$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = m.g(x,y) + n.h(x,y) + p.\phi(x,y)$$

$$x^2.\frac{\partial^2 t}{\partial x^2} + 2xy.\frac{\partial^2 f}{\partial x \partial y} + y^2.\frac{\partial^2 f}{\partial y^2} = m(m-1).g(x,y) + n(n-1).h(x,y) + p(p-1).\phi(x,y)$$

e. Concept of Maxima and Minima in Two Variables

If $f(x, y)$ is a two-variable differentiable function, then to find the maxima (or) minima.

Step-1: Calculate $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$ and equate $p = 0$, $q = 0$

Let (x_0, y_0) be a stationary point.

Step-2: Calculate r, s, t where $r = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)}$; $s = \left.\frac{\partial^2 f}{\partial x.\partial y}\right|_{(x_0,y_0)}$; $t = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)}$

Case (i): If $rt - s^2 > 0$ and $r > 0$, then the function $f(x, y)$ has minimum at (x_0, y_0) and the minimum value is $f(x_0, y_0)$.

Case (ii): If $rt - s^2 > 0$ and $r < 0$, then the function $f(x, y)$ has maximum at (x_0, y_0) and the maximum value is $f(x_0, y_0)$.

Case (iii): If $rt - s^2 \leq 0$; then we cannot comment on the existence of maxima and minima.

Such stationary points where $rt - s^2 \leq 0$ are called **saddle points**.

f. Concept of Constraint Maxima and Minima (Method of Lagrange's unidentified multipliers).

If $f(x, y, z)$ is a continuous and differentiable function, such that the variables x, y and z are related/constrained by the equation $\phi(x, y, z) = C$ then to calculate the extreme value of $f(x, y, z)$ using Lagrange's Method of unidentified multipliers.

Step-1: Form the function $F(x, y, z) = f(x, y, z) + \lambda\{\phi(x, y, z) - C\}$ where λ is a multiplier.

Step-2: Calculate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ and equate them to zero

Step-3: Equate the values of λ from the above 3 equations and obtain the relation between the variables x, y and z.

Step-4: Substitute the relation between x, y and z in $\phi(x, y, z) = C$ and get the values of x, y, z. Let they be (x_0, y_0, z_0).

Step-5: Calculate $f(x_0, y_0, z_0)$

The value $f(x_0, y_0, z_0)$ is the extreme value of $f(x, y, z)$.

g. Multiple Integrals

If $f(x, y)$ is continuous and differentiable at every point within a region 'R' bounded by some curves is given by

$$I = \iint_R f(x,y)\,dxdy$$

If there is a double integral,

$$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x, y) dydx \qquad \text{[Let } \psi(x) > \phi(x)\text{]}$$

Then I = area of the region bounded by the curves $y = \phi(x)$; $y = \Psi(x)$, $x = a$ and $x = b$ if $f(x, y) = 1$

Value of x – co-ordinate of centroid of the region bounded by $y = \phi(x)$; $y = \psi(x)$; $x = a$, $x = b$

if $f(x, y) = x$

h. Change of Orders of Integration

$$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(r)} f(x, y) dydx \rightarrow I = \int_{x=c}^{x=d} \int_{x=g(y)}^{x=h(y)} f(x, y) dxdy$$

In change of order of Integration, the order of the Integrating variables changes and the limits as well.

❑❑❑

2 DIFFERENTIAL EQUATIONS

2.1 Differential Equation

The equation involving differential coefficients is called a Differential Equation (DE).

1. $x^2.\frac{dy}{dx} + y^2 = 0$

2. $\frac{\partial^2 T}{\partial x^2} = k \cdot \frac{\partial T}{\partial t}$

3. $x^2.\frac{\partial^2 u}{\partial x^2} + y^2.\frac{\partial^2 u}{\partial y^2} = 0$

2.1.1 Ordinary Differential Equations (ODE)

The DEs involving only one independent variable is called ordinary differential equation.

Ex.

(1) $x^2 \frac{dy}{dx} + y^2 = 0;$

(2) $e^{-x} \cdot \frac{dy}{dx} + y^2 = e^x$

2.1.2 Partial Differential Equations

The DEs involving two (or) more independent variables are called Partial Differential Equations (PDEs).

Ex.

$$\frac{\partial^2 u}{\partial x^2} = C^2 \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{K} \cdot \frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2.1.3 Order and Degree of a Differential Equation

Order of a DE

The order of the highest derivative that occurs in a DE is called order of a DE.

Ex.

(1) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - y = 0$ → Order = 2

(2) $\frac{dy}{dx} + 2 \cdot \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} - 3x^2 = e^x$ → Order = 3

(3) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \cdot \frac{\partial^2 u}{\partial t^2}$ → Order = 2

(4) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t}$ → Order = 2

2.1.4 Degree of a Differential Equation

The Degree of the highest order derivative that occurs in a DE when the DE is free from fractional (or) radical powers.

Ex.

(1) The Degree of the DE $\left(\frac{d^2y}{dx^2}\right)^1 + 2\left(\frac{dy}{dx}\right)^3 - 3y = 0$ is 1.

(2) The Degree of the DE $\left(\frac{d^2y}{dx^2}\right)^1 + \sqrt{\left(\frac{dy}{dx}\right)^3 + 4y} = 0$ is 2

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^2 = \left(-\sqrt{\left(\frac{dy}{dx}\right)^3 + 4y}\right)^2$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3 + 4y$$

2.2 Formation of Differential Equations

If a solution $y = f(x)$ contains n arbitrary constants in it, then differentiate y for n times and calculate $y', y'', y''', \ldots . y^{(n)}$
So, from the $(n + 1)$ equations available,try to eliminate the arbitrary constants in $y = f(x)$

- The different equation formed for the solution $y = c_1 e^{K_1 x} + C_2 e^{K_2 x}$ where C_1, C_2 are arbitrary constants is

$$\frac{d^2y}{dx^2} - (K_1 + K_2)\frac{dy}{dx} + (K_1 . K_2)y$$

- If the solution is $y = C_1 e^{K_1 x} + C_2 e^{K_2 x} + C_3 e^{K_3 x}$ where C_1, C_2, C_3 are arbitrary constants, then the DE is $y''' - (K_1 + K_2 + K_3)y'' + (K_1K_2 + K_2K_3 + K_3K_1)y' - (K_1K_2K_3)y = 0$

2.2.1 First Order DE

The general form of a 1st order DE is given by $\frac{dy}{dx} = f(x, y)$

If $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x, y)$

- $N(x, y)dy + M(x, y)dx = 0$
- $Mdx + Ndy = 0$ where M, N are functions of x and y.

2.2.2 Linear ODE:

A DE is said to be linear if it does not contain the higher power terms of dependent variable $\left(y^2, y^3, y^4, \ldots, \left(\frac{dy}{dx}\right)^2, \left(\frac{dy}{dx}\right)^3, \ldots\right)$ and also the terms containing the product of dependent variable and its differential coefficient $\left(y.\frac{dy}{dx}, y^2\frac{dy}{dx}, y\left(\frac{dy}{dx}\right)^2, \ldots\right)$

Ex.

(1) $x^2.\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 6y = 0$

(2) $\frac{dy}{dx} = 5y = \sin x$

- $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + \sin y = 0$ → Non-linear DE

Here, $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} \ldots$

- $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + \sin x = 4y$ → Linear DE

2.3 Solving of Differential Equations

2.3.1 Solving of 1st Order DE

(i) Variable-separable form

If the 1st order DE is given by $\frac{dy}{dx} = \phi(x).\psi(y)$

$\Rightarrow \int \frac{1}{\Psi(y)} dy = \int \phi(x) dx$

On integrating we have solution of the given DE

(ii) Homogenous 1st Order

If the 1st order DE is of the form $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$

Such that both $M(x, y)$ and $N(x, y)$ are homogenous functions of same degree, then we say that the DE is homogeneous.

Ex.

(1) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

(2) $\frac{dy}{dx} = \frac{ax+by}{a'x+b'y}$

(a and b are not zero at the same time; a' and b' are not zero at the same time)

If the DE $\frac{dy}{dx} = \frac{M(r,y)}{N(x,y)}$ is a homogeneous DE, then the equation can be converted to Variable Separable form if we substitute $y = Vx$

2.3.2 Exact Differential Equations

The DE $Mdx + Ndy = 0$ where M, N are functions of x and y is said to be an Exact Differential Equation if there exist a function f(x, y) such that $Mdx + Ndy = d\big(f(x,y)\big)$

Mathematical condition to check the Exactness of a differential equation is

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

(i) Solution of an Exact DE

If $M(x, y)$ dx + N (x, y) dy = 0 is an Exact differential Equation, then the solution of the DE is given by $\int_{y=\text{const}} M(x,y)dx + \int(\text{terms not containing } x \text{ in } N)\, dy = C$

(ii) Integrating Factor

The function which, when multiplied to a non-exact DE converts the DE to exact DE.

Ex.

(1) $\frac{1}{y^2}$ is an integrating factor of $ydx - xdy = 0$

(2) $\frac{1}{y}$ is an integrating factor of $x^2dy - xydx = 0$

2.3.3 Methods of Writing the Integrating Factors (I.F.)

(i) If $M(x, y)dx + N(x, y)dy = 0$ is a homogeneous DE, then I.F. $= \frac{1}{Mx+Ny}$ $(Mx + Ny \neq 0)$

(ii)If $Mdx + Ndy = 0$ is of the form $yf(xy)dx + xg(xy)dy = 0$ then I.F. = $\frac{1}{Mx-Ny}$, $\left(M_x - N_y \neq 0\right)$

(iii)For a DE, $Mdx + Ndy = 0$, If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, then $e^{\int f(x)dx}$ is the integrating factor.

(iv)For the DE, $Mdx + Ndy = 0$, if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ then $e^{\int g(y)dy}$ is the integrating factor.

2.3.4 Leibnitz Linear Equation

The DE of the form $\frac{dy}{dx} + Py = Q$where P, Q are functions of x alone, is called Leibnitz Linear Equation

Integrating factor of the equation is $e^{\int Pdx}$

Solution of the Differential Equation is : $y.e^{\int Pdx} = \int Q.(e^{\int Pdx})\, dx$+ C where C is arbitrary constant.

2.3.5. Non-linear Equations Convertible to Leibnitz Linear Form

Bernoulli's Equation

Case-I

$$\frac{dy}{dx} + Py = Q.y^n (n > 1, n \neq 1)$$

(P, Q are functions of x alone)

$\Rightarrow \quad \frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^n}P_{y'} = \frac{Qy^n}{y^n}$

$\Rightarrow \quad y^{-n} \cdot \frac{dy}{dx} + y^{1-n} \cdot P = Q$

Let $\quad y^{1-n} = z$

$\Rightarrow \quad (1-n)y^{-n}\frac{dy}{dz} = \frac{dz}{dx}$

$\Rightarrow \quad y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dz}{dx}$

$\therefore \quad \frac{1}{(1-n)}\frac{dz}{dx} + zP = Q$

$\Rightarrow \quad \frac{dz}{dx} + (1-n)Pz = (1-n)Q$ [Leibnitz Linear Equation]

Case-II

$f'(y)\frac{dy}{dx} + Pf(y) = Q$

where P, Q are functions of x alone.

Let $f(y) = z$

$\Rightarrow \quad f'(y)\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore \quad \frac{dz}{dx} + Pz = Q$ [Leibnitz Linear Equation]

2.3.6 Applications of 1st order DE

Newton's Law of Cooling

The rate of change of temperature of a body placed in an ambience of temperature T_∞ is directional proportional to the temperature difference between the body and the ambient.

$\frac{dT}{dt} \propto -(T - T_\infty)$ where $T_\infty \to$ Ambient Temperature ($T > T_\infty$)

$\frac{dT}{dt} \propto (T_\infty - T)$

$\frac{dT}{dt} = -K(T - T_\infty)$

Radioactive Growth / Decay

The rate of growth/decay on any radioactive substance at any instant is directly proportional to concentration of the substance that is available at that instant.

- $\frac{dN}{dt} \propto N$ $\rightarrow$ For growth
  $\Rightarrow \frac{dN}{dt} = KN$
  $\Rightarrow \int \frac{1}{N} dN = \int K dt$
- $\frac{dN}{dt} \propto -N$ $\rightarrow$ For decay
  $\Rightarrow \frac{dN}{dt} = -KN$
  $\Rightarrow log_e N = Kt + C$
  $\Rightarrow N = e^{Kt} C$

2.4 Higher Order Differential Equations

The general form of Higher order Differential Equations is given by

$$K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + K_3 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + K_n y = X \qquad \ldots(1)$$

If $K_1, K_2, K_3, K_4, \ldots K_n, X$ are functions of x alone then (1) is called Linear Higher Order Linear DE with variable coefficients.

If $K_1, K_2, K_3, K_4, \ldots K_n$ are constants and X is a function of 'x' alone, then (1) is called Higher Order Linear DE with constant coefficients.

2.5 Higher Order Linear Differential Equations with Constant Coefficients

The DE $K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + K_3 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + K_n y = X$ …. (1) is said to be a higher order linear DE with constant coefficients if $K_1, K_2, K_3, K_4, \ldots K_n$ are constants and 'X' is a function of x alone.

If $X = 0$, then (1) is called Homogeneous DE

If $X \neq 0$, then (1) is called Non-Homogeneous DE.

2.5.1 Solution of Higher order Linear Differential Equation

$$Y = y_c + y_p$$

$y_c \rightarrow$ Complimentary function; $y_p \rightarrow$ Particular Integral

(Solution of homogeneous part; ($X = 0$); (Solution of Non-Homogeneous Part; ($X \neq 0$))

If $K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + K_{n-1} \frac{dy}{dx} + K_n y = X$ …. (1) is a linear DE with constant coefficients.

2.5.2 Rules for Writing the Complete Solution of $(f(D))y = X$:

(i) Form the auxiliary equation of $(f(D))y = X$ i.e. $f(M) = 0$

(ii) Depending on the roots of the auxiliary equation $(f(M) = 0)$, we write the complimentary function.

(iii) Calculate the Particular Integral $y_P = \frac{1}{(f(D))}X$.

(iv) Write the total solution of the equation $y = y_C + y_P$.

2.5.3 Rules for Writing the Complementary Function

(i) If the roots of $f(M) = 0$ are $M_1, M_2, M_3, \ldots.$ $(M_1, M_2, M_3, \ldots \in$ Rational)

Then $y_C = C_1e^{M_1x} + C_2e^{M_2x} + C_3e^{M_3x} + \ldots..$ where C_1, C_2, C_3,…. Are arbitrary constants)

(ii) If the roots of $f(M) = 0$ are $M_1, M_1, M_3 \ldots\ldots (M_1, M_3 \ldots\ldots \in$ Rational)

Then $y_C = (C_1x + C_2)e^{M_1x} + C_3.e^{M_3x} + \ldots\ldots$ Where C_1, C_2, C_3, …. are arbitrary constants).

(iii) If the roots of $f(M) = 0$ are $M_1, M_1, M_1, M_4 \ldots.$ (Where $\in M_1, M_4 \ldots.. \in$ Rational)

Then $y_C = (C_1x^2 + C_2x + C_3)e^{M_1x} + C_4e^{M_4x} + \ldots\ldots$ (where C_1, C_2, C_3 …. Are arbitrary constants)

(iv) If the roots of $f(M) = 0$ are $\alpha + i\beta, \alpha - i\beta, M_3, M_4 \ldots..$ then $y_C = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3e^{M_3x} + C_4e^{M_4x} + \ldots..$

(v) If the roots of $f(M) = 0$ are $\alpha + i\beta, \alpha - i\beta, \alpha + i\beta, \alpha - i\beta, M_5, M_6, \ldots.$ then

$y_c = e^{\alpha x}((C_1x + C_2)\cos\beta x + (C_3x + C_4)\sin\beta x) + C_5e^{M_5x} + C_6e^{M_6x} + \ldots..$

(vi) If the roots of $f(M) = 0$ are $\alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}$, M_3, M_4, …. then $y_C = e^{\alpha x}\{C_1 \sinh\sqrt{\beta}\,x + C_2 \cosh\sqrt{\beta}\,x\} + C_3e^{M_3x} + C_4e^{M_4x} + \ldots..$

(vii) If the roots of $f(M) = 0$ are $\alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, \alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, M_5, M_6, \ldots.$ then $y_C = e^{\alpha x}\{(C_1x + C_2)\sinh\sqrt{\beta}\,x + (C_3x + C_4)\cosh\sqrt{\beta}\,x\} + C_5e^{M_5x} + C_6.e^{M_6x} + \ldots.$

2.5.4 Rules for writing the particular Integral

(i) If $X = e^{ax}$,

$y_P = \frac{1}{f(0)}e^{ax} = \frac{1}{f(a)}e^{aX}$ (iff $(a) \neq 0$)

If $f(a) = 0$, then $y_P = x\frac{1}{f'(a)}e^{ax}$ (if $f'(a) \neq 0$)

If $f'(a) = 0$, then $y_p = x^2.\frac{1}{f''(a)}e^{ax}$ (if $f''(a) \neq 0$) and so on.

Solve $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + 6y = e^{2x}$

Sol. Aux. Eq^n $\rightarrow M^2 - 5M + 6 = 0 \Rightarrow M = 2,3$

$y_C = C_1e^{2x} + C_2e^{3x}$

$y_P = \frac{1}{D^2-5D+6}e^{2x}$ $since\ f(2) = 0$

$\Rightarrow y_P = x\frac{1}{(2D-5)}e^{2x} = x.\frac{1}{(2(2)-5)}e^{2x}$

$\frac{x}{-1}e^{2x} = -x.e^{2x}$

(ii) If $X = sin(ax+b)$ (or) $\cos(ax+b)$

$y_P = \frac{1}{f(D)} sin(ax+b)$

Replace D^2 by $-a^2$ in $f(D)$

If the denominator is the form $CD + d$ then rationalize the denominator and replace D^2 by $-a^2$

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = sin(2x+3)$

$y_p = \frac{1}{D^2-5D+6} \cdot sin(2x+3)$

$a = 2 \Rightarrow -a^2 = -4$

$\Rightarrow \quad y_p = \frac{1}{-4-5D+6} sin(2x+3)$

$\Rightarrow \quad y_p = \frac{1}{2-5D} \times \frac{2+5D}{2+5D} . sin(2x+3)$

$\Rightarrow \quad y_p = \frac{2+5D}{4-25D^2} sin(2x+3) = \frac{2+5D}{4-25(-4)} \sin(2x+3) = \frac{1}{104}(2.\sin(2x+3) + 10.\cos(2x+3))$

(iii) If $X = x^m$

$y_P = \frac{1}{f(D)} x^m$

$\Rightarrow \quad y_P = [f(D)]^{-1} x^m$

Calculate y_P for the DE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

$y_P = \frac{1}{D^2-5D+6} x^2$

$= \frac{1}{6\left(1+\left(\frac{D^2-5D}{6}\right)\right)} x^2$

$= \frac{1}{6}\left(1 + \left(\frac{D^2-5D}{6}\right)\right)^{-1} x^2$

$= \frac{1}{6}.\left\{1 - \left(\frac{D^2-5D}{6}\right) + \left(\frac{D^2-5D}{6}\right)^2 - \left(\frac{D^2-5D}{6}\right)^3 + \ldots\ldots\right\} x^2$

$= \frac{1}{6}\left\{x^2 - \frac{1}{6}(2 - 5(2x)) + \frac{1}{36}\{25(2)\}\right\}$

$= \frac{1}{6}x^2 - \frac{1}{18} + \frac{5x}{18} + \frac{25}{108}$

$= \frac{1}{6}x^2 + \frac{5x}{18} + \frac{19}{108}$

(iv) If $X = e^{ax}V$, then

$y_P = \frac{1}{f(D)} \cdot e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} V$

2.6 Method of Variation of Parameters

If the second order linear DE with constant coefficients is given by $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+qy=X$, and if $y_C=C_1y_1+C_2y_2$ then y_P (Particular integral of the DE) is given by

$$y_P=-y_1\int\frac{y_2X}{W}dx+y_2\int\frac{y_1X}{W}dx$$

Where W → Wronskian of the solution, $W=\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2\end{vmatrix}$

2.7 Euler Cauchy Equation (Higher order linear DE with Variable Coefficients)

The DE of the form $x^n\frac{d^ny}{dx^n}+K_1\cdot x^{n-1}\cdot\frac{d^{n-1}y}{dx^{n-1}}+k_2\cdot x^{n-2}\cdot\frac{d^{n-2}y}{dx^{n-2}}+\ldots..+K_{n-1}\cdot x\cdot\frac{dy}{dx}+K_ny=X$ Where $K_1,K_2,K_3,\ldots.K_n$ are constants is called Euler-Cauchy Equation

2.7.1 Procedure to solve Euler Cauchy Equations

Let $x^n\cdot\frac{d^ny}{dx^n}+K_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+K_2\cdot x^{n-2}\cdot\frac{d^{n-2}y}{dx^{n-2}}+\ldots..+K_{n-1}\cdot x\cdot\frac{dy}{dx}+K_ny=X$ ...(1)

$\left(x^n\cdot\frac{d^n}{dx^n}+K_1x^{n-1}\frac{d^{n-1}}{dx^{n-1}}+K_2\cdot x^{n-2}\cdot\frac{d^{n-2}}{dx^{n-2}}+\ldots..+K_{n-1}\cdot x\cdot\frac{d}{dx}+K_n\right)y=X$

Let $x=e^z \Rightarrow z=\log_e x$

$$x\frac{d}{dx}=\frac{d}{dz}=D$$

$$x^2\cdot\frac{d^2}{dx^2}=\frac{d}{dz}\left(\frac{d}{dz}-1\right)=D(D-1)$$

$x^3\cdot\frac{d^3}{dx^3}=D(D-1)(D-2)$ and so, on

$(1)\Rightarrow\{D(D-1)(D-2)\ldots\ldots.(D-(n-1)+K_1D(D-1)(D-2)\ldots..D-(n-2)+\ldots..K_{n-1}D+K_n)\}y=X$

Where $D=\frac{d}{dz}$

$\Rightarrow$ $(f(D))y=z\rightarrow$ Higher order linear DE with constant

$f(D)\rightarrow$ Polynomial in terms of D with constant coefficients.

2.8 Partial differential equation

2.8.1 The general form of a 2nd order Partial differential equation

The general form of a 2nd order Partial differential equation is given by

$$A.\frac{\partial^2u}{\partial x^2}+B\bullet\frac{\partial^2u}{\partial xdy}+\bullet C\frac{\partial^2u}{\partial y^2}+D\bullet\frac{\partial u}{\partial x}+E\bullet\frac{\partial u}{\partial y}+F\bullet u=G$$

For the nature of the above equation to be

(a) Elliptic $\rightarrow B^2 - 4AC < 0$

(b) Parabolic $\rightarrow B^2 - 4AC = 0$

(c) Hyperbolic $\rightarrow B^2 - 4AC > 0$

2.9 Heat Equation

The heat equation in 1 – D is of the form, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial u}{\partial t}$. Where 'c' is a constant.

Solution of heat equation is given by u (x, t) = (c_1 cos Px + c_2. s in Px). $e^{-c^2 pt^2}$

2. 10 Laplace equation

The Laplace equation is 2-D is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

2.10.1 Possible solution of Laplace equation

Possible solution of Laplace equation is given by

$$u(x, y) = \left(c_1 \cdot e^{Px} + c^2 \cdot e^{-px}\right).(c_3 \cos Py + c_u \cdot \sin Py)$$

$$u(x, y) = \left(c_1 \cos Px + c_2 \sin px\right).(c_3.e^{py} + cy.e^{-Py})$$

$$u(x, y) = \left(c_1 x + c_2\right).(c_3.y + c_u)$$

Where c_1, c_2, c_3, c_4 are arbitrary constants and the solution is picked depending on boundary conditions.

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3 VECTOR CALCULUS

3.1 Vector Product / Cross Product

If $\vec{a}$ and $\vec{b}$ are two vectors, then the cross product of the two vectors is denoted by $\vec{a} \times \vec{b}$ and it is given by $\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \sin\theta \cdot \hat{n}$

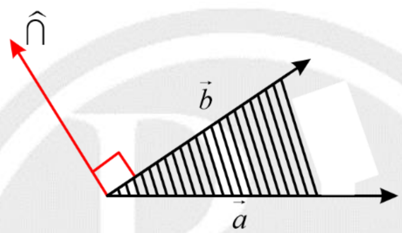

Fig. 3.1 Cross product

$\hat{n} \rightarrow$ unit vector passing through the point of intersection of $\vec{a}$ and $\vec{b}$ and lying perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

3.2 Dot / Scalar Product

If $\vec{a}$ and $\vec{b}$ are two vectors, then the dot /scalar product of the two vectors is denoted by $\vec{a}.\vec{b}$ and it is given by $\vec{a}.\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$ where $\theta$is the angle between the vectors $\vec{a}$ and $\vec{b}$.

Note:

$$|(\vec{a} \cdot \vec{b})|^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 \cdot |\vec{b}|^2 . \cos^2\theta + |\vec{a}|^2 \cdot |\vec{b}|^2 \cdot \sin^2\theta = |\vec{a}|^2 \cdot |\vec{b}|^2$$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\vec{b}\vec{c}] \Rightarrow [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

3.3 Differentiation of Vector Point functions

If $\vec{R}(t)$ is a vector point function, then the derivative of $\vec{R}(t)$ is given by

$$\frac{d\vec{R}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t}$$

If $\vec{R}(t) = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath}$ then $\frac{d\vec{R}(t)}{dt} = f'(t)\hat{\imath} + g'(t)\hat{\jmath}$

Ex.

If $\vec{R}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} \Rightarrow \frac{d\vec{R}(t)}{dt} = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$

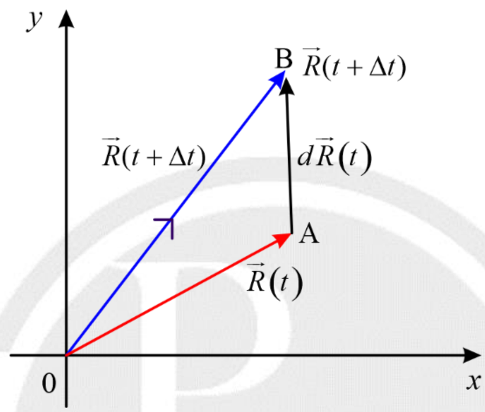

Fig. 3.2

3.3.1 Differentiation of Product of two vectors

$$\frac{d}{dt}\left(\vec{a}(t) \cdot \vec{b}(t)\right) = \vec{a}(t) \cdot \frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt} . \vec{b}(t)$$

$$\frac{d}{dt}\left(\vec{a}(t) \times \vec{b}(t)\right) = \vec{a}(t) \times \frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt} \times \vec{b}(t)$$

If $\vec{F}(t)$ is a vector point function with constant magnitude, then $\vec{F}(t) \cdot \frac{d}{dt}\vec{F}(t) = 0$.

If $\vec{F}(t)$ is a vector point function with constant direction, then $\vec{F}(t) \times \frac{d}{dt}\vec{F}(t) = \vec{0}$.

3.4 Del operator

The Vector operator $\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$ is called the differential operator in vector and it denoted as Del (or) ∇

$$\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}.$$

3.4.1 Gradient of a Scalar Point Function

If $\phi(x, y, z)$ is a Scalar Point function, then the gradient (change) of $\phi(x, y, z)$ is denoted by grad $\phi (or) \nabla\phi$ and it is given by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\vec{k}\right)\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k}.$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k}.$$

Note:

If $\vec{F}(x,y,z)$ is irrotational vector field ($\nabla \times \vec{F} = \vec{0}$), then definitely there exists a scalar point function $\emptyset(x,y,z)$ such that $\vec{F}(x,y,z) = grad\ \emptyset$.

If $\phi(x,y,z) = c$ is a level surface then $\nabla\phi|_{P(x_0,y_0,z_0)}$ gives the gradient of $\phi(x,y,z)$ at Point ' P '.

$$|\nabla\phi|_P| = \sqrt{\left(\frac{\partial\phi}{\partial x}\Big|_P\right)^2 + \left(\frac{\partial\phi}{\partial y}\Big|_P\right)^2 + \left(\frac{\partial\phi}{\partial z}\Big|_P\right)^2}.$$

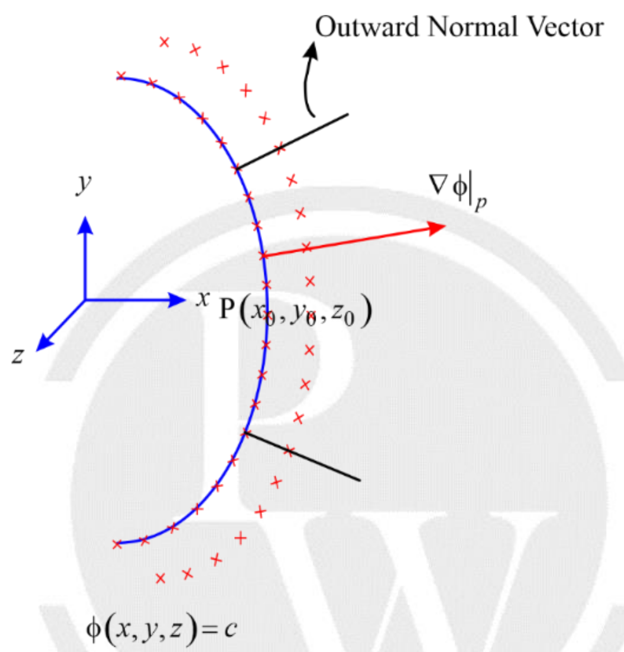

Fig. 3.3

→ $\nabla\phi|_p$ gives the change of $\phi(x,y,z)$ in the direction Normal to the surface $\phi(x,y,z) = c$ at P(x, y, z).

3.5 Directional Derivative

If $\phi(x,y,z) = c$ is a level surface, then the derivative of $\phi(x,y,z)$ at Point ' P ' in the direction of $\vec{a}$ is called Directional Derivative of $\phi(x,y,z)$ in the direction of $\vec{a}$.
It is given by

Direction Derivative = $\nabla\phi|_p \cdot \hat{a}$

$$= \nabla\phi\Big|_p \cdot \frac{\vec{a}}{|\vec{a}|} = \left|\nabla\phi|_p\right| \cdot |\vec{a}| \cdot \frac{\cos\theta}{|\vec{a}|} = \left|\nabla\phi|_p \cdot \cos\theta\right|$$

Directional Derivative of $\phi(x,y,z)$ at P in the direction of $\vec{a}$ is

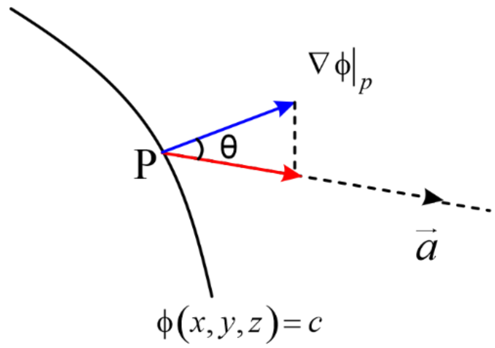

Fig.3.4

Directional derivative

$DD = |\nabla\phi|_p| \cdot \cos\theta$ where ' θ ' is angle between $\nabla\phi\ |_p$ and $\vec{a}$.

For Directional derivative to be maximum $\cos\theta = 1 \Rightarrow \theta = 0°$

⇒ The change of $\phi(x,y,z)$ at Point ' P ' is Maximum in the direction of Normal to $\phi(x,y,z)$

Maximum Change of $\phi(x,y,z)$ at $'p' = |\nabla\phi|_p|$

3.5.1 Del operated-on Vector Point functions

If Del is a differential operator and $\vec{F}(x,y,z)$ is a vector Point function then the Del operator is operated on $\vec{F}(x,y,z)$ in two Ways.

(i) $\nabla\cdot\vec{F}\rightarrow$ Divergence

(ii) $\nabla\times\vec{F}\rightarrow$Curl

(i) Divergence of a Vector Point function:

If $\vec{F}(x,y,z)$ is a Vector Point function, then the divergence of $\vec{F}(x,y,z)$ is denoted by div $\vec{F}$(or)$\nabla\cdot\vec{F}$ and for any $\vec{F}(x,y,z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ the divergence is given by

$$\text{div}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

If div $\cdot\vec{F} = 0$, then $\vec{F}(x,y,z)$ is called Solenoidal (or) Incompressible flow Vector.

(ii) Curl of a Vector Point Function:

If $\vec{F}(x,y,z)$ is a Vector Point function, then the curl of $\vec{F}(x,y,z)$ is denoted by $Curl\ \vec{F}(or)\nabla\times\vec{F}$ and for any $\vec{F}(x,y,z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$, the curl of $\vec{F}(x,y,z)$is given by

$$\text{curl}\ \vec{F} = \nabla\times\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\times\left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

If curl $\vec{F} = \vec{0}$; then $\vec{F}$ is called Irrotational Flow Vector.

(iii) Properties of div, Curl & Grad:

If $\phi(x,y,z)$ and $\vec{F}(x,y,z)$ are a scalar point function and a vector point function respectively, then

(a) curl (grad ϕ)=$\vec{0}$

(b) div(curl $\vec{F}$) $= 0$

(c) div(grad ϕ) $= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$

(iv) Angle between two Intersecting Surfaces:

If $\phi_1(x,y,z) = c_1$ & $\phi_2(x,y,z) = c_2$ are two surfaces intersecting at 'P', then the angle of Intersection 'θ' is given by

$$\cos\theta = \frac{\nabla\phi_1|_p \cdot \nabla\phi_2|_p}{|\nabla\phi_1|_p| \cdot |\nabla\phi_2|_p|}$$

3.6 Vector Integration

3.6.1 Line Integrals

If $\vec{F}(x,y,z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point along the path C , then the Integral of $\vec{F}(x,y,z)$ from Point 'A' to point 'B' along a path is given by $\int_{A,C}^{B} \vec{F} \cdot d\vec{r}$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\int_{A,C}^{B} \vec{F} \cdot d\vec{r} = \int_{A,C}^{B} (F_x\vec{i} + F_y\vec{j} + F_z\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

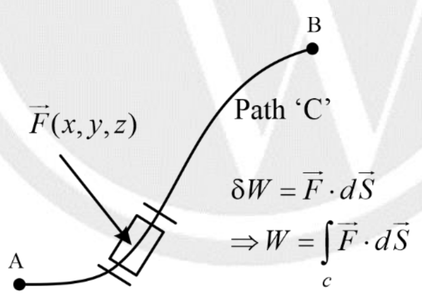

Fig.3.5 Line Integral

If $\vec{F}(x,y,z)$ is Irrotational Vector Point Function, (i.e., Curl $\vec{F} = \vec{0}$) then $\int_A^B \vec{F} \cdot d\vec{r}$ is independent of the path followed between the points A and B.

If $\vec{F}(x,y,z)$ is Irrotational Vector Point Function, then

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \nabla\phi \cdot d\vec{r} \text{ where} \vec{F} = \nabla\phi$$

$$= \phi|_B - \phi|_A$$

3.6.2 Surface Integral

If $\vec{F}(x,y,z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point on a surface 'S', then the surface integral of $\vec{F}(x,y,z)$ on the surface 'S' is given by $\int_S \vec{F}\cdot d\vec{s}$

Where $d\vec{s} = ds.\hat{n}$ and $\hat{n}$ is the outward unit normal vector to the surface at ds and

$$ds = \frac{dx.dy}{|\hat{n}.\hat{k}|} = \frac{dy.dz}{|\hat{n}.\hat{i}|} = \frac{dx.dz}{|\hat{n}.\hat{j}|}:$$

3.6.3 Volume Integral

If $\vec{F}(x,y,z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point over a volume V, then the volume integral of $\vec{F}(x,y,z)$ on the volume 'V' is given by $\int_V \vec{F}\cdot dv$.

3.6.4 Greens Theorem: (Connects closed line Integral to surface Integral)

If $\vec{F}(x,y) = F_x\vec{i} + F_y\vec{j}$ and if the first order derivatives of F_x& $F_y$are continuous at every point with in a region 'R' bounded by a closed path 'C', then

$$\oint_C \vec{F}d\vec{r} = \oint_C F_x dx + F_y dy = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy$$

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

3.6.5 Gauss – Divergence Theorem: (Connects closed surface integral to a Volume Integral)

If 'S' is a closed surface enclosing a volume 'V' and $\vec{F}$ is continuous and differentiable at every point on the closed surface 'S', then the closed surface integral $\oint_S \vec{F}.d\vec{s} = \iiint_V div\,\vec{F}.dV$

3.6.7 Stokes Theorem: (Connect Closed line integral to surface Integral)

If $\vec{F}$ is continuous and differentiable at every point within a region 'R' (on a surface S) bounded by a closed path 'C', then

$$\oint_C \vec{F}d\vec{r} = \iint_R curl\,\vec{F}.d\vec{S}$$

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4 LINEAR ALGEBRA

4.1 Matrix

An array of elements in horizontal lines (Rows) and Vertical Lines (Columns) is called a Matrix.

Ex.
$$A = \begin{bmatrix} i & n & d & i & a \\ j & a & p & a & n \end{bmatrix}$$

4.1.1 Size of Matrix

If a matrix has 'm' rows and 'n' columns, then we say that the size of the matrix is m × n (read as m by n)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots\ldots\ldots a_{2n} \\ . & . & .. \\ . & . & .. \\ a_{m1} & a_{m2} & a_{m3}\ldots\ldots\ldots a_{mn} \end{bmatrix}; \ A = [a_{ij}]_{m\times n} \text{ such that } 1 \le i \le m, 1 \le j \le n \text{ and } a_{ij} = f(i,j)$$

4.1.2 Addition of Matrices

(i) Two matrices $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{p\times q}$ can be added only if m = p & n = q.

(ii) Matrix Addition is commutative (A + B = B + A)

(iii) Matrix Addition is Associative. A + (B + C) = (A + B) + C

4.1.3 Multiplication of Matrices

The multiplication of two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ $(\Rightarrow AB_{m\times q})$ is feasible only if n = P.

$$A_{m\times n} \cdot B_{p\times q} = C$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

$$A_{3\times 3}B_{3\times 2} = \begin{bmatrix} a_{11}.b_{11} + a_{12}\cdot b_{21} + a_{13}.b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}.b_{11} + a_{32}\cdot b_{21} + a_{33}\cdot b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}_{3\times 2}$$

4.1.4 Properties of Multiplication of Matrices

(i) Matrix Multiplication Need not be commutative.

(ii) Matrix Multiplication is Associative (A(BC)) = ((AB)C)

(iii) Matrix Multiplication is distributive (A(B + C) = AB + AC)

(iv) The product of two Matrices $A_{m\times n}, B_{n\times q}$ (i.e. $AB_{m\times q}$) can be a zero matrix even if $A \neq O \& B \neq O$.

Ex. $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- For the multiplication of two matrices $A_{m\times n}$ & $B_{n\times q}$
  (i) The No. of Multiplications required = m n q
  (ii) The No. of Additions required = m (n –1) q

4.2 Types of Matrices

(1) Upper triangular Matrix: A matrix $A = [a_{ij}]; 1 \leq i, j \leq n$ is said to be an upper triangular matrix if

$$a_{ij} = 0 \ \forall \ i > j$$

(2) Lower Triangular Matrix: A matrix $A = [a_{ij}]_{n\times n}; 1 \leq i, j \leq n$ is said to be a lower Triangular Matrix

if $a_{ij} = 0 \forall i < j$

(3) Diagonal Matrix: A matrix $A = [a_{ij}], \forall\ 1 \leq i, j \leq n$ is said to be diagonal matrix if $a_{ij} = O \forall i \neq j$

Ex. $A = \begin{bmatrix} d_1 & O & O \\ O & d_2 & O \\ O & O & d_3 \end{bmatrix}$. Diagonal Matrix is also denoted as $A = diag[d_1, d_2, d_3]$

(4) Scalar Matrix: A Matrix 'A' = $[a_{ij}]$; $1 \leq i, j \leq n$is said to be a scalar Matrix if $a_{ij} = \begin{cases} k; i = j \\ O; 1 \neq j \end{cases}$

If K = 1, then A → Identity Matrix,
If K = O, then A → Null Matrix.

(5) Idempotent Matrix:

A Matrix '$A_{n\times n}$' is said to be an idempotent matrix if $A^2 = A$.

Ex. $A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

$\Rightarrow A^2 A \cdot A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A$

(6) Nilpotent Matrix: A non-zero matrix '$A_{n\times n}$' is said to be Nilpotent Matrix. if ∃ a value 'n' such that $n \in Z^+$ and $A^n = O$.

Ex. $\begin{bmatrix} 4 & -1 \\ 16 & -4 \end{bmatrix} = A \Rightarrow A^2 = \begin{bmatrix} 4 & -1 \\ 16 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = O$

The least of 'n' for which $A^n = 0$ is called Index of the Nilpotent Matrix.

(7) Orthogonal Matrix: A matrix A is said to be orthogonal if $A.A^T = I$

Ex. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A$

(8) Involutory Matrix: A matrix A is said to be involutory if $A^2 = I$

Ex. $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = A$

4.3 Transpose of a Matrix

For a given matrix $A = [a_{ij}]$; $1 \le i \le m, 1 \le j \le n$, we say that 'B' where $B = [b_{ij}]$, $i \le i \le n$ $i \le j \le m$ is transpose of the Matrix 'A' if $a_{ij} = b_{ji}$

4.3.1 Properties of Transpose of a Matrix

(i) $(A^T)^T = A$

(ii) $(AB)^T = B^T \cdot A^T$

(iii) $(KA)^T = KA^T$ where 'K' is a scalar.

4.4 Determinant

The summation of product of element of a row(or) column of a matrix with their corresponding Co-factors.

$$A \cdot adj(A) = |A| \cdot \mathrm{I}$$

4.4.1 Properties of Determinants

(i) If 'A' is a Square Matrix of size ' $n \times n$ ' and 'k' is a Scalar then

(a) $|K \cdot A_{n\times n}| = K^n \cdot |A_{n\times n}|$

(b) $|adj(A)| = |A|^{(n-1)}$

(c) $\left|adj\left(adj(A)\right)\right| = (|A)^{(n-1)^2}$

(ii) $|AB| = |A| \cdot |B|$

(iii) $|(AB)^T| = |B^T| \cdot |A^T|$

(iv) If two rows (or) two columns of a determinant are interchanged, then determinant changes its sign.

(v) The determinant of an upper triangular Matrix/a lower triangular Matrix/a diagonal Matrix is product of the principal diagonal elements of the Matrix.

(vi) The determinant of Every Skew-Symmetric Matrix of odd order $(A_{n\times n})('n' is odd)$ is zero

(vii) The determinant of an orthogonal Matrix $A_{n\times n}$ is ± 1

(viii) The determinant of an Idempotent Matrix is either 0 (or) 1.

(ix) The determinant of an Involuntary Matrix is ± 1

(x) The determinant of a Nilpotent Matrix is always zero.

(xi) If the product of two Non-zero Matrices $A_{n\times n} \ne O; B_{n\times n} \ne O$ is a zero Matrix $((AB)_{n\times n} = O)$, then both $|A| = 0$ & $|B| = 0$.

(xii) If two rows (or) two columns of a Matrix are either equal or Proportional, then the determinant of the Matrix is equal to zero.

(xiii) The number of terms in the general expansion of a 'n × n' determinant is $\underline{n!}$

4.5 Rank of a Matrix

A real Number 'r' is said to be rank of a matrix '$A_{m\times n}$' if

(1) All minors of order $(r+1)\times(r+1)$ and above are zeros and

(2) '∃' atleast one Non-zero minor of order '$r\times r$' of the matrix 'A'.

It is mathematically denoted by $\rho(A) = r$

4.5.1 Properties of Rank of a Matrix

(i) $\rho(A_{m\times n}) \leq (m, n)$

(ii) $\rho(AB) \leq min\ \{\rho(A), \rho(B)\}$

4.5.2 Row Echleon Form

A Matrix $A_{m\times n}$is said to be in row-echleon form if

(i) Number of zeroes before the 1st Non-zero element in any row is less then number of such zeroes in its succeeding row.

(ii) Zero rows (if any) should lie at the bottom of the Matrix.

$\rho(A_{m\times n})$ = Number of non-zero rows in Row-Echleon form of A.

4.6 System of Equations

The given system of equations

$$a_{11}x_1 + a_{12}x_{12} + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written in Matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \boxed{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Ax = B

↓ Coefficient Matrix ↓ Variable Matrix ↓ Constants Matrix

The system Ax = B is said to be homogeneous system if B = 0.

The system of Ax = B is said to be non-homogeneous system if $B \neq 0$.

4.6.1 Consistency of a non-homogeneous system of Equations

For above system of non – homogeneous equations, Ax = B; Augmented Matrix = [A/B] = $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

(i) If $\rho(A) = \rho(A/B) =$ Number of unknowns, then the system Ax = B has unique solution.

(ii) If $\rho(A) = \rho(A/B) <$ Number of unknowns, then the system has infinitely many solutions.

(iii) If $\rho(A) \neq \rho(A/B)$, then the system has no solution.

No. of linearly independent solutions for a system of 'n' equations given by Ax = B is $n - \rho(A)$

4.6.2 Consistency of Homogeneous System of Equations

$$a_{11}x + a_{12}y + a_{13}z = 0$$
$$a_{21}x + a_{22}y + a_{23}z = 0$$
$$a_{31}x + a_{32}y + a_{33}z = 0$$

Ax = 0 ⇒ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[A/B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix}_{3\times4}$

If $\rho(A) = \rho(A/B) = n$ $(i.e\ |A| \neq 0)$; the system has unique solution.

(Trivial solution; x = 0, y = 0, z = 0)

If $\rho(A) = \rho(A/B) < n(|A| = 0)$; the system has infinitely many solutions (Non-trivial solution exists for the system)

4.7 Linear Combination of Vectors

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' rows vectors, then the combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n$ is called linear combination of $x_1, x_2, \ldots., x_n (k_1, k_2, k_3, \ldots.. k_n$ are scalars)

(1) The linear combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n$ is said to be linearly dependent if $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n = 0$ when $k_1, k_2, k_3, \ldots.., k_n$ (NOT All zeroes).

If $x_1 = [a_1 \quad b_1 \quad c_1]; x_2[a_2 \quad b_2 \quad c_2]; x_3 = [a_3 \quad b_3 \quad c_3]$, then the vectors x_1, x_2, x_3 are said to be linearly dependent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

(2) The combination $k_1x_1 + k_2x_2 + \ldots\ldots + k_nx_n$ is said to be linearly independent if $k_1x_1 + k_2x_2 + \ldots\ldots.. + k_nx_n = 0$ when $k_1 = k_2 = k_3 = \ldots.. k_n = 0$

4.7.1 Eigen Values and Eigen Vectors

For any square Matrix $A_{n\times n}$, the equation $|A - \lambda I| = 0$ where 'λ' is a scalar is called the characteristic equation. The roots of the characteristic equation of a Matrix are called Eigen Values.

4.7.2 Properties of Eigen Values

(i) If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are 'n' Eigen Values of $A_{n\times n}$, then

(a) Sum of Eigen Values of 'A' = $\lambda_1, \lambda_2 + \lambda_3 + \ldots. + \lambda_n = \sum_{i=1}^{n} \lambda_i = tr(A)$ = Sum of Principal diagonal elements

(b) Product of all the Eigen Values of 'A' = $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots\ldots \lambda_n = \prod_{i=1}^{n} \lambda_i = |A|$

(c) Eigen Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots\ldots \lambda_n^m$

(d) Eigen Values of adj(A) are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots\ldots, \frac{|A|}{\lambda_n}$

(e) Eigen Values of A & A^T are same.

(f) Eigen Values of $k_1 A + k_2 I$ (Where k_1 and k_2 are scalar) are

$$k_1\lambda_1 + k_2, k_1\lambda_2 + k_2, k_1\lambda_3 + k_2, k_1\lambda_4 + k_2, \ldots\ldots.. k_1\lambda_n + k_2$$

(ii) '0' is always an Eigen Value of an odd order Skew-Symmetric Matrix.

(iii) Eigen Values of Real Symmetric Matrix are always real.

(iv) Eigen Values of Skew-Symmetric Matrix are either zero (or) purely Imaginary.

(v) The Eigen values of an Orthogonal Matrix are of unit modulus.

(vi) If sum of all the elements in a row (or Column) is constant (= k) for all the rows (or columns) in the matrix respectively, then 'k' is an Eigen Value of the Matrix.

Ex. If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and if $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3 = k,$

then 'k' is an Eigen Value of 'A'.

(vii) The Eigen Values of an upper triangular Matrix, a lower triangular Matrix, a diagonal Matrix are the Principal diagonal elements of the Matrix.

4.8 Eigen Vector

A non-zero column vector $X_{n\times 1}$ is said to be an Eigen Vector of $A_{n\times n}$ corresponding to the Eigen Value 'λ', if $AX = \lambda X (X \neq 0)$.

4.8.1 Properties of Eigen Vectors

(i) Eigen Vectors of A & A^T are not same.
(ii) Eigen Vectors of A & A^M are same.
(iii) The Eigen Vectors of a Real Symmetric Matrix are always orthogonal.
(iv) The number of linearly independent Eigen Vectors of '$A_{n\times n}$' is equal to number of distinct Eigen Values of '$A_{n\times n}$'.

4.8.2 Cayley Hamilton Theorem

Every Matrix satisfies its own characteristic equation.

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5 PROBABILITY AND STATISTICS

5.1 Random Experiment

The experiment in which the outcome is uncertain is called a Random Experiment (RE).

Ex. Flipping a coin, rolling a pair of dice, Picking a ball from a bag.

5.1.1 Sample Space

The set containing of all the possible outcomes of a random experiment. It is denoted by 'S'.

If RE is flipping a coin, S = {Head, Tail}

If RE is rolling a dice, S = {1,2,3,4,5,6}

5.2 Event

Any subset of sample space 'S' is called as Event.

Ex. If RE is flipping a coin, then occurring of a Head is an Event.

If RE is rolling a dice, then getting an odd number is an Event.

5.2.1 Probability of an Event

If 'A' is any event with in the sample space 'S' of a Random experiment, then the probability of event 'A' is given by

$$P(A) = \frac{No. of outcomes favouring event 'A' to happen}{Total\ number\ of\ elements\ in\ 'S'} = \frac{n(A)}{n(S)}$$

Probability of getting an Even Number when a dice is rolled.

P(Even Number) = $\frac{3}{6} = 0.5$

S = {1,2,3,4,5,6}, A = {2,4,6}

5.2.2 Axioms Probability

(i) If 'A' is any event with in the sample space 'S' of a RE, then $0 \leq P(A) \leq 1$

$$\frac{0}{n(S)} \leq \boxed{\frac{n(A)}{n(S)}} \leq \frac{n(S)}{n(S)}$$

$$\boxed{0 \leq P(A) \leq 1}$$

(ii) P(S) = 1

When a RE is conducted the experiment yields a possible outcome.

5.2.3 Types of Events

(i) Mutually Exclusive Events:

If A, B are two events within a sample space 'S', then A & B are said to be mutually exclusive if $A \cap B = \phi$.

Ex. If 'A' is the event of getting a prime number when a dice is rolled and 'B' is the event of getting a composite number when a dice is rolled then

$S = \{1,2,3,4,5,6\}$, $A = \{2,3,5\}$, $B = \{4,6\} \Rightarrow A \cap B = \phi \Rightarrow P(A \cap B) = 0$

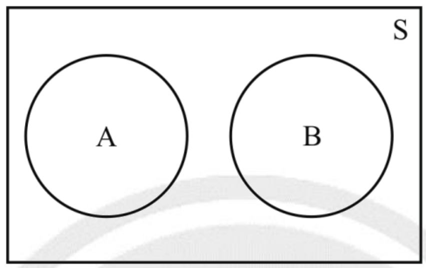

Fig. 5.1 Mutually exclusive event

(ii) Mutually Exhaustive Events:

If 'A', 'B' are two events with in a sample space 'S', then 'A' & 'B' are said to be mutually exhaustive if $A \cup B = S$

Ex. If 'A' is the event of getting an odd number when a dice is rolled and 'B' is the event of getting an Even Number, then

$A \cup B = S$

$S = \{1,2,3,4,5,6\}$

$A = \{1,3,5\}$, $B = \{2,4,6\}$

$\Rightarrow A \cup B = S$

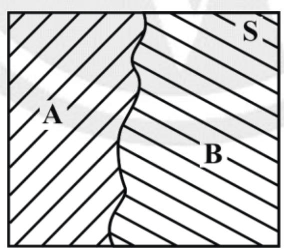

Fig. 5.2 Mutually exhaustive event

(iii) Independent Events:

Two events 'A' & 'B' with in the sample space 'S' (or) with in two different sample spaces 'S_1' & 'S_2' are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$.

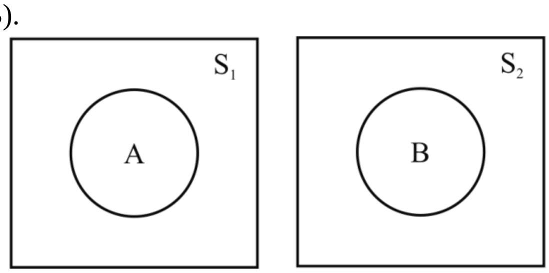

Fig. 5.3 Independent event

(iv) Impossible Event (ϕ):

The event with zero probability is called on Impossible Event P(ϕ) = 0.

5.3 Addition Theorem of Probability

If A, B are two events with a sample space 'S' of a Random Experiment, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

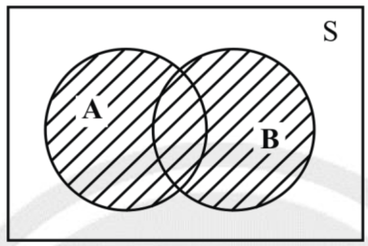

Fig. 5.4 Addition theorem

⇒ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

When A, B are mutually exclusive event, $A \cap B = \phi$.

⇒ $P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

- If $E_1, E_2, E_3, \ldots\ldots E_n$ are mutually exclusive events ($E_i \cap E_j = \phi$), then $P(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$ $= P(E_1) + P(E_2) + P(E_3) + \ldots\ldots p(E_n)$

5.3.1 Conditional Probability

The probability of heppening of event 'A' when it is known that event 'B' has already occurred is given by P(A/B)

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}$$

5.3.2 Multiplication Theorem of Probability

If A, B are two events within a sample space 'S', then P(A/B) · P(B) = P(B/A) · P(A)

$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A/B) \cdot P(B) \rightarrow (1)$

$P(B/A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B/A) \cdot P(A) \rightarrow (2)$

From (1) & (2)

$$P(A/B) \cdot P(B) = P(B/A) \cdot P(A)$$

5.3.3 Total Theorem of Probability

If $E_1, E_2, E_3, \ldots E_n$ are 'n' mutually exclusive $(E_i \cap E_j = \phi; \forall i \neq j)$ and collectively exhaustive event $(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S)$ and 'A' is any event with in the sample space 'S', then

$$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \ldots + P(E_n) \cdot P(A/E_n)$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i) \bullet P(A/E_i)}$$

5.3.4 Baye's Theorem

If $E_1, E_2, E_3, \ldots E_n$ are mutually exclusive $(E_i \cap E_j = \phi \forall i \neq j)$ and collectively exhaustive event $(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S)$ and 'A' is any event with in the sample space 'S', then

$$\boxed{P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}}$$

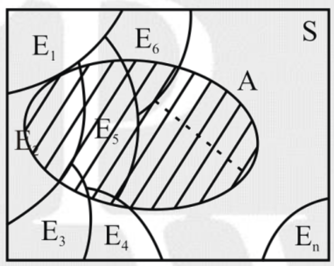

Fig. 5.5 Baye's theorem

5.4 Statistics

Statistics → Collection and Analysis of Data

Data
- → Ungrouped data / Row data
- → Grouped Data

5.4.1 Analysis of Ungrouped Data

If $x_1, x_2, x_3, \ldots, x_n$ are 'n' observations, then

(1) The range of the data = R = $\max\{x_1, x_2, \ldots, x_n\} - \min\{x_1, x_2, x_3, \ldots, x_n\}$

(2) Arithmetic mean **:** Mean of the data is equal to sum of observaions divided by the total number of observations.

$$\bar{x} (or) \mu = \frac{x_1 + x_2 + \ldots + x_n}{n} = \boxed{\frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} = \mu}$$

- The mean of 1^{st} 'n' natural numbers = $\frac{\left(\frac{n(n+1)}{2}\right)}{n} = \frac{n+1}{2}$
- The mean of 1^{st} 'n' odd numbers = $\frac{n^2}{n} = n$
- The mean of 1^{st} 'n' even numbers = n + 1

5.4.2 Median

The middle most observation of the data $(x_1, x_2, x_3, \ldots\ldots x_n)$ When the data is arranged in either ascending or descending order.

If $x_1, x_2, x_3, x_4, \ldots\ldots. x_n$ are 'n' observations that are arranged in ascending/descending order then

(i) Median of the Data = $\left(\frac{n+1}{2}\right)^{th}$ observation, if 'n' is odd.

(ii) Median of the Data = Mean of $\left(\frac{n}{2}\right)^{th}$ & $\left(\frac{n}{2}+1\right)^{th}$ observations, if 'n' is even.

5.4.3 Mode

The observation with highest frequency is called mode.

Any Data with two Modes is called → Bimodel Data

If $x_1, x_{2,} x_3, \ldots\ldots, x_n$ are 'n' data points, $\bar{x} = \mu = \frac{x_1+x_2+\ldots\ldots+x_n}{n}$

Mean Deviation of the observation $(x_i) = d_i = x_i - \bar{x}$

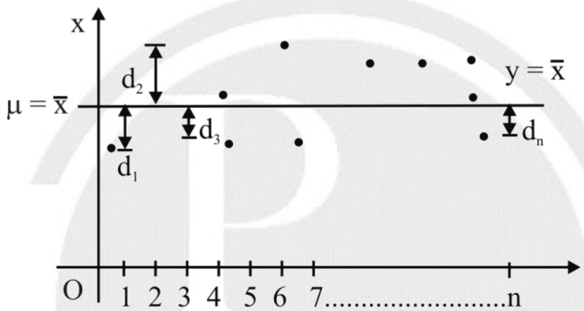

Fig. 5.6 Discrete data

Sum of derivations of all the observations = $\Sigma d_i = (x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots\ldots 0 + (x_n - \bar{x})$

$$= \Sigma d_i = (x_1 + x_2 + \ldots.. + x_n) - n\bar{x}$$

$$\boxed{\Sigma d_i = 0}$$

The sum of mean deviations of all the observations is equal to zero.

5.4.4 Absolute Mean Deviation

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' data points with Mean = $\bar{x}$, then the absolute mean deviation of x_i about $\bar{x}$ is given by $|d_i| = |x - \bar{x}|$

The sum of absolute mean derivations of given data is not zero.

$$(\Sigma|d_i| \neq 0) \Rightarrow (|x_1 - \bar{x}| + |x_2 - \bar{x}| + \ldots\ldots.. + |x_n - \bar{x}| \neq 0)$$

5.4.5 Standard Deviation

If $x_1, x_2, x_3, \ldots\ldots, x_n$ ('n' is very large), then the standard deviation of the data is given by

Population Standard Deviation $\sigma = \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}$, n → size of population

Sample Standard derivation: $\sigma = \sqrt{\frac{1}{(n-1)}\Sigma(x_i - \bar{x})^2}$, n→ size of sample

Generally (n > 29 → population) (n < 29 →sample)

5.5 Random Variables

The variable that connects the outcome of a Random Experiment to a real number.

Ex. 'x' is the value of the number that a dice shows when it is rolled.

Random Variable
- Discrete RV → The RV whose value is obtained by counting
- Continuous RV→ The RV whose value is obtained by Measuring

- If a data consists of 'f_1' datapoints with value $'x_i', 'f_2'$ data points with value $'x_2'......'f_n'$ data point with value $'x_n'$, then
  (i) Expectation of 'x' $= E(x) = \sum_{i=1}^{n} x_i P(x = x_i)$
  (ii) Variance of 'x' = $\sigma^2 = E(x^2) - (E(x))^2$ and σ is the standard deviation.

5.5.1 Continuous RV

The value of the Random Variable is obtained by Measuring.

5.6 Probability distribution Function (Pdf)

A continuous & differentiable function P(x) is said to be a probability distribution/density function of a continuous random variable 'x' if $P(a \le x \le b) = \int_a^b P(x)dx$

5.6.1 Mean (or) Expectation

If P(x) is a probability distribution/density function of a continuous Random Variable 'x' then the Mean of 'x' = E(x) = $\int_{-\infty}^{\infty} x \cdot P(x)dx$

5.6.2 Median

The value of 'x' for which the total probability is exactly divided into two equal halves is called Median.

5.6.3 Mode

The valueof 'x' at which P(x) is maximum is called mode.

5.6.4 Variance

V = $\sigma^2 = E(x^2) - (E(x))^2$

$$\Rightarrow \sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot P(x)dx - \left(\int_{-\infty}^{\infty} x \cdot P(x)dx\right)$$

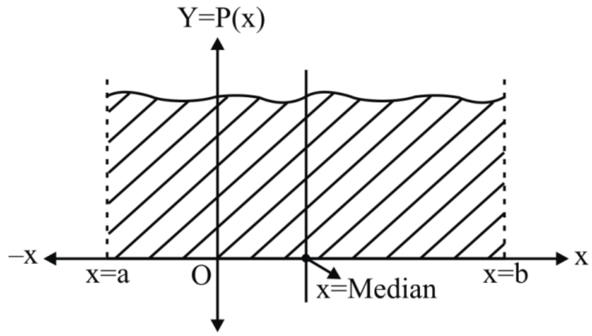

Fig.5.7 Continuous random variables

5.7 Continous RV Distributions

(1) Gaussian/Normal Distributon:

If 'x' is a continuous Random variable with mean 'μ' and standard deviation 'σ', then the probability distribution/density function of normally distributed variable 'x' is given by

$$P(x)=\frac{1}{\sigma\sqrt{2\pi}}.e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}$$

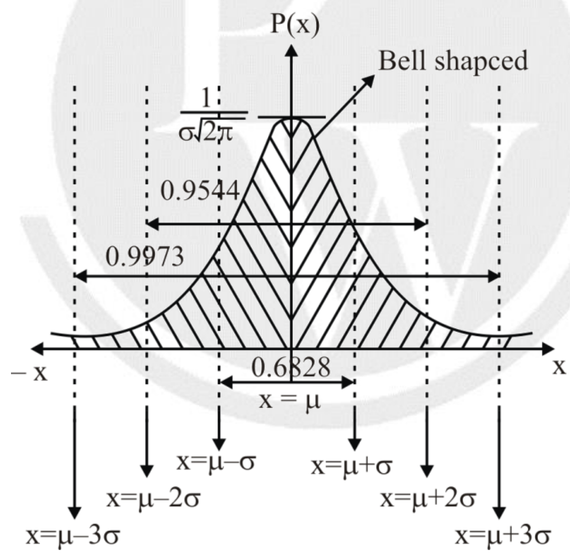

Fig.5.8 Normal distribution

Mean = Median = Mode = μ

$$P(\mu-\sigma\le x\le\mu+\sigma)=0.6828$$
$$P(\mu-2\sigma\le x\le\mu+2\sigma)=0.9544$$
$$P(\mu-3\sigma\le x\le\mu+3\sigma)=0.9973$$
$$P(x)=\frac{1}{\sigma\cdot\sqrt{2\pi}}.e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

(2) Standard Normal Distribution:

Assuming $z = \frac{x-\mu}{\sigma}; \mu = 0; \sigma = 1, P(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}}$

$$P(-1 \le z \le 1) = 0.6828$$
$$P(-2 \le z \le 2) = 0.9544$$
$$P(-3 \le z \le 3) = 0.9973$$

Note:

1. The normal distribution curve is bell shaped curve
2. The points of infelection of the normal distribution curve are at $x = \mu + \sigma \; and \; x = \mu - \sigma$.
3. The cumulative function graph is of 'S' Shape.
4. For a given normal distribution, Mean = median = Mode

(3) Uniform Distribution:

If 'x' is a uniformly distrbuted random variable such that $a \le x \le b$ then the Pdf is given by

$$P(x) = \frac{1}{(b-a)}$$

Mean = $\int_a^b x \cdot P(x)dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{(b-a)} \int_a^b x \cdot dx$

$$\boxed{\left(\frac{b+a}{2}\right) = \text{Mean}}$$

$\Rightarrow \quad Variance = \sigma^2 = \frac{(b-a)^2}{12}$

Std.deviation = $\sigma = \frac{(b-a)}{\sqrt{12}}$

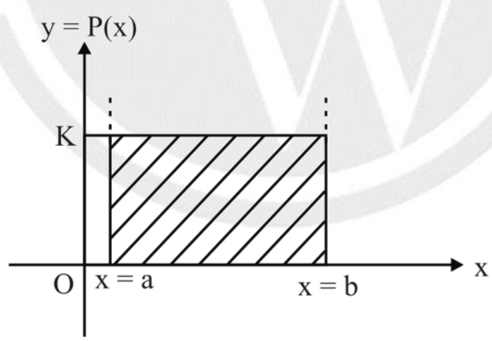

Fig.5.9 Uniform distribution

5.7.1 Properties of Mean and Variance

$E(ax + by) = a \cdot E(x) + b \cdot E(y)$

$V(ax + by) = a^2 \cdot V(x) + b^2 \cdot V(y) - 2abCOV(x, y) \; where \; COV(x, y) = E(xy) - E(x) \cdot E(y)$

If x,y are indpendent random variables, then $E(xy) = E(x) \cdot E(y) \Rightarrow COV(x, y) = 0$

(1) Exponential Dirtibution:

If 'x' is a contrinous random variable with mean as $\frac{1}{\lambda}$ then the exponential distribution of 'x' is given by the function

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & ; x \geq 0 \\ 0 & : otherwise \end{cases}$$

Mean = $\frac{1}{\lambda}$

$\sigma^2 = \frac{1}{\lambda^2}$

Mean = Standard Deviation = $\frac{1}{\lambda}$

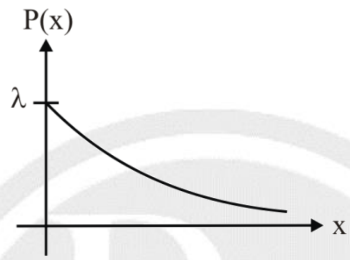

Fig.5.10 Exponential distribution

5.8 Discrete Random Variable Distributions

5.8.1 Binomial Distribution

If a Random experiment has **only two Possible outcomes**, (one is Success & other is failure) and the Probability of Success doesn't depend on time, then the probability of occuring of exactly 'r-successes' in 'n-trials' is given by

$$P(X = r) = ^n C_r \cdot P^r \cdot q^{n-r}$$

Where, P → Probability of Success,

q → Probability of Failure

p + q = 1

Mean = np,Variance = npq = σ^2, standard deviation = $\sigma = \sqrt{npq}$

5.8.2 Poisson Distribution

If a random experiment has only two possible outcomes, and the average number of successes in a given time 't' is λ, then the probability that exactly 'r' successes occur within the same time 't' given by

$P(x = r) \frac{e^{-\lambda}.\lambda^r}{r!}$

Mean = λ.

Mean = $\sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}.\lambda^x}{x!(x-1)!}$ $\qquad E(x^2) = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}.\lambda^x}{x!}$

$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!} \Rightarrow E(x^2)$

$= e^{-\lambda} \cdot \lambda \cdot \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$

$= e^{-\lambda} \cdot \lambda \left\{1 + \frac{\lambda}{1!} + \frac{\lambda}{2!} + \frac{\lambda}{3!} + \ldots\ldots\right\}$

$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda = E(x)$

$\sigma^2 = E(x^2) - (E(x))^2$

$= \lambda^2 + \lambda - \lambda^2 = \lambda$

$\Rightarrow \sigma^2 = \lambda$

$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{x^2 \cdot \lambda^x}{x!(x-1)!}$

$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{\lambda \cdot x \cdot \lambda^{x-1}}{(x-1)!}$

$= e^{-\lambda} \cdot \lambda \cdot \sum_{x=0}^{\infty} \left\{\frac{(x-1)\lambda^{x-1}}{(x-1)!} + \frac{\lambda^{x-1}}{(r-1)!}\right\}$

$= e^{-\lambda} \cdot \lambda\left\{\lambda \cdot e^{\lambda} + e^{\lambda}\right\} = \lambda^2 + \lambda$

For Poisson distribution,

Mean = Variance = λ

$\Rightarrow \sigma = \sqrt{\lambda}$

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6 NUMERICAL METHODS

6.1 Solving System of Linear Equations

6.1.1 Gauss Elimination Method

Let the given system of linear equations be $\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{matrix}\right\} \Rightarrow AX = B$

Augmented Matrix = $[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

By Row transformations, we convert the above matrix to row echelon form and then we go backward substitution.

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & 0 & a_{33}^{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^1 \\ b_3^{11} \end{bmatrix}$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{22}^1 x_2 + a_{23}^1 x_3 = b_2$$
$$a_{33}^{11} x_3 = b_3$$

By solving the above equations, we get the values of the variables.

Note:

While applying row transformations, zero pivot elements are avoided by swapping the rows of augmented matrix.

LU Decomposition Method:

Let the given system of linear equations be $\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{matrix}\right\} \Rightarrow AX = B$

The coefficient matrix A, is split into the product of an upper triangular and a lower triangular Matrix.

A = LU

⇒ LUX = B

Taking UX = Y we get AY = B

On Backward substitution , we get the elements of the matrix Y.

Since $UX = Y$, on forward substituion,we get the value of X.

6.2 Solution of Algebraic Equations (Non-Linear)

6.2.1 Bracketing Methods

In all the bracketing methods, to find the root of an equation $f(x) = 0$, we assume an interval [a, b] such that $f(a).f(b) < 0$, then we develop an iterative scheme for evaluating the iterates.

Method	Iteration formulae
(1) Bisection Method	(1) $c = \frac{a+b}{2}$
(2) Secant Method	(2) $c = \frac{a.f(b)-b.f(a)}{f(b)-f(a)}$

6.2.2 Newton Raphson Method

To find the root of the equation $f(x) = 0$, we assume an initial guess $x = x_0$ and the iterative scheme for the iterates is given by the equation

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Note

For the convergence of Newton Raphson method, $|f(x). f''(x)| < |f'(x)|^2$.

6.2.3 Rate of Convergence Values of iterative Methods

(1) Bisection Method -1

(2) Regula – Falsi Method -1

(3) Secant Method – 1.618

(4) Newton – Raphson Method -2

6.3 Numerical Integration

6.3.1 Trapezoidal Rule

If f(x) is continuous on [a,b] and differentiable on (a,b) then the value of $\int_a^b f(x)dx$ evaluated using 'n' sub intervals is given by $\int_a^b f(x) = dx = \frac{h}{2}[(f(a) + f(b)) + 2(f(a + h) + f(a + 2h) + \ldots\ldots + f(a + (n - 1))h)]$

$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1})]$$

Salient Points

(i) The order of the fitting polynomial is 1.

(ii) The error involved in Trapezoidal Rule is = $\frac{nh^3}{12}$| Max. of f"(x) in [a, b] |= $\frac{n(b-a)^3}{12n^3}$ | max. of $f''(x)$ in [a, b] |

(iii) Trapezoidal Rule gives exact results for Polynomials of order ≤ 1

(iv) As number of subintervals increases, the accuracy of the result increases.

6.3.2 Simpson's 1/3rd Rule

If f(x) is continuous on [a,b] and differentiable on (a,b) then the value of $\int_a^b f(x)dx$ evaluated using 'n' even number of sub intervals then the value of the integration is given by

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_2 + 4y_4 + y_6 + \ldots\ldots + y_{n-2})]$$

Salient Points:

(i) The order of fitting polynomial → 2 (Quadratic)

(ii) Simpons rule gives the exact results for all the polynomials of degree ≤ 2.

(iii) Error in volved in the integration = $\frac{h^4}{100}|\max\ of\ f^4(x) in[a,b]|$ where f^4 (x) is the fourth derivative of f (x)

6.4 Numerical Solutions of a 1st Order DE

6.4.1 Explicit Euler Method (or) Forward Euler Method

For a given differential Equation, $\frac{dy}{dx} = f(x,y)$

$\left.\frac{dy}{dx}\right|_{x_i} = f'(x)|_{x_i} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$ (Forward difference)

We have $\left.\frac{dy}{dx}\right|_i = f(x,y)|_{(x_i,y_i)}$

$\Rightarrow \quad \frac{y_{i+1}-y_i}{h} = f(x_i, y_i)$

$\boxed{\Rightarrow y_{i+1} = y_i + h \cdot f(x_i, y_i)}$

↘ Explicit Euler Method

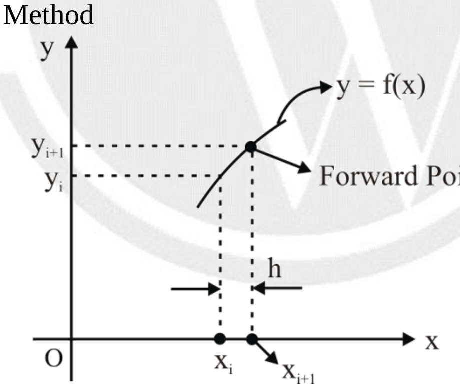

Fig.6.1 Forward Euler method

6.4.2 Implicit Euler Metohd (or) Backward Euler Method

$\left.\frac{dy}{dx}\right|_{x_{i+1}} = f(x_{i+1}, y_{i+1})$

$\left.\frac{dy}{dx}\right|_{x_{i+1}} = f(x,y)|_{(x_{i+1},y_{i+1})} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$

$\Rightarrow \frac{y_{i+1}-y_i}{h} = f(x_{i+1}, y_{i+1})$

$\boxed{\Rightarrow y_{i+1} = y_i + h \cdot \{f(x_{i+1}, y_{i+1})\}}$

↘ Implicit Euler Method

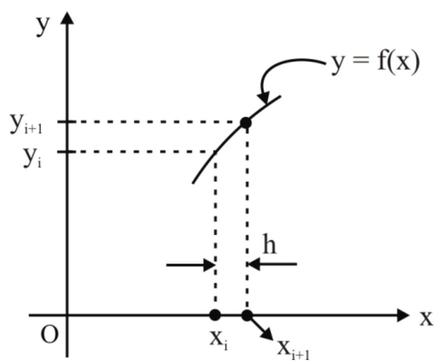

Fig.6.2 Backward Euler method

6.4.3 Modified Euler Method

For a given differential Equation, $\frac{dy}{dx} = f(x, y)$

$y_{i+1}^{(c)} = y_i + \frac{h}{2} \cdot \left\{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(p)})\right\}$ where $y_{i+1}^{(p)} is\ the$ predicted value.

And the predicted value is calclated using one of the Explicit (or) Implicit Euler Methods. (Mostly Forward method id used)

6.4.4 Runge – Kutta (R-K) Methods

For a given differential Equation, $\frac{dy}{dx} = f(x, y)$ with the condition $f(x_0) = y_0$,

(i) **R-K 1st order Method** : Forward Euler Method

(ii) **R-K 2nd order Method** : Modified Euler Method

(iii) **R-K 3rd order Method** :The iterative Scheme is given by $y_{i+1} = y_i + \frac{h}{6}(k_1 + 4k_2 + k_3)$

Where

$k_1 = f(x_i, y_i)$

$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$

$k_3 = f(x_i + h, y_i + k_2)$

(iv) **R-K 4th order Method :**

The iterative Scheme is given by $y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Where

$k_1 = f(x_i, y_i)$

$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$

$k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$

$k_4 = f(x_i + h, y_i + k_3)$

Multi step methods Includes Adams- Bashforth Methods

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7 COMPLEX CALCULUS

A number of the form $z = x + iy$ where $x, y \in R$ is called a complex number.

x is called real part of $z.x = Re(z)$

y is called real part of $zy = Im(z)$

7.1 Modulus – Amplitude form of a Complex Number

Every Complex number $z = x + iy$ can be written as $z = r.e^{i\theta}$ where

$r = \sqrt{x^2 + y^2}$ is called the modulus of the complex number and

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called the amplitude (or) argument of the complex number.

$$e^{i\theta} = \cos\theta + i.\sin\theta = \cos\theta \text{ and}$$

$$e^{-i\theta} = \cos\theta - i.\sin\theta$$

7.2 Arithmetic Operations with Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers then

(i) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

(ii) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1y_2 + x_2y_1)$

(iii) $\frac{z_1}{z_2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$

(iv) $|z_1 + z_2| \le |z_1| + |z_2|$

(v) $|z_1 - z_2| \ge ||z_1| - |z_2||$

(vi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

If r_1, θ_1 are modulus and amplitude of a complex number z_1 and r_2, θ_2 are modulus and amplitude of a complex number z_2 respectively, then

(i) The modulus of $z_1 \cdot z_2$ is $r_1 \cdot r_2$ and the amplitude of $z_1 \cdot z_2$ is $\theta_1 + \theta_2$

(ii) The modulus of $\frac{z_1}{z_2}$ is $\frac{r_1}{r_2}$ and the amplitude of $\frac{z_1}{z_2}$ is $\theta_1 - \theta_2$.

If $z = x + iy$ is a complex number, then the conjugate of the complex number is given by z^* (or) $\bar{z} = x - iy$.

$$Re(z) = \frac{z + z^*}{2}$$

and

$$Im(z) = \frac{z - z^*}{2i}$$

$$z \cdot z^* = |z|^2$$

7.3 De-moivers Theorem

If $z = r \cdot \cos\theta$ is a complex number, then

(i) $z^n = r^n \cdot \cos n\theta$ if 'n' is an integer.

(ii) One of the values of $z^n = r^n \cdot \cos n\theta$ if 'n' is a fraction.

If $n = \frac{p}{q}$, then the n values of z^n are given by $r^n \cdot \cos(2n\pi + \theta)\left(\frac{p}{q}\right)$ where $n = 0,1,2,3\ldots\ldots q\text{-}1$.

The cube roots of unity are given by 1, ω, ω^2 where $\omega = -\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$

The cube roots of unity when plotted on an argand plane form an equilateral triangle.

7.4 Standard Complex Functions

If $z = x + iy$ is a complex number, then

(i) $\ln z = \frac{1}{2} \cdot \ln(x^2 + y^2) + i \cdot \tan^{-1}\left(\frac{y}{x}\right)$

(ii) $\exp(z) = e^x \cdot (\cos y + i \sin y)$

7.5 Periodic function

A complex function $f(z)$ is a periodic function if there exists a complex number 'k' such that $f(z) = f(z + k)$

Ex. The function $f(z) = e^z$ is a periodic function with period $2\pi i$.

7.5.1 Analytic Functions

A function $f(z)$ is said to analytic at a point $z = z_0$ if the fucntion $f(z)$ is differentiable at the pount $z = z_0$ and also at every point in the neighbourhood of z_0.

The mathematical conditions for a function $f(z) = u(x, y) + i.v(x, y)$ to be analytic at a point $z_0 = x_0 + i\, y_0$ is

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous and differentiable at (x_0, y_0)

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These set of euations are called Cauchy – Riemann (C-R) Equations .

Note:

If the function $f(z) = u(x, y) + i.v(x, y)$ is analytic then

(i) Both $u(x, y)$ and $v(x, y)$ satisfy laplace equation.

i.e. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(ii) The family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal to each other.

Cauchy – Riemann Equations in polar form for the function $f(z) = u(x, y) + i.v(x, y)$ are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}$$

and

$$\frac{\partial u}{\partial \theta} = -r\cdot\frac{\partial v}{\partial r}$$

7.6 Complex Integration

If $f(z) = u + iv$ is continuous and differentiable at every point along a path 'C' , then the evaluation of $f(z)$ along the pathe 'C' is given by

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (u\,dy + v\,dx)$$

Note:

If the function $f(z)$ is analytic, then the integral $\int_{z_1}^{z_2} f(z)dz$ is independent of the path connecting the complex numbers z_1 and z_2.

7.6.1 Cauchy Integral Theorem

If the function $f(z)$ is analytic at everypoint with in a closed path 'C' then $\oint_C f(z)dz = 0$.

Note:

If the function $f(z)$ is analytic at everypoint with in a closed path 'C', except at the point $z = z_0$, then

$$\oint_C (z-z_0)^n dz = \begin{cases} 0, \text{ if } n \neq -1 \\ 2\pi i, \text{ if } n = -1 \end{cases}$$

7.6.2 Cauchy Integral formula

If $f(z) = \dfrac{\phi(z)}{(z-z_0)^{n+1}}$ is analytic at every point with in a closed path 'C' except at the point $z = z_0$, then

$$\oint_C f(z)dz = \oint_C \frac{\phi(z)}{(z-z_0)^{n+1}}dz = 2\pi i\cdot\frac{(\phi^n(z_0))}{n!}$$

Where $(\phi^n(z_0))$ is the n^{th} derivative of $\phi(z)$ at the point $z = z_0$.

7.7 Taylor Series and Laurentz Series

(i) Taylor series: If the function $f(z)$ is analytic at every point with in a circle with centre at $z = z_0$, then for any point z with in the circle,

$$f(z) = \sum_{n=0}^{\infty} a_n\cdot(z-z_0)^n$$

where

$$a_n = \left(\frac{1}{2\pi i}\right)\cdot\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

(ii) Laurentz Series: If the function $f(z)$ is analytic at every point with in a region bounded by two concetric circles C and C_1 with radii r, r_1 respectively $(r > r_1)$ with centre at $z = z_0$, then for any point z with in the region,

$$f(z) = \sum_{n=0}^{\infty} a_n\cdot(z-z_0)^n + \sum_{n=1}^{\infty} b_n\cdot(z-z_0)^{-n}$$

where $$a_n = \left(\frac{1}{2\pi i}\right) \cdot \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

and $$b_n = \left(\frac{1}{2\pi i}\right) \cdot \oint_{C_1} \frac{f(z)}{(z-z_0)^{-n+1}}\, dz$$

Note:

All the formulaes above for the cyclic integrals are for counter clockwise sense by default, if the questions are asked for clockwise sense, the answer evaluted using above formulaes should be written with sign change.

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