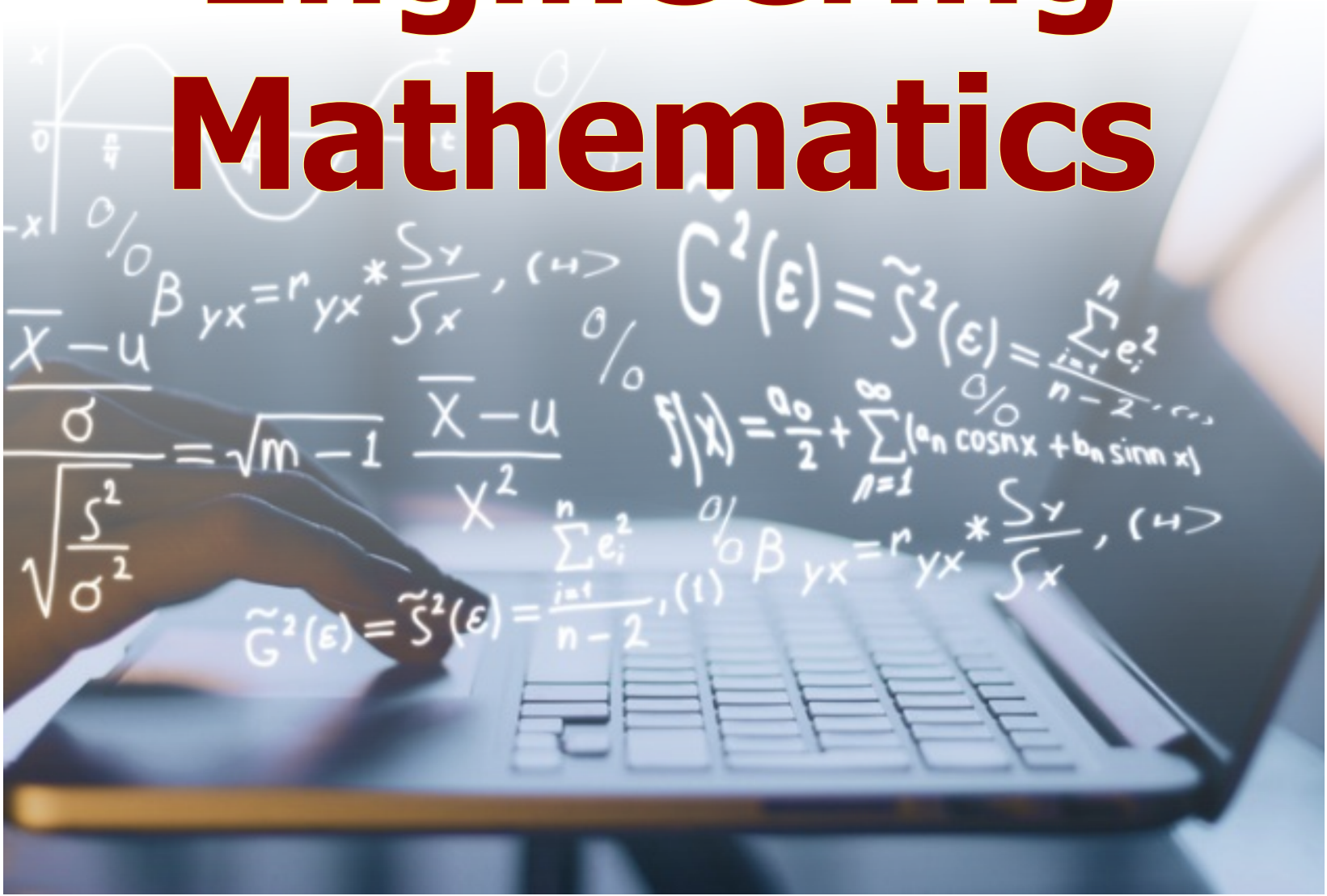
Engineering Mathematics



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(This Module shall only be Used for Educational Purpose.)

Engineering Mathematics

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1 BASIC CALCULUS

1.1. Introduction

1.1.1 Limits, Continuity and Differentiability

(a) As x tends to a ($x \to a$) $\Rightarrow$ x is moving towards a

A value l is said to be limit of a function $f(x)$ at $x \to a$ if $f(x) \to l$ as $x \to a$.

It is mathematically defined as

$$\lim_{x \to a} f(x) = l = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

That is, Limit exist at any point, if LHL = RHL

A function f(x) is said to be continuous at x = a, if

$$\lim_{x \to a} f(x) = l = f(a) = f(x)|_{x=a}$$

That is, for a function to be continuous at any point, RHL = LHL = Value of function at point $x = a$.

Note:
- For $\lim_{x \to a} f(x)$ to exist, the function need not be continuous at $x = a$.
- But for $f(x)$ to be continuous at x = a, $\lim_{x \to a} f(x)$ should exist.

- Continuity from Left : $\lim_{x \to a^-} f(x) = f(a)$
- Continuity from Right : If $\lim_{x \to a^+} f(x) = f(a)$

Continuity in an Open Interval

A function 'f' is said to be continuous in open interval (a, b), if it is continuous at each point of open interval.

Continuity in a Closed Interval

Let 'f' be a function defined on the closed interval (a, b) then 'f' is said to be continuous on the closed interval [a, b], if it is :

1. Continuous from the right at a and
2. Continuous from the left at b and
3. Continuous on the open interval (a, b).

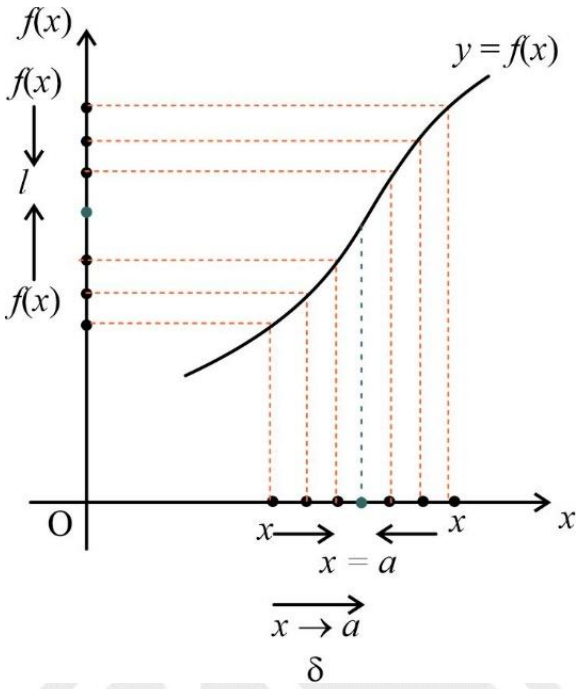


Fig. 1.1

(b) Concept of differentiability

A continuous function $f(x)$ is said to be differentiable at $x = a$, if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists, that is, RHL and LHL exist at a point under consideration in $f'(x)$.

$f'(x)|_{x=a} = f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$f'(a) = \tan\theta$, where θ is the angle made by the tangent to the curve at x=a with x – axis.

(c) Some Standard Derivatives

(i) $\frac{d}{dx}(x^n) = n.x^{n-1}$

(ii) $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\frac{d}{dx}(\cos x) = -\sin x$

(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$

(v) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vi) $\frac{d}{dx}(\sec x) = \sec x\,.\tan x$

(vii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; -1 < x < 1$

(ix) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

(x) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xi) $\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1+x^2}$

(xii) $\frac{d}{dx}(sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

(xiii) $\frac{d}{dx}(cosec^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

(xiv) $\frac{d}{dx}(log_a x) = \frac{1}{x\, log_e a}$

(xv) $\frac{d}{dx}(log_e x) = \frac{1}{x}$

(xvi) $\frac{d}{dx}(a^x) = a^x.log_e a$

(xvii) $\frac{d}{dx}(e^x) = e^x$

(xviii) $\frac{d}{dx}(|x|) = \frac{|x|}{x}$, $(x \neq 0)$

(xix) $\frac{d}{dx}(x^x) = x^x(1 + log_e x)$

(xx) $\frac{d}{dx}(sinh\, x) = cos\, h\, x$

(d) Product rule of differentiation

(i) $\frac{d}{dx}\big(f(x).g(x)\big) = f(x).g'(x) + f'(x).g(x)$

(ii) $d(uvw) = uvw' + uv'w + u'vw$

(e) Quotient rule of differentiation

$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).f'(x)-f(x).g'(x)}{(g(x))^2}, (g(x) \neq 0)$

(f) Logarithmic differentiation:

Taking log might help in differentiation of a function. For example if $y = v^u$ then we can take log both side and differentiable to get $\frac{dy}{dx}$

(g) Differentiation in parametric from :

If we write x and y in term of find variable 't' that is x = f(t), y = ω(t), then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

(h) Greatest Integer function / step function / integer part function

$f(x) = [x] = n, \forall\, n \leq x < n+1 \text{where}, n \in Z$

$\lim_{x\to a}[x] = \nexists$ if a is an integer ($\therefore \nexists$ = do not exist)

L.H.L. = $\lim_{x\to a^-}[x] = a - 1$

R.H.L. = $\lim_{x\to a^+}[x] = a$

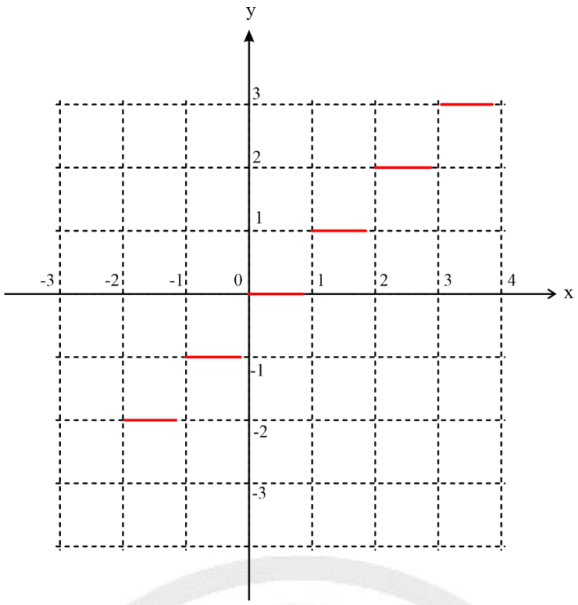


Fig.1.2. Greatest Integer

(i) Properties of Limits

(i) $\lim_{x\to a}(f(x)\pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}\big(f(x).g(x)\big) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \left(\lim_{x\to a} g(x) \neq 0\right)$

(iv) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x) = \nexists$, then $\lim_{x\to a} f(x).g(x)$ **may exist**

Example: Let $f(x) = \sin x$, $g(x) = \frac{1}{x}$, $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0}\frac{1}{x} = \nexists$

But $\lim_{x\to 0} \sin x.\frac{1}{x} = 1$

(v) Indeterminate form III (0^0, 1^∞, ∞^0)

If $y = \lim_{x\to a}\left[f(x)\right]^{\phi(x)}$

Then, $\log y = \lim_{x\to a}\phi(x)\log\left[f(x)\right]$

Thus 0^0, 1^∞, ∞^0 will convert into $\infty\times 0$ from which can be solved easily.

(vi) If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \neq \left(\frac{0}{0}\right)$

If $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$ and so on

(vii) If $\lim_{x\to a}\big(f(x).g(x)\big) = 0\times\infty \Rightarrow \lim_{x\to a}\frac{f(x)}{\left(\frac{1}{g(x)}\right)} = \frac{0}{0}$ (Apply L- Hospital Rule again)

(j) Some Standard Limits

(i) $\lim_{x\to 0}\frac{\sin x}{x}=1$

(ii) $\lim_{x\to 0}\frac{\tan x}{x}=1$

(iii) $\lim_{x\to 0}\frac{1-\cos ax}{x^2}=\frac{a^2}{2}$

(iv) $\lim_{x\to \infty}\frac{\sin x}{x}=0$

(v) $\lim_{x\to \infty}\frac{\cos x}{x}=0$

(vi) $\lim_{x\to 0}(1+ax)^{b/x}=e^{ab}$

(vii) $\lim_{x\to \infty}\left(1+\frac{a}{x}\right)^{bx}=e^{ab}$

(viii) $\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x}=\sqrt{ab}$

(ix) $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+....+n^x}{n}\right)^{1/x}=\sqrt[n]{n!}$

(x) $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a\,;\lim_{x\to 0}\frac{e^x-1}{x}=1$

(xi) $\lim_{x\to 0}x.\sin\left(\frac{1}{x}\right)=0$

1.2 Mean Value Theorems

1.2.1 Lagrange's Mean Value Theorem (LMVT)

If $f(x)$ is continuous in [a, b] and it is differentiable in (a, b) then ∃ at least one point 'c' such that c ∈ (a, b) and

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Here $f'(c)$ slope of tangent to $f(x)$ at $x = c$.

Tangent at x = c is parallel to the line connecting the points A and B

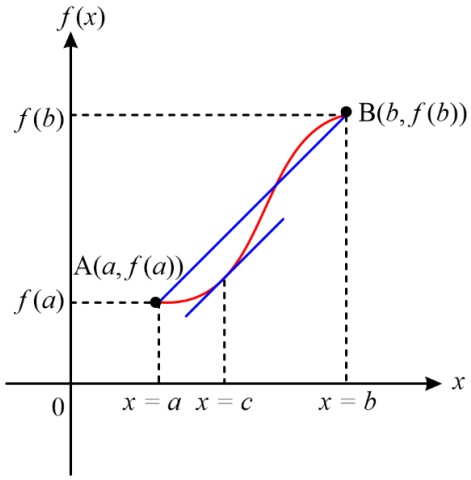


Fig.1.3. LMVT

1.2.2 Rolle's Mean Value Theorem

If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and $f(a) = f(b)$ then $\exists$ at least one-point c $\in (a, b)$ such that $f'(c) = 0$.

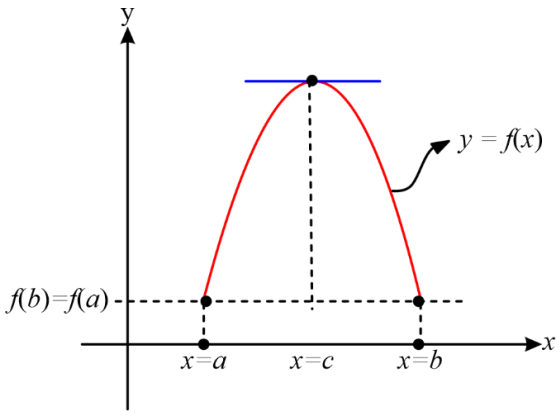


Fig. 1.4. Rolle's mean value

1.2.3 Cauchy's Mean Value Theorem

If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b) then $\exists$ at least one value of 'c' such that $c \in (a, b)$ and $\frac{g'(C)}{f'(C)} = \frac{g(b)-g(a)}{f(b)-f(a)}$

1.3 Increasing and Decreasing Functions

1.3.1 Increasing Functions

A function $f(x)$ is said to be increasing, if $f(x_1) < f(x_2)\ \forall\ x_1 < x_2$

Or

A function $f(x)$ is said to be increasing, if $f(x)$ increases as x increases.

For a function $f(x)$ to be increasing at the point x=a, $f'(a) > 0$.

Example:

e^x, $\log_e x$ $\rightarrow$ Monotonically increasing functions

sin x in $(0, \pi/2) \rightarrow$ non-monotonic functions

1.3.2 Decreasing Functions

A function $f(x)$ is said to be a decreasing function, if $f(x_1) > f(x_2) \forall x_1 < x_2$

A function $f(x)$ is said to be decreasing function, if $f(x)$ decreases as x increases.

Example: e^{-x} $\rightarrow$Monotonically decreasing function, sin x in $\left(\frac{\pi}{2}, \pi\right)$

1.4. Concept of Maxima and Minima

Let $f(x)$ be a differentiable function, then to find the maximum (or) minimum of $f(x)$.

(1) Find $f'(x)$ and equate to zero.

(2) Solve the resulting equation for x. Let its roots be $a_1, a_2, \ldots$ then $f(x)$ is stationary at $x = a_1, a_2, \ldots\ldots$ Thus $x = a_1, a_2, \ldots \ldots$ are the only points at which $f(x)$ can be maximum or a minimum.

(3) Find $f''(x)$ and substitute in it by terms $x = a_1, a_2, \ldots\ldots$ wherever $f''(x)$ is negative, we have a maximum and wherever $f''(x)$ is positive, we have a minimum.

(4) If $f''(a_1) = 0$, find $f'''(x)$ put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. If $f'''(a_1) = 0$, find $f^{iv}(x)$ and put $x = a_1$ in it. If $f^{iv}(a_1)$ is negative, we have maximum at $x = a_1$, if it is positive there is a minimum at $x = a_1$. If $f^{iv}(a_1)$ is zero, we must find $f^{v}(x)$, and so on. Repeat the above process for each root of the equation $f'(x) = 0$.

Example: $x = 0$ is a critical point of $f(x) = x^3$

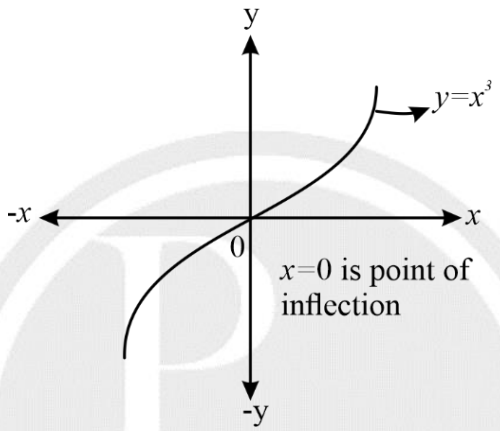


Fig. 1.5. Graph of x^3

$$f(x) = x^3$$

$$\Rightarrow \quad f'(x) = 3x^2 = 0 \quad \Rightarrow \quad x = 0$$

$$f''(x) = 6x \Rightarrow f''(0) = 6(0) = 0$$

- **Global maxima and minima :**
  We first find local maxima and minima and then calculate the value of 'f' at boundary points of interval given e.g. (a, b) we find f(a) and f(b) and compare it with the values of local maxima and minima. The absolute maxima and minima can be decided then.

1.5. Taylor Series

If $f(x)$ is continuously differentiable $(f'(x), f''(x), f'''(x), \ldots\ldots \text{exists})$ then the Taylor series expansion of $f(x)$ about the point $x = a$ is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3. + \ldots\infty$$

If $a = 0$, then $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots\ldots\infty$ (Remember that Mc-Lauren Series is same as Taylor Series if a = 0)

The coefficient of $(x - a)^n$ in the Taylor series expansion of $f(x)$ is $\frac{f^n(a)}{n!}$.

The general expansion of Taylor series is given by $f(x+h) = f(x) + h.\frac{f'(x)}{1!} + h^2.\frac{f''(x)}{2!} + h^3.\frac{f'''(x)}{3!} + \ldots\ldots\infty$

- Finding the expansion of e^x about $x = 0$

$f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1; f"(0) = f'''(0) = f''''(0) = \ldots = 1$

$f(x) = e^x = 1 + (x-0)\frac{1}{1!} + (x-0)^2.\frac{1}{2!} + (x-0)^3.\frac{1}{3!} + \ldots..$

$\Rightarrow \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$

1.6 Integral Calculus

If $F(x)$ is anti-derivative of $f(x)$. That is, continuous and differentiable in (a, b), then we write $\int_{x=a}^{x=b} f(x)\,dx = F(b) - F(a)$. Here $f(x)$ is integrand

If $f(x) > 0\ \forall a \le x \le b, then\ \int_a^b f(x)\,dx$ represents the shaded area in the given figure.

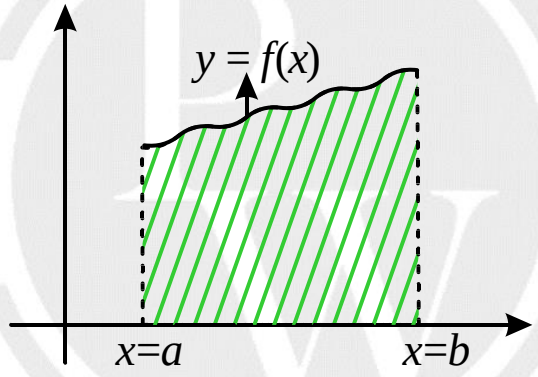


Fig.1. 6. Integration of continuous function

1.6.1 Mean Value Theorem of Integration

If f(x) is continuous in $[a, b]$ and differentiable in (a, b) then '$\exists$' atleast one-point $c \in (a, b)$ such that

$f(c) = \frac{\int_a^b f(x)dx}{(b-a)}$

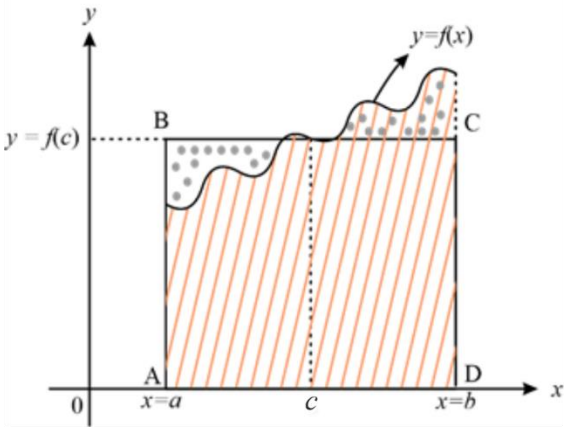


Fig. 1.7. Mean value of integration

1.7. Newton-Leibnitz Rule

If $f(x)$ is continuously differentiable and $\phi(x)$, $\Psi(x)$ are two functions for which the 1st derivative exists, then

$$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(x)dx\right) = f\big(\psi(x)\big).\psi'(x) - f\big(\phi(x)\big).\phi'(x)$$

Example: $\frac{d}{dx}\left(\int_{x}^{x^2} \sin x\, dx\right) = \sin(x^2).2x - \sin x.1 = 2x\sin(x^2) - \sin x$

1.8. Some Standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ (n \neq -1)$

2. $\int \frac{1}{x} dx = \log_e |x| + C$

3. $\int \sin x\, dx = -\cos x + C$

4. $\int \cos x\, dx = \sin x + C$

5. $\int \frac{f'(x)}{f(x)} dx = \log_e |f(x)| + C$

6. $\int \tan x\, dx = -\int -\frac{\sin x}{\cos x} dx = -\log_e |\cos x| + C$

$\Rightarrow \int \tan x\, dx = \log_e |\sec x| + C$

7. $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \log_e |\sin x| + C = -\log_e |\operatorname{cosec} x| + C$

8. $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \log_e |\sec x + \tan x| + C$

9. $\int \operatorname{cosec} x dx = \log_e |\operatorname{cosec} x - \cot x| + C$

10. $\int a^x dx = \frac{a^x}{\log_e a} + C$

11. $\int \frac{1}{x.\log_e a} dx = \log_a x + C$

12. $\int x^x (1 + \log_e x)\, dx = x^x + C$

13. $\int f(x).f'(x) dx = \frac{1}{2}\big(f(x)\big)^2 + C$

14. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2.\sqrt{f(x)} + C$

15. If $f(x)$, $g(x)$ are two functions. that are differentiable, then

$\int f(x)\, g(x) dx = f(x).\int g(x)\, dx - \int [f'(x)\, g(x)] dx + C$

Before integrating the product, the functions $f(x)$ and $g(x)$ are to be arranged according to the ILATE Principle.

Here, ILATE stands for INVERSE LOGARITHMIC ALGEBRAIC TRIGONOMETRIC EXPONENTIAL.

1.9 Properties of Definite Integrals

1. If $f(x)$ is differentiable in interval (a, b), then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. If $\exists$ a point c $\in (a, b)$ such that $f(x)$ is not differentiable, then

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

3. If $f(x)$ is continuously differentiable function,

$\int_{-a}^{a} f(x)\,dx = 2 \times \int_0^a f(x)\,dx$; if $f(-x) = f(x)$, ("$f(x)$ is even function")

$= 0$; if $f(-x) = -f(x)$, ("$f(x)$ is odd function")

4. $\int_0^{2a} f(x)\,dx = 2 \times \int_0^a f(x)\,dx$, if $f(2a - x) = f(x)$

5. $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

6. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \left(\frac{b-a}{2}\right)$

Example:

(i) $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} = \frac{\pi}{4}$

(ii) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx = \int_0^{\pi/2} \frac{1}{1+\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = \frac{\pi}{4}$

(iii) $\int_2^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}} = \left(\frac{3-2}{2}\right) = \frac{1}{2}$

(iv) $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx = \frac{\pi}{4}$

7. $\int_0^{\pi/2} \sin^m x\,dx = \int_0^{\pi/2} \cos^m x\,dx = \frac{(m-1)\times(m-3)\times(m-5)}{m\times(m-2)\times(m-4)} \times \ldots \left(\frac{1}{2}\right)$ (or) $\frac{2}{3} \times K$

Where $K = \pi/2$ if m is even

$= 1$ if m is odd.

8. $\int_0^{\pi} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{\pi}{ab}$

9. $\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{\pi}{2ab}$

1.10 Length of a Curve

(a) The length of the arc of the curve $y = f(x)$ between the points where x = a and x = b is $s = \int_a^b \sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx$

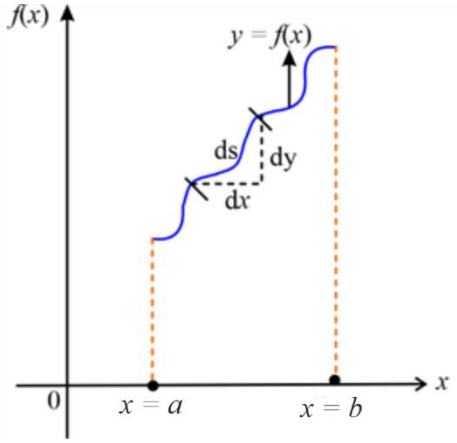


Fig.1.8. Length of the curve

(b) The length of the arc of the curve $x = f(y)$ between the points where y = a and y = b, is $s = \int_a^b \sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}dy$

(c) The length of the arc of the curve $x = f(t), y = f(t)$ between the points where t = a and t = b, is $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

(d) The length of the arc of the curve $r = f(\theta)$, between the points where $\theta = \alpha$ and $\theta = \beta$, is $s = \int_\alpha^\beta \sqrt{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]}d\theta$

1.11 Surface Area of Solid generated by revolving a curve about a fixed axis

Elemental Surface Area

$dA = 2\pi y \times ds = 2\pi y ds$

$\Rightarrow$ Total surface area = A = $\int_{x=a}^{x=b} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

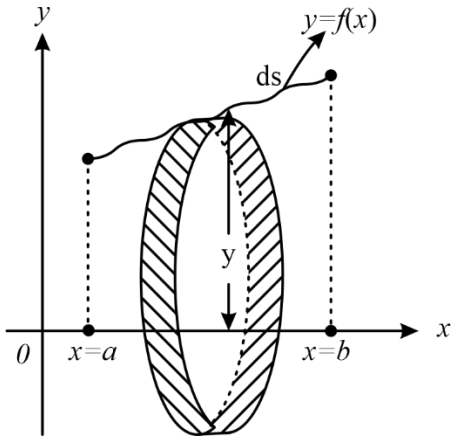


Fig.1.9. Surface area

1.12 Volume of the solid

A. The volume of the solid obtained by revolving the curve y = $f(x)$ between the lines $x = a$ and $x = b$ is given by

$\Rightarrow \quad V \approx \int_{x=a}^{x=b} \pi y^2 dx$

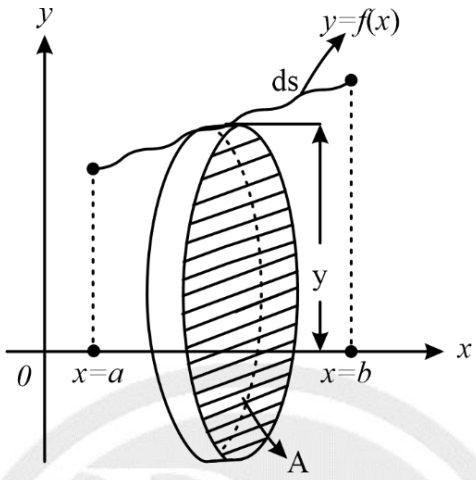


Fig. 1.10. Volume of the solid

B. **Revolution about the y-axis.** Interchanging x and y in the above formula, we see that the volume of the solid generated by the revolution, about y-axis, of the area, bounded by the curve $x = f(y)$, the y-axis and the abscissa y = a, y = b is $\int_a^b \pi x^2 dy$.

1.13 Gamma Function

The integral $\int_0^\infty e^{-x}.x^{n-1}\,dx, (n > 0)$ is called Gamma function of n. It is denoted by $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$.

Note : $\int_0^{\pi/2} \sin^m x \cos^n x dx = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left[\frac{m+n+2}{2}\right]}$

Where Γ(x) is called the gamma function.

1.13.1 Properties of Gamma Function

(i) $\Gamma n = (n-1)!$

(ii) $\Gamma(n+1) = (n)!$

(iii) $\Gamma(n+1) = n\Gamma n$

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

1.14 Beta Function

The function β (m, n) = $\int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ (m, $n > 0$) is called β function of m and n.

1.14.1 Properties of β function

(i) $\beta(m,n) = \frac{\Gamma m.\Gamma n}{\Gamma(m+n)}$

(ii) $\beta(m,n) = \beta(n,m)$

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$\beta(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

(iv) $sin^p\theta.cos^q\,\theta\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right), (p,q > -1)$

1.15 Area between the curves

If the function f(x) > g(x) for all values of x between x=a and x=b then

$A = \int_a^b f(x)dx - \int_a^b g(x)dx \quad \Rightarrow \quad A = \int_a^b \big(f(x) - g(x)\big)\,dx$

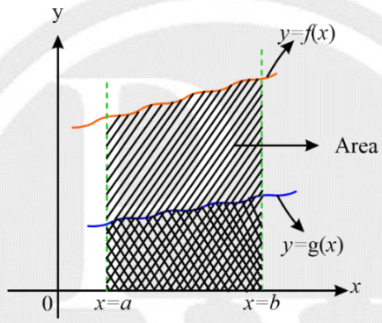


Fig. 1.11. Area under curve

Note : Area bounded by curve $r = f(\theta)$ between $\theta = \alpha$ and β is $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$

1.16 Multi Variable Calculus

(a) Continuity of a function

A function *f*(x, y) is said to be continuous at (*a*, *b*), if $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$

(b) Differentiation of a two-variable function

If *f*(*x*, *y*) is a continuous function, then the derivative of *f*(*x*, *y*) with respect to *x* treating *y* as constant is given by

$$p = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

The derivative of *f*(*x*, *y*) with respect to *y* treating *x* as constant is given by

$$q = \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

(c) Homogenous Function

A function *f* (x, y) is said to be homogenous function of degree 'n' if $f(kx,ky) = k^n.f(x,y)$.

Example: $f(x,y) = x^3 - 3x^2y + 3xy^2 + y^3$

$\Rightarrow f(kx,ky) = (kx)^3 - 3(kx)^2(ky) + 3(kx).(ky)^2 + (ky)^3$

$= k^3(x^3 - 3x^2y + 3xy^2 + y^3)$

$= k^3.f(x,y) \Rightarrow f(x,y)$ is a homogenous function of degree '3'.

(d) Euler's Theorem

If $f(x, y)$ is a homogeneous function of degree 'n' then

(i) $x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nf$

(ii) $x^2.\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

If $f(x, y) = g(x, y) + h(x, y) + \phi(x, y)$ where $g(x, y)$, $h(x, y)$ and $\phi(x, y)$ are homogenous functions of degrees m, n and p respectively, then

$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = m.g(x,y) + n.h(x,y) + p.\phi(x,y)$

$x^2.\frac{\partial^2 f}{\partial x^2} + 2xy.\frac{\partial^2 f}{\partial x \partial y} + y^2.\frac{\partial^2 f}{\partial y^2} = m(m-1).g(x,y) + n(n-1).h(x,y) + p(p-1).\phi(x,y)$

(e) Total derivative:

(i) If u = f(x, y) and if x = ω(t), y = v(t) then $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$

(ii) If u be a function of x and y, where y is a function of x, then $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$

(iii) If u = f(x, y) and $x = f_1(t_1, t_2)$ and $y = f_2(t_1, t_2)$, then

$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_1}$ and $\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}.\frac{\partial y}{\partial t_2}$

(iv) If x and y are connected by an equation of the form f(x, y) = 0, then $\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$

(f) Concept of Maxima and Minima in Two Variables

If $f(x, y)$ is a two-variable differentiable function, then to find the maxima (or) minima.

Step-1: Calculate $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$ and equate $p = 0$, $q = 0$

Let (x_0, y_0) be a stationary point.

Step-2: Calculate r, s, t where $r = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)}$; $s = \left.\frac{\partial^2 f}{\partial x.\partial y}\right|_{(x_0,y_0)}$; $t = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)}$

Case (i): If $rt - s^2 > 0$ and $r > 0$, then the function $f(x, y)$ has minimum at (x_0, y_0) and the minimum value is $f(x_0, y_0)$.

Case (ii): If $rt - s^2 > 0$ and $r < 0$, then the function $f(x, y)$ has maximum at (x_0, y_0) and the maximum value is

$f(x_0, y_0)$.

Case (iii): If $rt - s^2 < 0$; then we cannot comment on the existence of maxima and minima.

Such stationary points where $rt - s^2 = 0$ are called **saddle points**.

(g) Concept of Constraint Maxima and Minima (Method of Lagrange's unidentified multipliers).

If $f(x, y, z)$ is a continuous and differentiable function, such that the variables x, y and z are related/constrained by the equation $\phi(x, y, z) = C$ then to calculate the extreme value of $f(x, y, z)$ using Lagrange's Method of unidentified multipliers.

Step-1: Form the function $F(x, y, z) = f(x, y, z) + \lambda\{\phi(x, y, z) - C\}$, where λ is a multiplier.

Step-2: Calculate $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ and equate them to zero

Step-3: Equate the values of λ from the above 3 equations and obtain the relation between the variables x, y and z.

Step-4: Substitute the relation between x, y and z in $\phi(x, y, z) = C$ and get the values of x, y, z. Let they be (x_0, y_0, z_0).

Step-5: Calculate $f(x_0, y_0, z_0)$

The value $f(x_0, y_0, z_0)$ is the extreme value of $f(x, y, z)$.

(h) Multiple Integrals

If $f(x, y)$ is continuous and differentiable at every point within a region 'R' bounded by some curves is given by

$$I = \iint_R f(x,y)\,dxdy$$

If there is a double integral,

$$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x,y)dydx \quad [\text{Let } \psi(x) > \phi(x)]$$

Then I = area of the region bounded by the curves, $y = \phi(x)$; $y = \Psi(x)$, $x = a$ and $x = b$ if $f(x, y) = 1$

Value of x – co-ordinate of centroid of the region bounded by $y = \phi(x)$; $y = \psi(x)$; $x = a$, $x = b$ if $f(x, y) = x$

(i) Change of Orders of Integration

$$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x,y)dydx \rightarrow I = \int_{y=c}^{y=d} \int_{x=g(y)}^{x=h(y)} f(x,y)dxdy$$

In change of order of Integration, the order of the Integrating variables changes and the limits as well.

Note : When limits are constants, the order of integration does not matter,

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)dxdy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)dydx$$

1.17 Jacobian of the Transformation

(i) The Jacobian of the transformation, $x = f_1(u,v), y = f_2(u,v)$ is defined as,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

(ii) The Jacobian of the transformation,

$x = f_1(u,v,w), y = f_2(u,v,w), z = f_3(u,v,w)$ is defined as

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

1.18 Change of Variables Formula

(i) $\iint\limits_R f(x,y)dx\,dy = \iint\limits_R f(f_1(u,v), f_2(u,v))\,|J|\,du\,dv$

(ii) $\iiint\limits_R f(x,y,z)dxdydz = \iiint\limits_R f(f_1(u,v,w), f_2(u,v,w), f_3(u,v,w))\,|J|\,du\,dv\,dw$

1.19 Change of Variables

(i) Cartesian to polar co-ordinates :

$x = r\cos\theta$

$y = r\sin\theta$

$J = r$

$dx\,dy = rdrd\theta$

(ii) Cartesian to cylindrical polar co-ordinate :

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$J = r$

$dxdydz = rdr\,d\theta\,dz$

(iii) Cartesian to spherical polar co-ordinates :

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$J = \rho^2\sin\phi$

$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

❑❑❑

2 LINEAR ALGEBRA

2.1 Matrix

An array of elements in horizontal lines (Rows) and Vertical Lines (Columns) is called a Matrix.

Example: $A = \begin{bmatrix} i & n & d & i & a \\ j & a & p & a & n \end{bmatrix}$

2.1.1 Size of Matrix

If a matrix has 'm' rows and 'n' columns, then we say that the size of the matrix is m × n (read as m by n)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23}\ldots\ldots\ldots a_{2n} \\ \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot\cdot \\ a_{m1} & a_{m2} & a_{m3}\ldots\ldots\ldots a_{mn} \end{bmatrix}; \; A = [a_{ij}]_{m\times n} \text{ such that } 1 \le i \le m, 1 \le j \le n \text{ and } a_{ij} = f(i,j)$$

2.1.2 Addition of Matrices

(i) Two matrices $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{p\times q}$ can be added only if m = p & n = q.

(ii) Matrix Addition is commutative (A + B = B + A)

(iii) Matrix Addition is Associative. A + (B + C) = (A + B) + C

(iv) Existence of additive identity : If O be m × n matrix each of whose elements are zero. Then, A + O = A = O + A for every m × n matrix A.

(v) Existence of additive inverse : Let $A=[a_{ij}]_{m\times n}$ then the negative of matrix A is defined as matrix $[-a_{ij}]_{m\times n}$ and is denoted by –A.

⇒ Matrix –A is additive inverse of A. Because (–A) + A = O = A + (–A). Here O is null matrix of order m × n.

(vi) Cancellation laws holds good in case of addition of matrices, which is X = –A.

⇒ A + X = B + X ⇒ A = B

⇒ X + A = X + B ⇒ A = B

(vii) The equation A + X = 0 has a unique solution in the set of all m × n matrices.

2.1.3 Multiplication of Matrices

The multiplication of two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{p\times q}$ $(\Rightarrow AB_{m\times q})$ is feasible only if n = P.

$A_{m\times n} \cdot B_{p\times q} = C$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times2} \quad A_{3\times3} \times B_{3\times2}$$

$$\Rightarrow \begin{bmatrix} a_{11}.b_{11} + a_{12}\cdot b_{21} + a_{13}.b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}.b_{11} + a_{32}\cdot b_{21} + a_{33}\cdot b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}_{3\times2}$$

2.1.4 Properties of Multiplication of Matrices

(i) Matrix Multiplication Need not be commutative.

(ii) Matrix Multiplication is Associative (A(BC)) = ((AB)C)

(iii) Matrix Multiplication is distributive A(B + C) = (AB + AC)

(iv) The product of two Matrices $A_{m\times n}, B_{n\times q}$ (i.e. $AB_{m\times q}$) can be a zero matrix even if $A \neq O \& B \neq O$.

Example: $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- For the multiplication of two matrices $A_{m\times n}$ & $B_{n\times q}$

  (i) The No. of Multiplications required = m n q

  (ii) The number of Additions required = m (n –1) q

2.2 Types of Matrices

(1) Upper triangular Matrix: A matrix $A = [a_{ij}]; 1 \leq i, j \leq n$ is said to be an upper triangular matrix if

$$a_{ij} = 0 \; \forall \; i > j$$

(2) Lower Triangular Matrix: A matrix $A = [a_{ij}]_{n\times n}; 1 \leq i, j \leq n$ is said to be a lower Triangular Matrix

if $a_{ij} = 0 \; \forall \; i < j$

(3) Diagonal Matrix: A matrix $A = [a_{ij}], \forall \; 1 \leq i, j \leq n$ is said to be a diagonal matrix if $a_{ij} = O \; \forall \; i \neq j$

Example: $A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$. The diagonal Matrix is also denoted as $A = diag\,[d_1, d_2, d_3]$

(4) Scalar Matrix: A Matrix 'A' = $[a_{ij}]$; $1 \leq i, j \leq n$ is said to be a scalar Matrix if $a_{ij} = \begin{cases} K; i = j \\ 0; 1 \neq j \end{cases}$

If K = 1, then A → Identity Matrix,

If K = 0, then A → Null Matrix.

(5) Idempotent Matrix:

A Matrix '$A_{n\times n}$' is said to be an idempotent matrix if $A^2 = A$.

Example: $A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

$\Rightarrow A \cdot A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A$

(6) Nilpotent Matrix: A matrix A is said to be nilpotent of class x or index if $A^x = 0$ and $A^{x-1} \neq 0$ i.e. x is the smallest index which makes $A^x = 0$.

Example: The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent class 3, since $A \neq 0$ and $A^2 \neq 0$, but $A^3 = 0$.

(7) Orthogonal Matrix: A matrix A is said to be orthogonal if $A.A^T = I$

Example: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(8) Involutory Matrix: A matrix A is said to be involutory if $A^2 = I$

Example: $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$

2.3 Transpose of a Matrix

For a given matrix $= [a_{ij}]; 1 \leq i \leq m, 1 \leq j \leq n$, we can say that 'B' where $B = [b_{ij}], i \leq i \leq n \;\; i \leq j \leq m$ is the transpose of the Matrix 'A' if $a_{ij} = b_{ji}$

2.3.1 Properties of Transpose of a Matrix

(i) $(A^T)^T = A$

(ii) $(AB)^T = B^T \cdot A^T$

(iii) $(KA)^T = KA^T$ where 'K' is a scalar.

2.4 Conjugate of a matrix

The matrix obtained by replacing each element of matrix by its complex conjugate.

2.4.1 Properties of conjugate matrix

(a) $(\bar{\bar{A}}) = A$ (b) $(\overline{A+B}) = \bar{A} + \bar{B}$

(c) $(\overline{KA}) = \bar{K}\bar{A}$ (d) $(\overline{AB}) = \bar{A}\bar{B}$

(e) $\bar{A} = A$ if A is real matrix

$\bar{A} = -A$ if A is purely imaginary matrix

2.5 Transposed Conjugate of a Matrix

The transpose of conjugate of a matrix is called transposed conjugate. It is represented by A^θ.

(a) $(A^\theta)^\theta = A$

(b) $(A+B)^\theta = A^\theta + B^\theta$

(c) $(KA)^\theta = \bar{K}A^\theta$ (K : Complex number)

(d) $(AB)^\theta = B^\theta A^\theta$

2.6 Trace of a Matrix

Trace is simply sum of all diagonal elements of a matrix.

2.6.1 Properties of Trace of a matrix

Let A and B be two square matrices of order η and λ is scalar then

1. $Tr(\lambda A) = \lambda Tr(A)$
2. $Tr(A+B) = Tr(A) + Tr(B)$
3. $Tr(AB) = Tr(BA)$ [If both AB and BA are defined]

2.7 Type of Real Matrix

(a) Symmetric matrix : $(A)^T = A$

(b) Skew symmetric matrix : $(A^T) = -A$

(c) Orthogonal matrix : $(A^T = A^{-1}, AA^T = I)$

Note : (a) If A and B are symmetric, then (A + B) and (A – B) are also symmetric.

(b) For any matrix AA^T is always symmetric.

(c) For any matrix, $\left(\frac{A+A^T}{2}\right)$ is symmetric and $\left(\frac{A-A^T}{2}\right)$ is skew symmetric.

(d) For orthogonal matrices, $|A| = \pm 1$

(e) We can write any matrix A as a sum of symmetric and skew symmetric matrix $A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$

2.8 Type of complex matrix

(a) Hermitian matrix : $(A^\theta = A)$

(b) Skew-Hermitian matrix: $A^\theta = -A$

(c) Unitary matrix : $(A^\theta = A^{-1}, AA^\theta = I)$

Note : (a) $\frac{A+A^\theta}{2}$ is Hermitian and $\frac{A-A^\theta}{2}$ is skew Hermitian matrix.

(b) We can write any matrix as a sum of Hermitian and skew Hermitian matrix $A = \frac{A+A^\theta}{2} + \frac{A-A^\theta}{2}$

2.9 Determinant

The summation of the product of elements of a row(or) column of a matrix with their corresponding Co-factors.

$$A \cdot adj(A) = |A| \cdot I$$

Determinant can be calculated only if matrix is a square matrix.

Suppose, we need to calculate a 3 × 3 determinant,

$$\Delta = \sum_{j=1}^{3} a_{1j} cof(a_{1j}) = \sum_{j=1}^{3} a_{2j} cof(a_{2j}) = \sum_{j=1}^{3} a_{3j} cof(a_{3j})$$

We can calculate determinant along any row or column of the matrix.

2.9.1 Properties of Determinants

(i) If 'A' is a Square Matrix of size ' $n \times n$ ' and 'k' is a Scalar then

$|K \cdot A_{n\times n}| = K^n \cdot |A_{n\times n}|$

(ii) $|adj(A)| = |A|^{(n-1)}$

(iii) $\left|adj\left(adj(A)\right)\right| = (|A|)^{(n-1)^2}$

(iv) $|AB| = |A| \cdot |B|$

(v) $|(AB)^T| = |B^T| \cdot |A^T|$

(vi) If two rows (or) two columns of a determinant are interchanged, then the determinant changes its sign.

(vii) The determinant of an upper triangular Matrix/a lower triangular Matrix/a diagonal Matrix is the product of the principal diagonal elements of the Matrix.

(viii) The determinant of Every Skew-Symmetric Matrix of odd order $(A_{n\times n})\left('n' is\ odd\right)$ is zero.

(ix) The determinant of an orthogonal Matrix $A_{n\times n}$ is ± 1

(x) The determinant of an Idempotent Matrix is either 0 (or) 1.

(xi) The determinant of an Involuntary Matrix is ± 1

(xii) The determinant of a Nilpotent Matrix is always zero.

(xiii) If the product of two Non-zero Matrices $A_{n\times n} \neq 0; B_{n\times n} \neq 0$ is a zero Matrix $((AB)_{n\times n} = 0)$, then both $|A| = 0$ & $|B| = 0$.

(xiv) If two rows (or) two columns of a Matrix are either equal or Proportional, then the determinant of the Matrix is equal to zero.

(xv) The number of terms in the general expansion of an 'n × n' determinant is $\underline{n!}$

(xvi) Value of the determinant is invariant under row and column interchange i.e., $\left|A^T\right| = |A|$

(xvii) If any row or column is completely zero, then |A| = 0.

(xviii) If any single row or column of the matrix is multiplied by k then the determinant the of new matrix = K|A|

(xix) In a determinant the sum of the product of the element of any row or column with its cofactor gives a determinant of the matrix.

(xx) In determinant the sum of the product of the element of any row or column with a cofactor of another row or column will give zero.

(xxi) |AB| = |A| × |B|

(xxii) Elementary operations don't effect the determinant that is $A \xrightarrow{R_i = R_i + KR_j} B$ then |A| = |B|

$A \xrightarrow{C_i = C_i + KC_j} B$ then |A| = |B|

2.10 Minors, Cofactor and Adjoint of a Matrix

Minor of an element is equal to the determinant of the remaining elements of the matrix, after excluding the row and column containing the particular element. The cofactor of an element can be calculated from the minor of the element. The cofactor of an element is equal to the product of the minor of the element, and -1 to the power of position values of row and column of the element.

$$\text{Cofactor of an Element} = (-1)^{i+j} \times \text{Minor of an Element}$$

Here i and j are the positional values of the row and column of the element.

Example :

If $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of element a_{21}: $M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Co-factor of an element, $a_{ij} = (-1)^{i+j} M_{ij}$

- To design co-factor matrix, we replace each element by its co-factor.
- Adjoint of a matrix = transpose of cofactor matrix
- $A^{-1} = \dfrac{Adj(A)}{|A|}$

2.11 Inverse of a matrix

Inverse of a matrix only exists for square matrices.

$$\left(A^{-1}\right) = \frac{Adj(A)}{|A|} \text{ and } |A| \neq 0$$

Properties:

(a) $AA^{-1} = A^{-1}A = I$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(d) $(A^T)^{-1} = (A^{-1})^T$

(e) The inverse of 2 × 2 matrix should be remembered,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(i) Interchange the diagonal elements and put negative sign on the rest.

(ii) Divide by determinant.

2.12 Rank of a Matrix

- The rank of the matrix refers to the number of linearly independent rows or columns in the matrix. ρ(A) is used to denote the rank of matrix A.
- A matrix is said to be of rank zero when all of its elements become zero.
- The rank of the matrix is the dimension of the vector space obtained by its columns.
- The rank of a matrix cannot exceed more than the number of its rows or columns. The rank of the null matrix is zero.
- The nullity of a matrix is defined as the number of vectors present in the null space of a given matrix. In other words, it can be defined as the dimension of the null space of matrix A called the nullity of A. Rank + Nullity is the number of all columns in matrix A.

A real Number 'r' is said to be the rank of a matrix '$A_{m\times n}$' if

(1) There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.

(2) If the matrix A contains any square sub-matrix of order $(r + 1)$ and above, then the determinant of such a matrix should be zero.

It is mathematically denoted by $\rho(A) = r$

2.12.1 Properties of Rank of a Matrix

(i) $\rho(A_{m\times n}) \leq (m, n)$

(ii) $\rho(AB) \leq min\{\rho(A), \rho(B)\}$

(iii) Rank of transpose of matrix is equal to rank of matrix

(iv) Elementary operations do-not affect the rank the matrix

(v) $\rho(A + B) \leq \{\rho(A) + \rho(B)\}$

2.12.2 Row Echelon Form

A Matrix $A_{m\times n}$ is said to be in row-echelon form if

(i) Number of zeroes before the 1st Non-zero element in any row is less than the number of such zeroes in its succeeding row.

(ii) Zero rows (if any) should lie at the bottom of the Matrix.

$\rho(A_{m\times n})$ = Number of non-zero rows in the Row-Echelon form of A.

2.13 System of Equations

The given system of equations

$$a_{11}x_1 + a_{12}x_{12} + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written in Matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \boxed{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Ax = B

Coefficient Matrix — Variable Matrix — Constants Matrix

The system Ax = B is said to be a homogeneous system if B = 0.

The system of Ax = B is said to be a non-homogeneous system if $B \neq 0$.

2.13.1 Consistency of a non-homogeneous system of Equations

For the above system of non – homogeneous equations, Ax = B; Augmented Matrix = [A/B] = $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

(i) If $\rho(A) = \rho(A/B) =$ Number of unknowns, then the system Ax = B has a unique solution.

(ii) If $\rho(A) = \rho(A/B) <$ Number of unknowns, then the system has infinitely many solutions.

(iii) If $\rho(A) \neq \rho(A/B)$, then the system has no solution.

Number of linearly independent solutions for a system of 'n' equations given by Ax = B is $n - \rho(A)$

2.13.2 Consistency of Homogeneous System of Equations

$a_{11}x + a_{12}y + a_{13}z = 0$

$a_{21}x + a_{22}y + a_{23}z = 0$

$a_{31}x + a_{32}y + a_{33}z = 0$

$$\text{Ax} = 0 \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} [A/B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix}_{3\times4}$$

If $\rho(A) = \rho(A/B) = n$ $(i.e\ |A| \neq 0)$; the system has a unique solution.

(Trivial solution; x = 0, y = 0, z = 0)

If $\rho(A) = \rho(A/B) < n(|A| = 0)$; the system has infinitely many solutions (Non-trivial solution exists for the system).

2.14 Linear Combination of Vectors

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' rows vectors, then the combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots\ldots + k_nx_n$ is called a linear combination of $x_1, x_2, \ldots\ldots, x_n (k_1, k_2, k_3, \ldots\ldots k_n$ are scalars)

(1) The linear combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots\ldots + k_nx_n$ is said to be linearly dependent if $k_1x_1 + k_2x_2 + k_3x_3 + \ldots\ldots + k_nx_n = 0$ when $k_1, k_2, k_3, \ldots\ldots, k_n$ (NOT All zeroes).

If $x_1 = [a_1 \quad b_1 \quad c_1]; x_2[a_2 \quad b_2 \quad c_2]; x_3 = [a_3 \quad b_3 \quad c_3]$, then the vectors x_1, x_2, x_3 are said to be linearly dependent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

(2) The combination $k_1x_1 + k_2x_2 + \ldots\ldots + k_nx_n$ is said to be linearly independent if $k_1x_1 + k_2x_2 + \ldots\ldots\ldots + k_nx_n = 0$ when $k_1 = k_2 = k_3 = \ldots\ldots k_n = 0$

2.14.1 Eigen Values and Eigen Vectors

For any square Matrix $A_{n\times n}$, the equation $|A - \lambda I| = 0$ where 'λ' is a scalar is called the characteristic equation.

The roots of the characteristic equation of a Matrix are called Eigen Values.

2.14.2 Properties of Eigen Values

(i) If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are 'n' Eigen Values of $A_{n\times n}$, then

(a) Sum of Eigen Values of 'A' = $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n = \sum_{i=1}^{n} \lambda_i = trace(A)$ = Sum of Principal diagonal elements

(b) Product of all the Eigen Values of 'A' = $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots\ldots \lambda_n = \prod_{i=1}^{n} \lambda_i = |A|$

(c) Eigen Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots\ldots \lambda_n^m$

(d) Eigen Values of adj(A) are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots\ldots, \frac{|A|}{\lambda_n}$

(e) Eigen Values of A & A^T are the same.

(f) Eigen Values of $k_1A + k_2I$ (Where k_1 and k_2 are scalar) are

$$k_1\lambda_1 + k_2, k_1\lambda_2 + k_2, k_1\lambda_3 + k_2, k_1\lambda_4 + k_2, \ldots\ldots\ldots k_1\lambda_n + k_2$$

(ii) '0' is always an Eigen Value of an odd-order Skew-Symmetric Matrix.

(iii) Eigen Values of a Real Symmetric Matrix are always real.

(iv) Eigen Values of the Skew-Symmetric Matrix are either zero (or) purely Imaginary.

(v) The Eigen values of an Orthogonal Matrix are of unit modulus.

(vi) If the sum of all the elements in a row (or Column) is constant (= k) for all the rows (or columns) in the matrix respectively, then 'k' is an Eigen Value of the Matrix.

Example: If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and if $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3 = k,$

then 'k' is an Eigen Value of 'A'.

(vii) The Eigen Values of an upper triangular Matrix, a lower triangular Matrix, a diagonal Matrix are the Principal diagonal elements of the Matrix.

2.15 Eigen Vector

A non-zero column vector $X_{n\times1}$ is said to be an Eigen Vector of $A_{n\times n}$ corresponding to the Eigen Value 'λ', if $AX = \lambda X (X \neq 0)$.

2.15.1 Properties of Eigen Vectors

(i) Eigen Vectors of A & A^T are not the same.

(ii) Eigen Vectors of A & A^M are same.

(iii) The Eigen Vectors of a Real Symmetric Matrix are always orthogonal.

(iv) The number of linearly independent Eigen Vectors of '$A_{n\times n}$' is equal to the number of distinct Eigen Values of '$A_{n\times n}$'.

2.15.2 Cayley Hamilton Theorem

Every Matrix satisfies its characteristic equation.

This means that, if $c_0\lambda^n + c_1\lambda^{-1} + ... + c_{n-1}\lambda + c_n = 0$ is the characteristic equation of a square matrix *A* of order *n*, then

$c_0A^n + c_1A^{n-1} + ... + c_{n-1}A + c_nI = 0$ …(i)

Note: When λ is replaced by A in the characteristic equation, the constant term c_n should be replaced by c_nI to get the result of the Cayley-Hamilton theorem, where *I* is the unit matrix of order *n*.

Also, 0 in the R.H.S. of (i) is a null matrix of order *n*.

2.16. Subspace (Basis of Dimensions)

2.16.1 Vector

An ordered n-tuple of numbers is called an n-vector.

2.16.2 Linearly Independent and Dependent Vector

Let X_1 and X_2 be the non-zero vectors:

- $\{x_1, x_2, …., x_k\}$ are linearly independent if $r_1x_1 + r_2x_2 + … + r_k x_k = 0$ only for $r_1 = r_2 = … + r_k = 0$.
- The vectors $x_1, r_2, …., x_k$ = are linearly dependent if they are not linearly independent; that is, if there exist scalars $r_1, r_2, … , r_k$ which are not all zero such that

  $r_1x_1 + r_2x_2 + …. + r_k x_k = 0$

Note: Let $X_1, X_2…….X_n$ be 'n' vector of matrix *A*.

- If rank (*A*) = number of vectors then vector $X_1, X_2…..X_n$ are linearly independent.
- If rank (*A*) ≠ number of vectors then vector $X_1, X_2 ….. X_n$ are linearly dependent.

2.16.3 Vector Space R^n

If n positive integer, then an ordered n-tuple is a sequence of n real numbers ($\alpha_1, \alpha_2, \ldots \alpha_n$). the set of all ordered *n*-tuples is called n-space and is denoted by $\mathbb{R}^n$.

2.16.4 Subspaces of an N-vector space V_n

A non-empty set S, of vectors of $V_n(F)$,is called a subspace of $V_n(F)$, if

- ξ_1, ξ_2 are any two members of S, then $\xi_1 + \xi_2$ is also a member of S; and
- ξ is a member of S, and k is a scalar then $k\xi$ is also a member of S.

  Briefly, we may say that a set S of vectors $V_n(F)$ is a subspace of $V_n(F)$ it closed w.r.t. the compositions of "addition" and "multiplication with scalars".

  Every subspace of V_n contains the zero vector; being the product of any vector with the scalar zero.

2.16.5 Construction of Subspaces

- **A subspace Spanned by a Set of Vectors:** A subspace that arises as a set of all linear combinations of any given set of vectors is said to be spanned by the given set of vectors.
- **Basis of a subspace:** A set of vectors is said to be a basis of a subspace, if
  - The subspace is spanned by the set, and
  - The set is linearly independent.

Note: If we have N vectors and they are independent then they span N-dimension space. But if they are dependent then they span only a subspace of N-dimension space.

2.16.6 Orthogonality of Vectors

- Two vectors are orthogonal if each is non-zero and $X_1^T X_2 = 0$
- If n vectors $X_1, X_2 \ldots X_n$ each of n dimensions is orthogonal then they are surely linearly independent and form the basis for n-dimension space.
- The set of the vector is orthonormal if they are orthogonal and have unit magnitude.

2.17 Similar Matrices

- Two matrix A and B are similar if there exist a non singular matrix P such that $B = P^{-1}AP$
- Similar matrix has same eigen valves
- If A is similar to B then B is also similar to A
- If A is similar to B and B is similar to C then A is similar to C.

2.18 Diagonalization of a matrix

Finding a matrix D which is a diagonal matrix and which is similar to A is called diagonalization i.e., we wish to find a non-singular matrix M such that $A = M^{-1}DM$ where D is a diagonal matrix.

2.18.1 Condition for a Matrix to be Diagonalizable

1. A necessary and sufficient condition for a matrix $A_{n \times n}$ to be diagonalizable is that the matrix must have n linearly independent eigen vectors.

2. A sufficient (but not necessary) condition for a matrix $A_{n \times n}$ to be diagonalizable is that the matrix must have n linearly independent eigen values.

   This is because if a matrix has n linearly independent eigen values then it surely has n linearly independent eigen vectors (although the converse of this is not true).

❑❑❑

3 PROBABILITY AND STATISTICS

3.1 Random Experiment

The experiment in which the outcome is uncertain is called a Random Experiment (RE).

Example: Flipping a coin, rolling a pair of dice, Picking a ball from a bag.

3.1.1 Sample Space

The set contains all the possible outcomes of a random experiment. It is denoted by 'S'.

If RE is flipping a coin, S = {Head, Tail}

If RE is rolling a dice, S = {1,2,3,4,5,6}

3.2 Event

Any subset of sample space 'S' is called an Event.

Example: If RE is flipping a coin, then the occurring of a Head is an Event.

If RE is rolling a dice, then getting an odd number is an Event.

3.2.1 Probability of an Event

If 'A' is any event with in the sample space 'S' of a Random experiment, then the probability of event 'A' is given by

$$P(A) = \frac{\text{No. of outcomes favouring event 'A' to happen}}{\text{Total number of elements in 'S'}} = \frac{n(A)}{n(S)}$$

Probability of getting an Even Number when a dice is rolled.

$$P(\text{Even Number}) = \frac{3}{6} = 0.5$$

S = {1,2,3,4,5,6},

A = {2,4,6}

Note: Probability can also be expressed as odds if favour and odds against an event:

- **Odds is favour of an event:**
  Odds in **favour** of an event = Number of successes : Number of failures = m: (n – m).
- **Odds against an event:**
  Odds against an event = Number of failures : Number of successes = (n – m) : m.

3.2.2 Axioms Probability

(i) If 'A' is any event with in the sample space 'S' of a RE, then $0 \le P(A) \le 1$

$$\frac{0}{n(S)} \le \boxed{\frac{n(A)}{n(S)}} \le \frac{n(S)}{n(S)}$$

$$\boxed{0 \le P(A) \le 1}$$

(ii) P(S) = 1

When a RE is conducted the experiment yields a possible outcome.

3.2.3 Types of Events

(i) Mutually Exclusive Events:

If A, B are two events within a sample space 'S', then A & B are said to be mutually exclusive if $A \cap B = \phi$.

Example: If 'A' is the event of getting a prime number when a dice is rolled and 'B' is the event of getting a composite number when a dice is rolled then

$S = \{1,2,3,4,5,6\}$, $A = \{2,3,5\}$, $B = \{4,6\} \Rightarrow A \cap B = \phi \Rightarrow P(A \cap B) = 0$

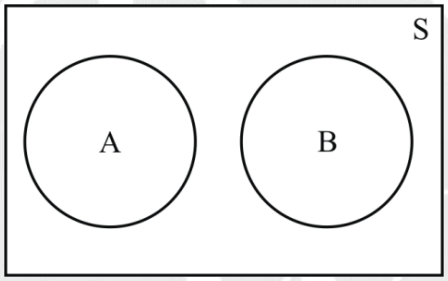


Fig. 5.1. Mutually exclusive event

(ii) Mutually Exhaustive Events:

If 'A', and 'B' are two events within a sample space 'S', then 'A' & 'B' are said to be mutually exhaustive if $A \cup B = S$

Example: If 'A' is the event of getting an odd number when a dice is rolled and 'B' is the event of getting an Even Number, then

$A \cup B = S$

$S = \{1,2,3,4,5,6\}$

$A = \{1,3,5\}$, $B = \{2,4,6\}$

$\Rightarrow$ $A \cup B = S$

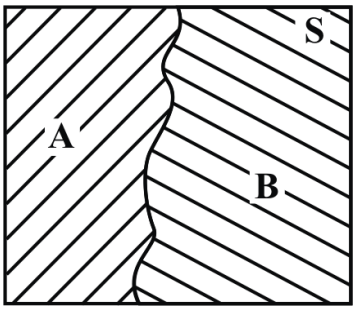


Fig. 5.2. Mutually exhaustive event

(iii) Independent Events:

Two events 'A' & 'B' within the sample space 'S' (or) within two different sample spaces 'S_1' & 'S_2' are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$.

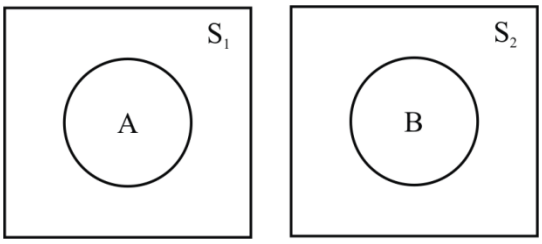


Fig. 5.3. Independent Event

(iv) Impossible Event (ϕ):

The event with zero probability is called an Impossible Event $P(\phi) = 0$.

3.3 Addition Theorem of Probability

If A, and B are two events with a sample space 'S' of a Random Experiment, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

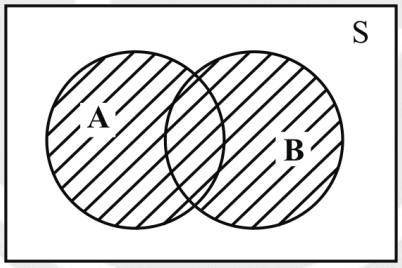


Fig. 5.4. Addition theorem

$\Rightarrow$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

When A, and B are mutually exclusive events, $A \cap B = \phi$.

$\Rightarrow$ $P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

- If $E_1, E_2, E_3, \ldots\ldots E_n$ are mutually exclusive events ($E_i \cap E_j = \phi$), then $P(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$

  $= P(E_1) + P(E_2) + P(E_3) + \ldots\ldots p(E_n)$

3.3.1 De Morgan's Law

- $(A \cup B)^C = A^C \cap B^C$
- $(A \cap B)^C = A^C \cup B^C$

3.3.2 Union and Intersection properties

For any two events A and B:

(a) $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

(b) $P(A^c\cap B^c)=1-P(A\cup B)$

For any three events A, B and C:

(a) $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$

(b) $P(A^c\cap B^c\cap C^c)=1-P(A\cup B\cup C)$

3.3.3 Conditional Probability:

The probability of the happening of event 'A' when it is known that event 'B' has already occurred is given by P(A/B)

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{n(A\cap B)}{n(B)}$$

3.3.4 Joint Probability:

- $f(x, y)$ is the joint probability of two RV'S x, y.
- If the two RV are Independent then
  $f(x, y) = f(x) \cdot f(y)$
- $P(a\le x\le b, c\le y\le d)=\int_a^b\int_c^d f(x,y)dydx$
- $f(x)=\int_{-\infty}^{\infty} f(x,y)dy$
- $f(y)=\int_{-\infty}^{\infty} f(x,y)dx$

3.3.5 Multiplication Theorem of Probability:

If A, and B are two events within a sample space 'S', then P(A/B) · P(B) = P(B/A) · P(A)

P(A/B) = $\frac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B)=P(A/B)\cdot P(B) \rightarrow (1)$

P(B/A) = $\frac{P(B\cap A)}{P(A)} \Rightarrow P(B\cap A)=P(B/A)\cdot P(A) \rightarrow (2)$

From (1) & (2)

$P(A/B)\cdot P(B)=P(B/A)\cdot P(A)$

3.3.6 Total Theorem of Probability:

If $E_1, E_2, E_3, \ldots\ldots E_n$ are 'n' mutually exclusive ($E_i\cap E_j=\phi;\forall i\neq j$) and collectively exhaustive event ($E_1\cup E_2\cup E_3\cup\ldots\ldots\cup E_n = S$) and 'A' is any event with in the sample space 'S', then

$P(A)=P(E_1)\cdot P(A/E_1)+P(E_2)\cdot P(A/E_2)+\ldots\ldots+P(E_n)\cdot P(A/E_n)$

$$\boxed{P(A)=\sum_{i=1}^{n}P(E_i)\bullet P(A/E_i)}$$

3.3.7 Baye's Theorem

If $E_1, E_2, E_3, \ldots\ldots E_n$ are mutually exclusive ($E_i \cap E_j = \phi \forall i \neq j$) and collectively exhaustive event ($E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n = S$) and 'A' is any event with in the sample space 'S', then

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}$$

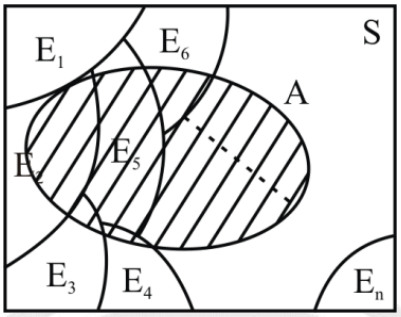


Fig. 5.5. Baye's theorem

3.3.8 Use of permutation and combination

What is combination?

A combination of 'n' objects taken 'r' at a time (r-combination of 'n' objects is an unordered selection of 'r' of the objects).

Number of ways of combining of 'r' object out of 'n' objects without repetition

$$^{\eta}C_r = \frac{n!}{(n-r)!r!}$$

What is permutation?

A combination of 'n' objects taken 'r' at a time (r-combination of 'n' objects is an ordered selection of 'r' of the objects).

Number of ways of selection of r object out of n objects without repetition

$$^{\eta}P_r = \frac{n!}{(n-r)!}$$

Result:

(i) $^{n}C_r = {}^{n}C_{n-r}$

(ii) $^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \ldots. + {}^{n}C_n = 2^n$

(iii) $^{n}C_0 + {}^{n}C_2 + {}^{n}C_4 + \ldots. += 2^{n-1}$

(iv) $^{n}C_1 + {}^{n}C_3 + {}^{n}C_5 + \ldots. += 2^{n-1}$

(v) $0.{}^{n}C_0 + 1.{}^{n}C_1 + 2.{}^{n}C_2 + \ldots. + n.{}^{n}C_n = n.2^{n-1}$

Permutations with Repetition

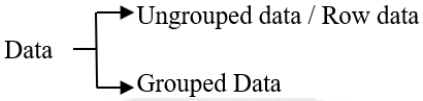
The number of permutations of n objects, where *p* objects are of one kind, q objects are of another kind and the rest, if any, are of a different kind is $\frac{{}^{n}P_{r}}{p!q!}$.

Combination with Repetition

Number of combinations of 'n' distinct things taking 'r' at a time when each thing may be repeated any number of times is given by ${}^{n-1+r}C_{r}$.

3.4 STATISTICS

Statistics → Collection and Analysis of Data



3.4.1 Analysis of Ungrouped Data

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' observations, then

(1) The range of the data = R = max{$x_1, x_2, \ldots\ldots, x_n$} – min{$x_1, x_2, x_3, \ldots\ldots, x_n$}

(2) Arithmetic mean **:** Mean of the data is equal to sum of observaions divided by the total number of observations.

$$\bar{x}(or)\mu = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \boxed{\frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} = \mu}$$

- The mean of 1st 'n' natural numbers = $\frac{\left(\frac{n(n+1)}{2}\right)}{n} = \frac{n+1}{2}$
- The mean of 1st 'n' odd numbers = $\frac{n^2}{n} = n$
- The mean of 1st 'n' even numbers = n +1

3.4.2 Median

The middle most observation of the data $(x_1, x_2, x_3, \ldots\ldots x_n)$ When the data is arranged in either ascending or descending order.

If $x_1, x_2, x_3, x_4, \ldots\ldots x_n$ are 'n' observations that are arranged in ascending/descending order then

(i) Median of the Data = $\left(\frac{n+1}{2}\right)^{th}$ observation, if 'n' is odd.

(ii) Median of the Data = Mean of $\left(\frac{n}{2}\right)^{th}$ & $\left(\frac{n}{2}+1\right)^{th}$ observations, if 'n' is even.

3.4.3 Mode

The observation with highest frequency is called mode.

Any Data with two Modes is called → Bimodel Data

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' data points, $\bar{x} = \mu = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n}$

Mean Deviation of the observation $(x_i) = d_i = x_i - \bar{x}$

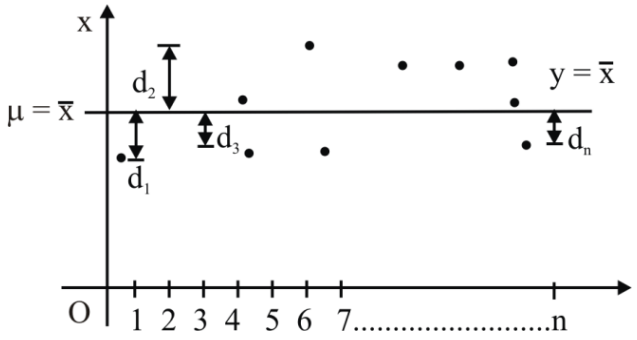


Fig. 5.6. Discrete data

Sum of derivations of all the observations = $\Sigma d_i = (x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots\ldots 0 + (x_n - \bar{x})$

$$= \Sigma d_i = (x_1 + x_2 + \ldots\ldots + x_n) - n\bar{x}$$

$$\boxed{\Sigma d_i = 0}$$

The sum of mean deviations of all the observations is equal to zero.

3.4.4 Absolute Mean Deviation

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' data points with Mean = $\bar{x}$, then the absolute mean deviation of x_i about $\bar{x}$ is given by $|d_i| = |x - \bar{x}|$

The sum of absolute mean derivations of given data is not zero.

$$(\Sigma |d_i| \neq 0) \Rightarrow (|x_1 - \bar{x}| + |x_2 - \bar{x}| + \ldots\ldots\ldots + |x_n - \bar{x}| \neq 0)$$

3.4.5 Standard Deviation

If $x_1, x_2, x_3, \ldots\ldots, x_n$ ('n' is very large), then the standard deviation of the data is given by

Population Standard Deviation $\sigma = \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}$, n $\rightarrow$ size of population

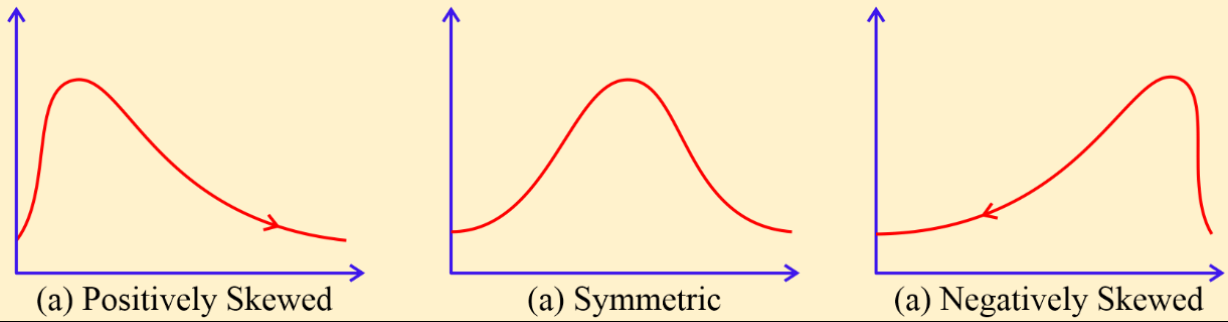
Sample Standard derivation: $\sigma = \sqrt{\frac{1}{(n-1)}\Sigma(x_i - \bar{x})^2}$, n$\rightarrow$ size of sample

Generally (n > 29 $\rightarrow$ population) (n < 29 $\rightarrow$sample)

Note: **Measures of skewness (The degree of asymmetry)**

A lack of symmetry is skewness.

- For symmetric distribution mean (M) = Median (M_d) = Mode (M_e)
- For negatively skewed distribution mean (M) < Median (M_d) < Mode (M_e)
- For positively skewed distribution Mean (M) > Median (M_d) > Mode M_e).



(a) Positively Skewed (a) Symmetric (a) Negatively Skewed

3.5 Random Variables

The variable that connects the outcome of a Random Experiment to a real number.

Example: 'x' is the value of the number that a dice shows when it is rolled.

Discrete RV → The RV whose value is obtained by counting, defined by PMF

Random Variable

Continuous RV → The RV whose value is obtained by Measuring, defined by PDF

- If a data consists of 'f_1' data points with value $'x_1'$, $'f_2'$ data points with value $'x_2'$……$'f_n'$ data point with value $'x_n'$, then
  (i) Expectation of 'x' $= E(x) = \sum_{i=1}^{n} x_i P(x = x_i)$
  (ii) Variance of 'x' = $\sigma^2 = E(x^2) - (E(x))^2$ and σ is the standard deviation.

3.5.1 Probability Mass Function (PMF)

The PMF p(x) of a discrete random variable X taking values $x_1, x_2, ...x_n$ is defined such that,

(i) $p(x_i) \geq 0$

(ii) $\sum_{i=1}^{n} p(x_i) = 1$

(iii) $p(x_i) = p(X = x_i)$

3.5.2 Probability Density Function (PDF)

The pdf f(x) of a continuous random variable X is defined such that,

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a < X < b) = \int_{a}^{b} f(x)dx$

3.5.3 Expected Value

1. Expected value of a random variable X, E [X], is defined as, $E[X]\begin{cases} \sum xp(x); & \text{X is discrete rv} \\ \int_{-\infty}^{\infty} xf(x)dx; & \text{X is continuous rv} \end{cases}$

2. Expected value of X^2 is,

$$E[X^2] = \begin{cases} \sum x^2 p(x); & \text{X is discrete rv} \\ \int_{-\infty}^{\infty} x^2 f(x)dx; & \text{X is continuous rv} \end{cases}$$

Note: $E[X^n]$ is called nth moment.

3.5.4 Mean of Random Variable 'X'

Mean $= \mu = E[X]$

3.5.5 Variance of a Random Variable 'X'

Var $(X) = E[(X - \mu)^2]$

Or, Var $(X) = E[X^2] - \mu^2$

3.5.6 Properties of Expectation

(i) E[c] = c, c is a constant.

(ii) E[ax] = aE[X]

(iii) E(aX + b) = aE(X) + b

(iv) If X and Y are random variable E[X ± Y] = E(x) ± E(Y).

(v) If X and Y are random variables E(X. Y) = E(X). E(Y / X).

(vi) If X and Y independent random variables E(X. Y) = E(X). E (Y).

3.5.7 Properties of Variance

(i) Var[C] = 0, C is constant.

(ii) Var (aX) = a^2V(X) where X is random variable and 'a' constant.

Var(–X) = $(-1)^2$ Var(X) = Var(X) Variance is always positive.

(iii) Var(ax + b) = a^2 Var(X) + 0

(iv) If X and Y are independent random variables.

Var(X + Y) = Var(X) + Var(Y)

Var(X – Y) = Var(X) + Var(Y)

(v) Var(ax + by) = a^2 v(x) + b^2 v(y) + 2ab Cov (x, y)

(vi) Cov (x, y) = E(x, y) – E(x) E(y)

(vii) For independent random variables Cov(x, y) = 0

3.5.8 Continuous RV

The value of the Random Variable is obtained by Measuring.

3.6 Probability Distribution Function (PDF)

A continuous & differentiable function P(x) is said to be a probability distribution/density function of a continuous random variable 'x' if $P(a \leq x \leq b) = \int_a^b P(x)dx$

3.6.1 Mean (or) Expectation

If P(x) is a probability distribution/density function of a continuous Random Variable 'x' then the Mean of 'x' = E(x) $= \int_{-\infty}^{\infty} x \cdot P(x)dx$

3.6.2 Median

The value of 'x' for which the total probability is exactly divided into two equal halves is called Median.

3.6.3 Mode

The valueof 'x' at which P(x) is maximum is called mode.

3.6.4 Variance

$= \sigma^2 = E(x^2) - (E(x))^2$

$$\Rightarrow \sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot P(x)dx - \left(\int_{-\infty}^{\infty} x \cdot P(x)dx\right)^2$$

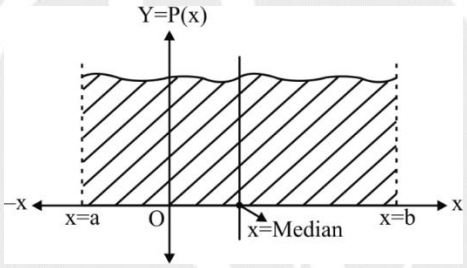


Fig. 5.7. Continuous random variables

3.7 Continous RV distributions

(1) Gaussian/Normal Distributon:

If 'x' is a continuous Random variable with mean 'μ' and standard deviation 'σ', then the probability distribution/density function of normally distributed variable 'x' is given by

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}$$

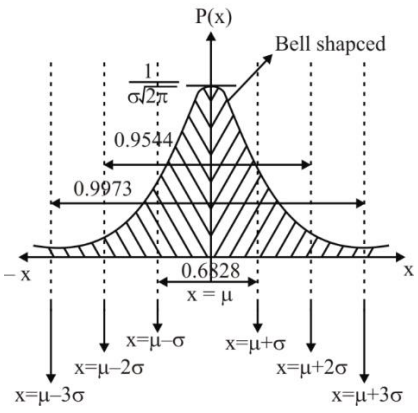


Fig. 5.8. Normal distribution

Mean = Median = Mode = μ

$$P(\mu - \sigma \le x \le \mu + \sigma) = 0.6828$$

$$P(\mu - 2\sigma \le x \le \mu + 2\sigma) = 0.9544$$

$$P(\mu - 3\sigma \le x \le \mu + 3\sigma) = 0.9973$$

$$P(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} . e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

(2) Standard Normal Distribution:

Assuming $z = \frac{x-\mu}{\sigma}; \mu = 0; \sigma = 1, P(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}}$

$$P(-1 \le z \le 1) = 0.6828$$

$$P(-2 \le z \le 2) = 0.9544$$

$$P(-3 \le z \le 3) = 0.9973$$

Note:

1. The normal distribution curve is bell shaped curve
2. The points of infelection of the normal distribution curve are at $x = \mu + \sigma \; and \; x = \mu - \sigma$.
3. The cumulative function graph is of 'S' Shape.
4. For a given normal distribution, Mean = median = Mode

(3) Uniform Distribution:

If 'x' is a uniformly distrbuted random variable such that $a \le x \le b$ then the Pdf is given by

$$P(x) = \frac{1}{(b-a)}$$

Mean = $\int_a^b x \cdot P(x)dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{(b-a)} \int_a^b x \cdot dx$

$$\boxed{\left(\frac{b+a}{2}\right) = \text{Mean}}$$

$\Rightarrow \; Variance = \sigma^2 = \frac{(b-a)^2}{12}$

Std. deviation = $\sigma = \frac{(b-a)}{\sqrt{12}}$

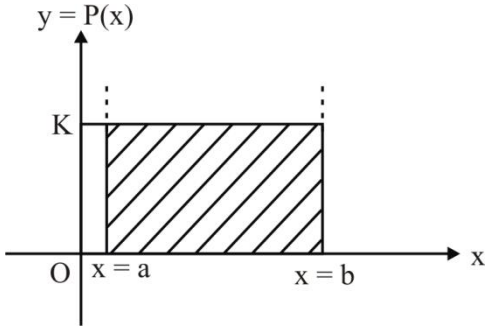


Fig. 5.9. Uniform Distribution

3.7.1 Properties of Mean and Variance:

$E(ax + by) = a \cdot E(x) + b \cdot E(y)$

$$V(ax + by) = a^2 \cdot V(x) + b^2 \cdot V(y) - 2abCOV(x, y)$$

where $COV(x, y) = E(xy) - E(x) \cdot E(y)$

If x,y are indpendent random variables, then $E(xy) = E(x) \cdot E(y) \Rightarrow COV(x,y) = 0$

(1) Exponential Dirtibution:

If 'x' is a continous random variable with mean as $\frac{1}{\lambda}$ then the exponential distribution of 'x' is given by the function

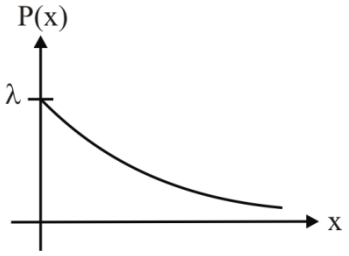


Fig. 5.10. Exponential distribution

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & ; x \geq 0 \\ 0 & : otherwise \end{cases}$$

Mean = $\frac{1}{\lambda}$

$\sigma^2 = \frac{1}{\lambda^2}$

Mean = Standard Deviation = $\frac{1}{\lambda}$

3.8 Discrete Random Variable Distributions

If a Random experiment has **only two Possible outcomes**, (one is Success & other is failure) and the Probability of Success doesn't depend on time, then the probability of occuring of exactly 'r-successes' in 'n-trials' is given by

$$P(X = r) = {}^{n}C_r \cdot P^r \cdot q^{n-r}$$

Where, P $\rightarrow$ Probability of Success,

q $\rightarrow$ Probability of Failure

p + q = 1

Mean = np,Variance = npq = σ^2, standard deviation = $\sigma = \sqrt{npq}$

3.8.1 Poisson Distribution

If a random experiment has only two possible outcomes, and the average number of successes in a given time 't' is λ, then the probability that exactly 'r' successes occur within the same time 't' given by

$P(x = r)\frac{e^{-\lambda} \cdot \lambda^r}{r!}$

Mean = λ.

Mean = Variance = λ

$\Rightarrow \sigma = \sqrt{\lambda}$

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