Engineering Mechanics

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INDEX

1 BASIC CONCEPTS

1.1 Force

- Force is the interaction between two bodies.
- It is the action of one body on another body that tries to change the motion of the other body.

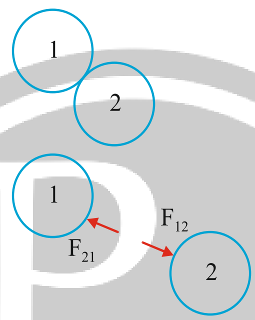


Fig. 1.1 Interaction of force between two bodies

Force is a vector quantity. It is characterized by –

(1) Point of application

(2) Line of action (direction)

(3) Magnitude

1.1.1 Principle of transmissibility of force

The point of application of force can be moved anywhere along the line of action of force without changing the external effects of the force on the body.

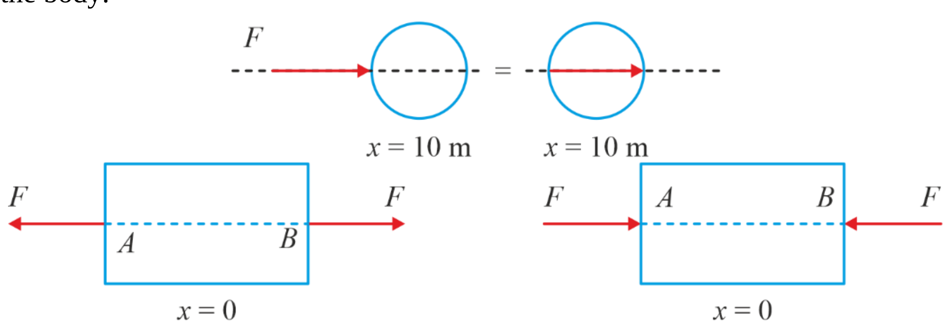


Fig. 1.2 Transmission of force along its line of action

1.2 Moment of a force

Moment of a force is the measure of tendency of the force to rotate a body about a particular point or axis.

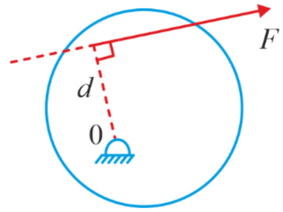


Fig. 1.3 Force at a distance from center of rotation

Moment of F about O

$$M_0 = F \times d \text{ (cw)}$$

d = Perpendicular distance of F from O

1.3 Moment of a couple

Couple is caused by two non-concurrent parallel forces of same magnitude and having opposite directions. Couple is the product of Force and the perpendicular distance between them.

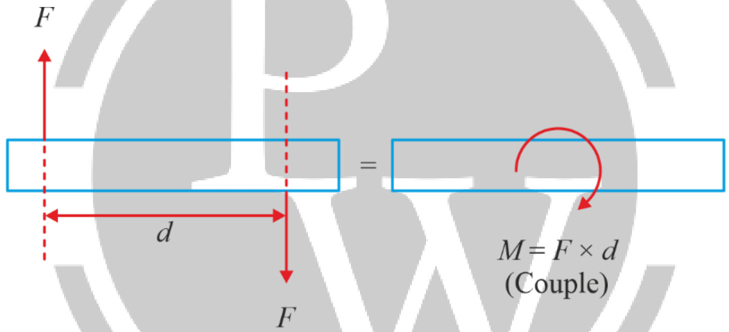


Fig. 1.4 Moment of couple

1.4 Resolution of a Force

1.4. 1 Resolution of a force into two components

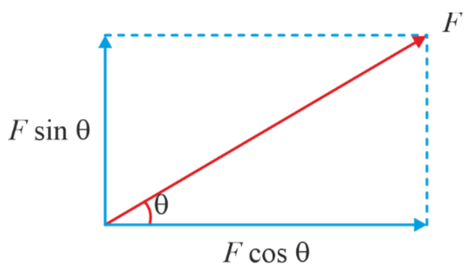


Fig. 1.5 Resolution of force into two components

1.4. 2 Resolution of a force into a force and a couple

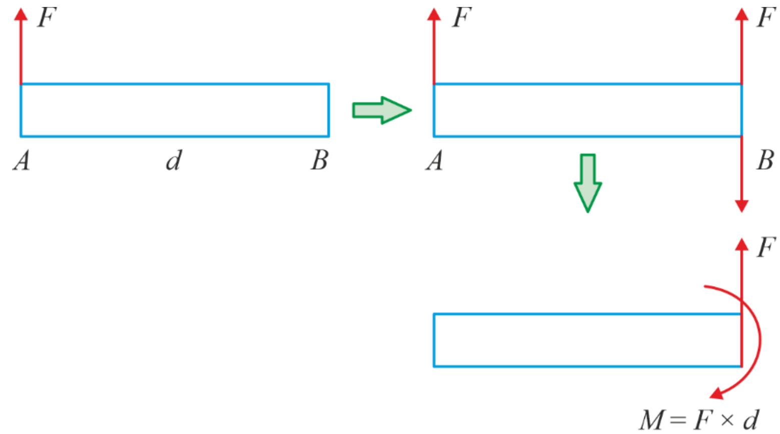


Fig. 1.6 Resolution of force into a force and a couple

1.5 Varignon's theorem

Moment of a force about any point is equal to the sum of the moments of the components of that force about the same point.

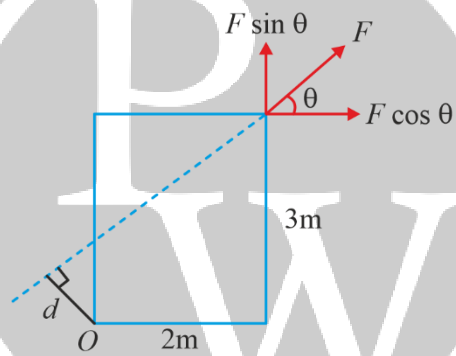


Fig. 1.7 Moment of Force about a point

$$M_0 = F \times d = F \cos\theta \times 3 - F \sin\theta \times 2$$

1.6 Newton's Three Laws of Motion

1.6.1 First Law

If the resultant force acting on a body is zero, the body will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion)

1.6.2 Second Law

If the resultant force acting on a body is not zero, the body will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.

$$F \propto a$$
$$F = ma$$

1.6.3 Third Law

The forces of action and reaction between bodies in contact have the same magnitude, same line of action, and opposite sense.

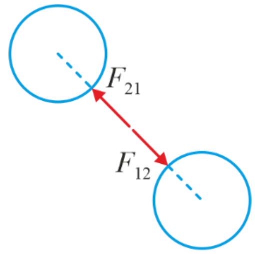


Fig. 1.8 The forces of action and reaction

$$F_{12} = -F_{21}$$

1.7 Resultant of a force system

1.7.1 Resultant of a Parallel force system:

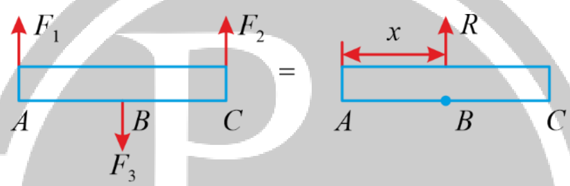


Fig. 1.9 Equivalent force of a parallel force system

$$R = F_1 + F_2 - F_3$$

Note:

We can find the distance x by equating the moment of the resultant R with the moment of all the forces at any point.

1.7.2 Resultant of a concurrent force system:

Two forces: (Parallelogram law)

If two forces acting a point, are represented in magnitude and direction by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from the same point.

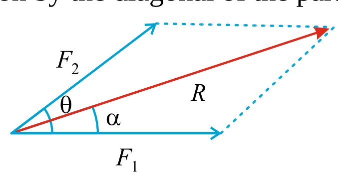


Fig. 1.10 Resultant of two forces and its direction

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

$$\alpha = \tan^{-1}\left(\frac{F_2\sin\theta}{F_1 + F_2\cos\theta}\right)$$

Several forces: (Polygon law)

If more than two forces acting at a point are represented by consecutive sides of a polygon taken in order then their resultant will be represented by closing side of polygon taken in opposite order.

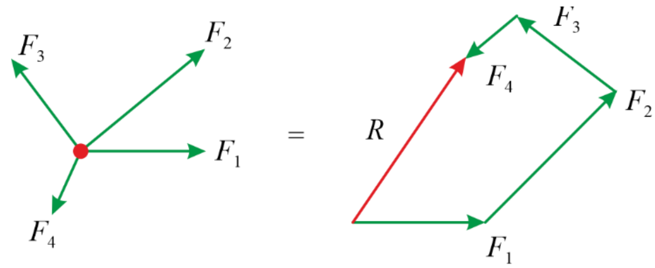


Fig. 1.11 Representation of forces in a polygon

Several forces: (Method of Projections)

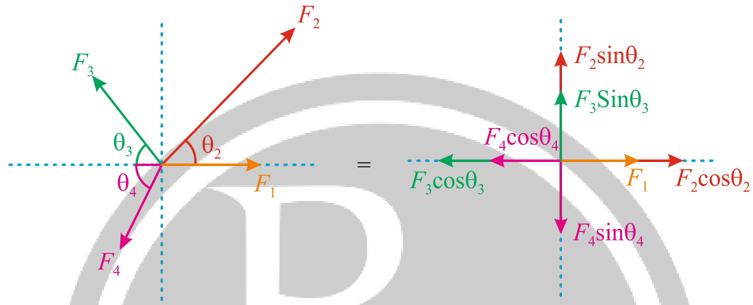


Fig. 1.12 Projection of forces on two perpendicular axes

$$\Sigma F_x = F_1 + F_2 \cos\theta_2 - F_3 \cos\theta_3 - F_4 \cos\theta_4$$

$$\Sigma F_y = F_2 \sin\theta_2 + F_3 \sin\theta_3 - F_4 \sin\theta_4$$

$$R = \sqrt{\Sigma F_x^2 + \Sigma F_y^2}$$

$$\alpha = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x}$$

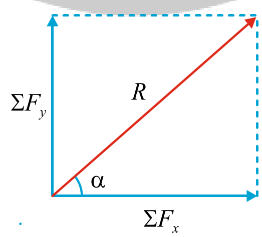


Fig. 1.13 Resultant of two perpendicular forces and its direction

1.7.3 Resultant of a non-concurrent force system

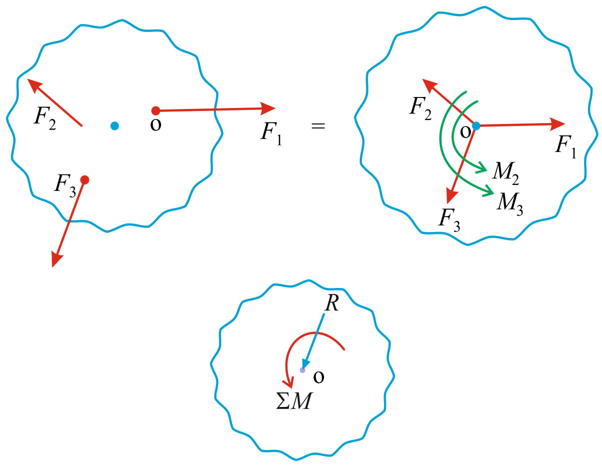


Fig. 1.14 A non-concurrent force system and its resultant

R is the resultant of forces F_1, F_2 and F_3 and $\sum M$ is the net moment about O.

1.8 Free Body Diagram

An FBD is a sketch of the selected system consisting of a body, part of body or a collection of interconnected bodies completely isolated from other bodies showing the interaction of all other bodies by force.

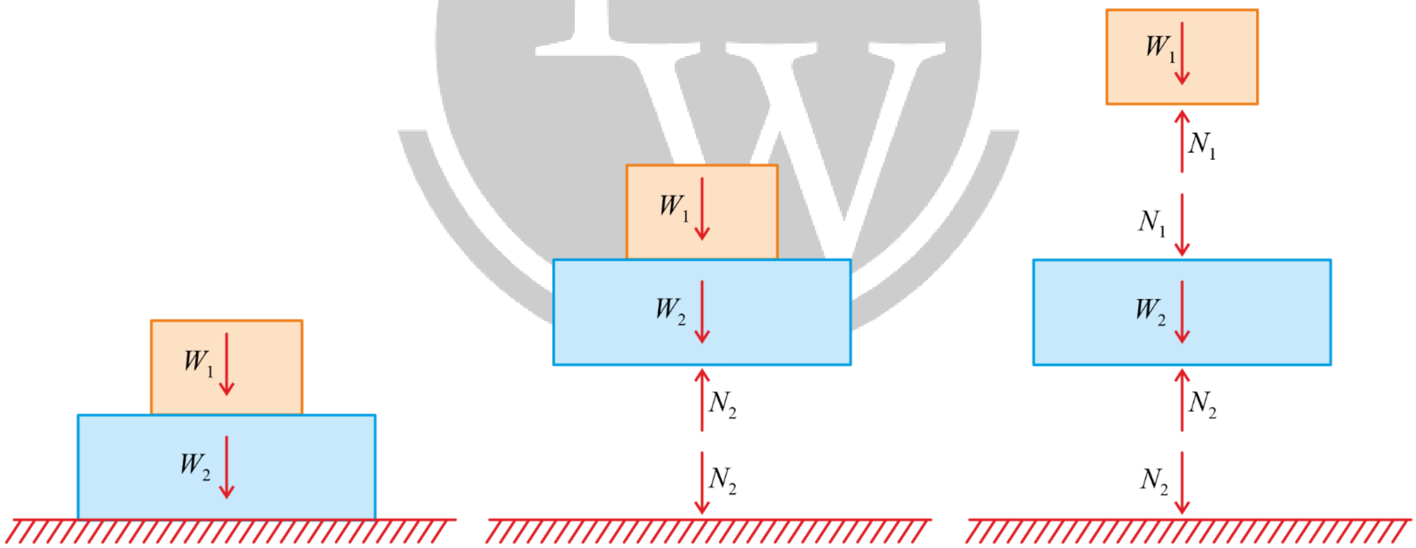


Fig. 1.15 Free body diagrams

1.9 Types of supports and their reactions

1.9.1 Roller/Frictionless surface

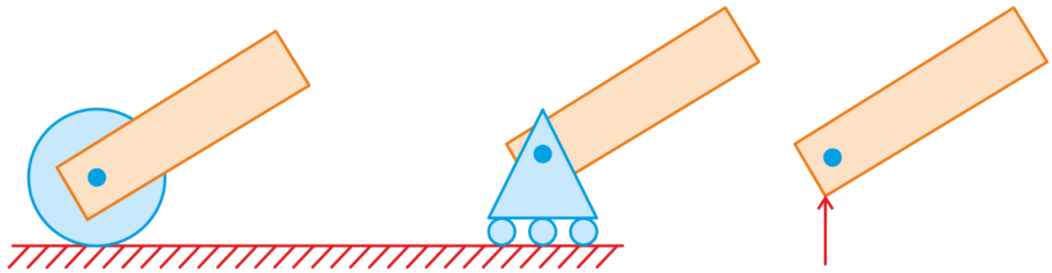


Fig. 1.16 Roller support and reaction force

Note:
We can assume any direction of reaction force (Upward or downward). If the calculated value of reaction is (+ve), the assumed direction is right otherwise wrong.

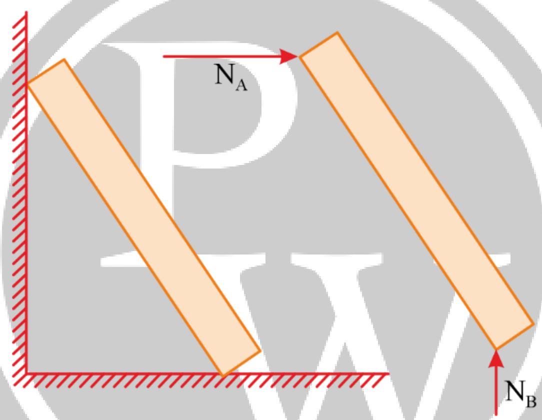
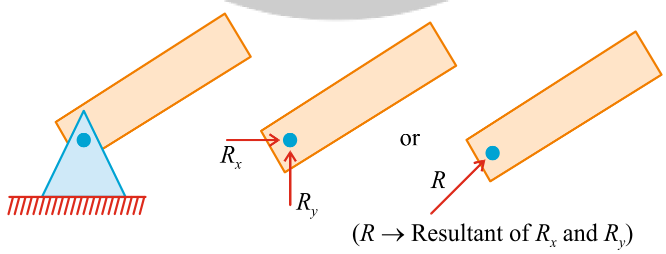


Fig. 1.17 A block leaning against wall and its FBD

1.9.2 Hinge/Friction surface



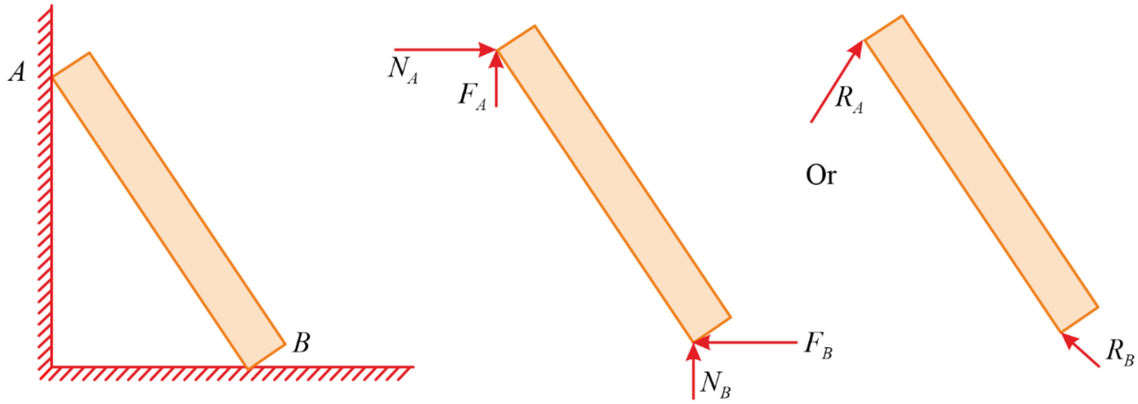


Fig. 1.18 Hinge support and FBD of block leaning against rough wall

1.9.3 Fixed support

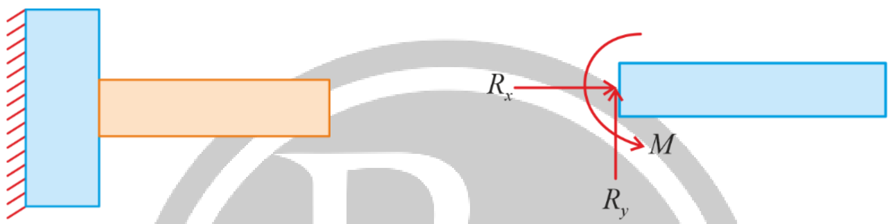


Fig. 1.19 Fixed support and its reactions

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2 EQUILIBRIUM OF RIGID BODIES

2.1 Equilibrium of a rigid body

When a body is subjected to forces and the body is either at rest or moving with constant velocity in the same direction, the body is said to be in equilibrium.

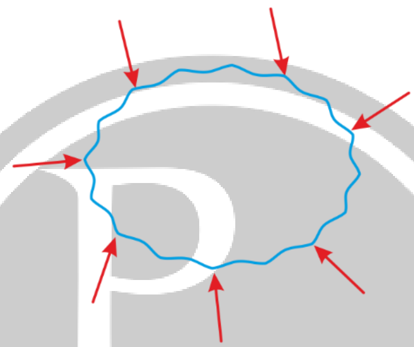

Fig. 2.1 A body in equilibrium under the application of loads

For the rigid body to be in equilibrium, necessary conditions are -

$$\begin{Bmatrix} \Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0 \\ \Sigma M_x = 0, \Sigma M_y = 0, \Sigma M_z = 0 \end{Bmatrix}$$

For coplanar force system

$$\Sigma F_x = 0, \Sigma F_y = 0$$

$$\Sigma M_z = 0$$

The above equations are known as equilibrium equations.

2.1.1 Statically Determinate System

If number of unknowns = number of equilibrium equation, then system is statically determinate.

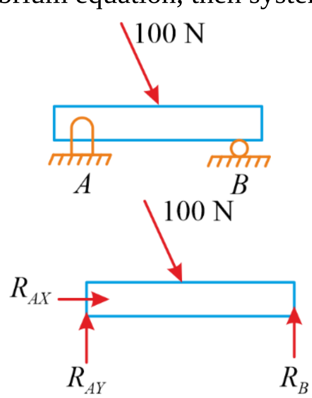

Fig. 2.2 Statically determinate system

Number of unknowns (R_{AX}, R_{AY}, R_B) = 3

Number of equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_z = 0$) = 3

Note:
All the problems of engineering mechanics are statically determinate.

2.1.2 Statically Indeterminate System

If number of unknowns > number of equilibrium equations, then system is statically indeterminate.

Note:
Statically indeterminate problems are solved in strength of materials.

2.1.3 Equilibrium of a two-force system:

If a body, subjected to two forces, is in equilibrium, the two forces must have the same magnitude, the same line of action, and opposite sense.

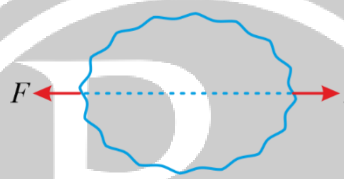

Fig. 2.3 A body in equilibrium under the application of two forces

2.1.4 Equilibrium of a three-force system:

If a body subjected to three forces is in equilibrium, the lines of action of the three forces must be either parallel or concurrent.

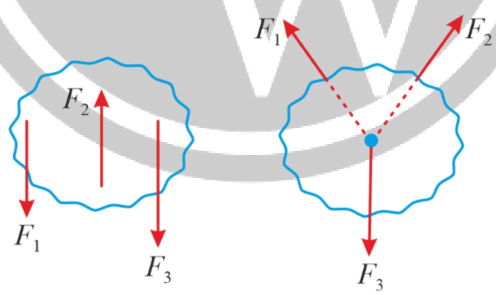

Fig. 2.4 A body in equilibrium under the application of three forces

Lami's Theorem

It states that, when a body under the action of three concurrent and coplanar forces, is in equilibrium, then each force is proportional to the sine of the angle between the other two forces.

$$\frac{F_1}{\text{Sin}\,\theta_1} = \frac{F_2}{\text{Sin}\,\theta_2} = \frac{F_3}{\text{Sin}\,\theta_3}$$

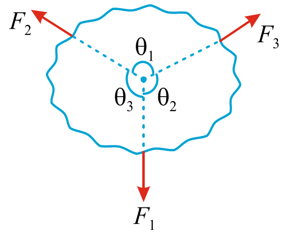

Fig. 2.5 Three concurrent forces on a stationary body

> **Note:**
> For calculating angles between forces, draw all the forces either diverging from a point or converging to a point.

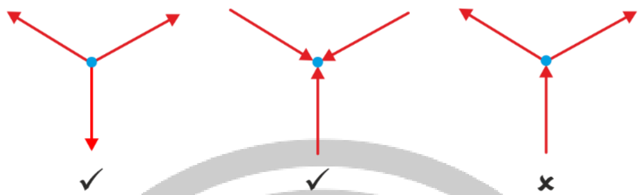

Fig. 2.6 Method to calculate angles between forces in Lami's Theorem

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3 FRICTION

3.1 Friction

Friction is a force that resists the relative motion between two contacting surfaces.

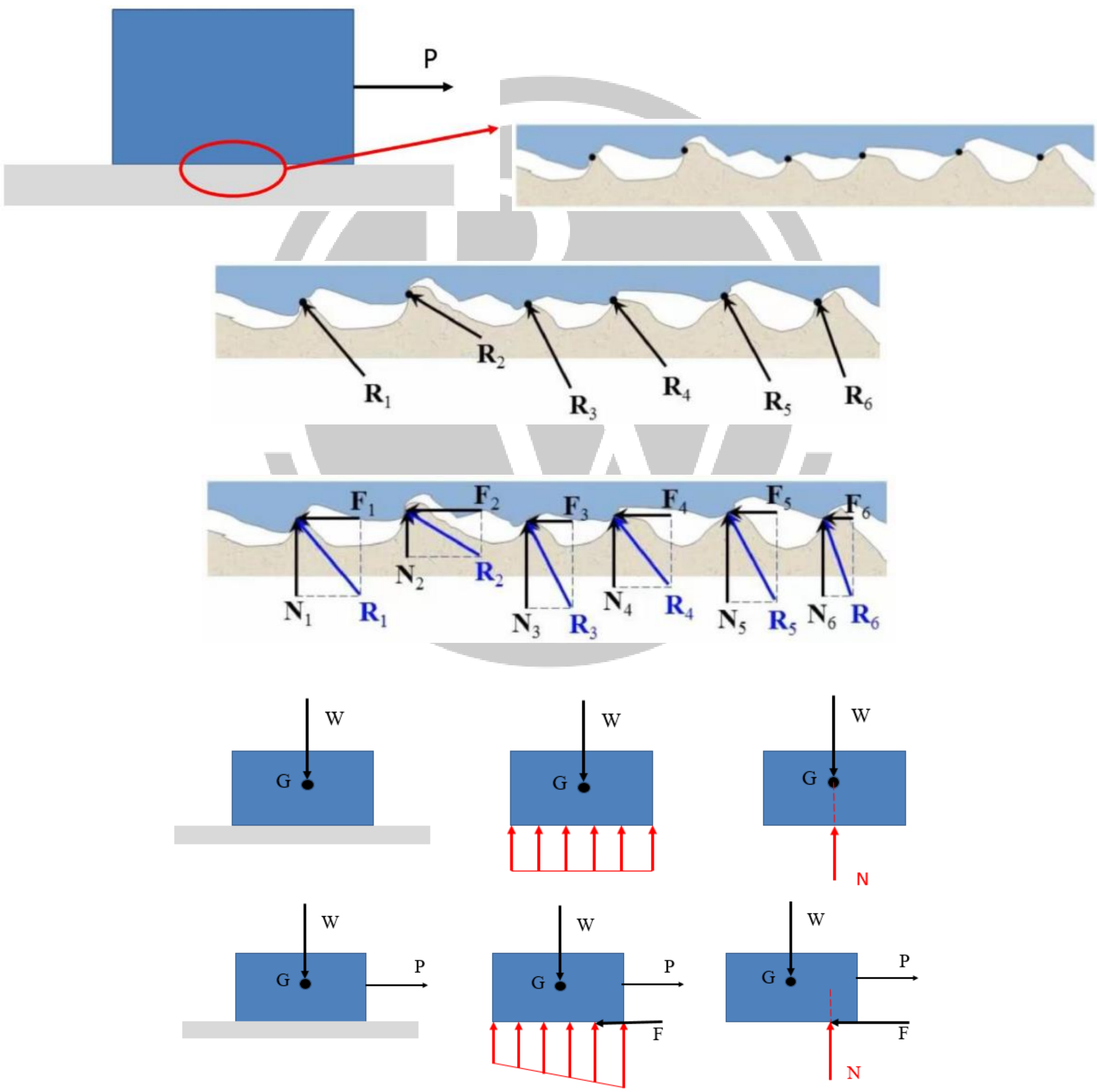

Fig. 3.1 Understanding of friction on microscopic level

Note:

The normal reaction N does not pass through the center of gravity G.

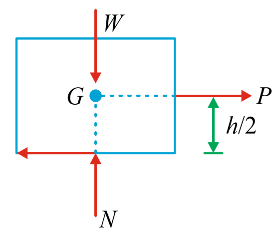

Fig. 3.2 FBD of a block moving on a rough surface

$$\Sigma M_G = F \times \frac{b}{2} \neq 0$$

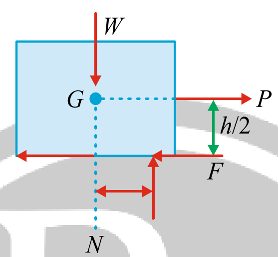

Fig. 3.3 FBD of a block moving on a rough surface

$$\Sigma M_G = F \times \frac{h}{2} - N \times x = 0$$

3.1.1 Limiting Friction:

It is the maximum static friction between two dry surfaces.

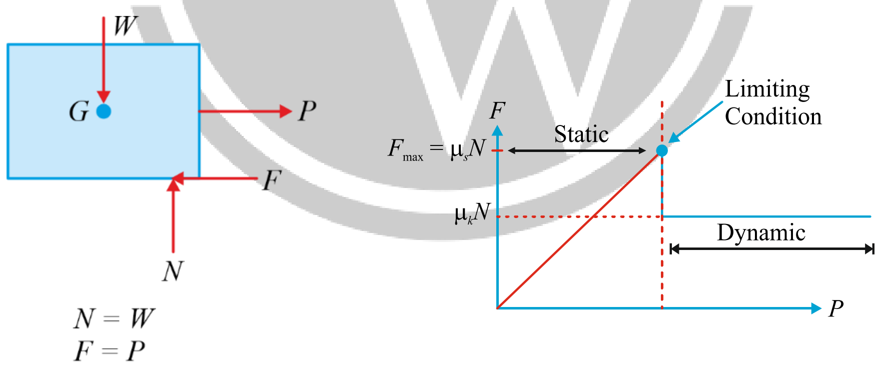

Fig. 3.4 Load and friction force diagram

$\mu_s \rightarrow$ static coefficient of friction

$\mu_k \rightarrow$ kinetic coefficient of friction

$\mu_k < \mu_s$

$F_{max} = \mu_s N$ (Maximum static friction force)

- If $P < \mu_s N$ — Body remains static.
- If $P = \mu_s N$ — Body is on the range of moving. (static)
- If $P > \mu s N$ — Body starts moving.

3.1.2 Laws of dry friction

- Static friction force is directly proportional to applied load.
- Friction force is independent of area of contact.
- Kinetic friction force is independent of velocity.

3.2 Sliding vs Tipping

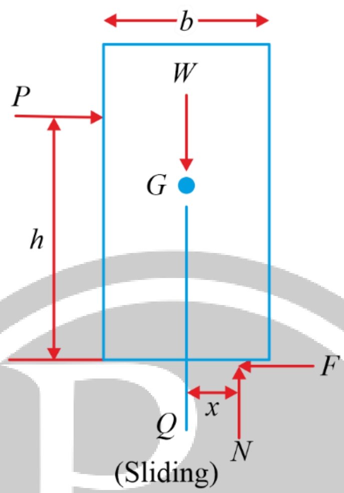

Fig. 3.5 Block sliding on a rough surface

$$\sum M_Q = 0$$

$$P \times h - N \times x = 0$$

$$P \times h = N \times x$$

$$x = \frac{Ph}{W}$$

As the distance x increases, normal force N moves towards the edge. At the time of tipping, $x = b/2$ and N is at the edge.

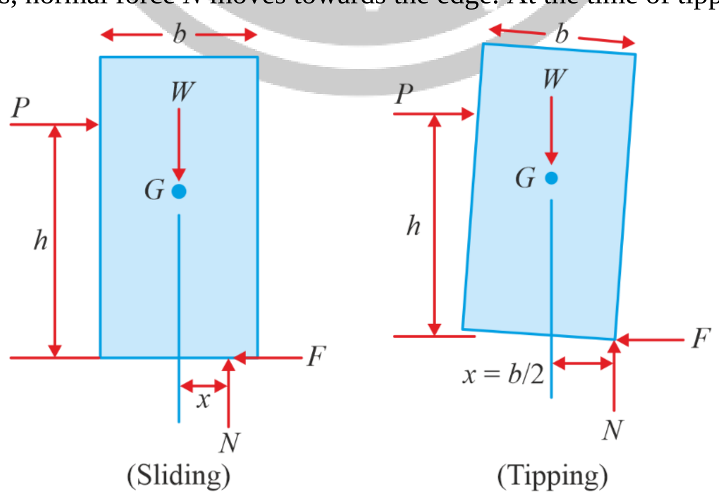

Fig. 3.6 Block sliding and tipping on a rough surface

3.3 Friction Angle (ϕ)

It is the angle between **normal force** and **resultant of normal and friction force** when the body is on the verge of moving.

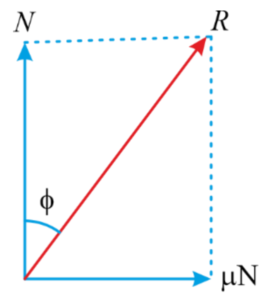

Fig. 3.7 Resultant of normal and friction force

$$\tan\phi = \frac{\mu N}{N}$$

$$\phi = \tan^{-1}\mu$$

3.4 Angle of Repose

If a body is resting on an inclined surface, then the angle of repose is the maximum angle at which the body can be at rest without slipping down.

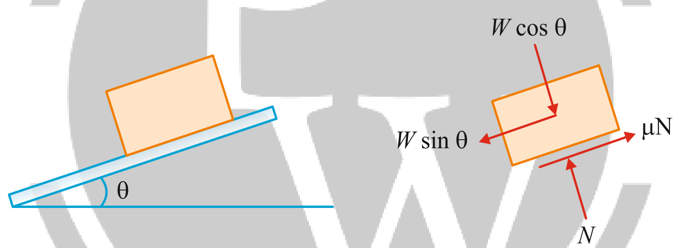

Fig. 3.8 Block sliding on an inclined surface

$$W\sin\theta = \mu N$$

$$W\cos\theta = N$$

$$\tan\theta = \mu$$

$$\tan\theta = \tan\phi$$

$$\theta = \phi$$

3.5 Belt Friction

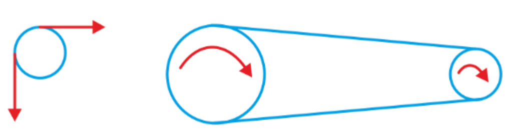

Fig. 3.9 Tensions in the belt of a belt-pulley system

$\theta \rightarrow$ angle of contract

$T_1 \rightarrow$ maximum tension in belt

$T_2 \rightarrow$ minimum tension in belt

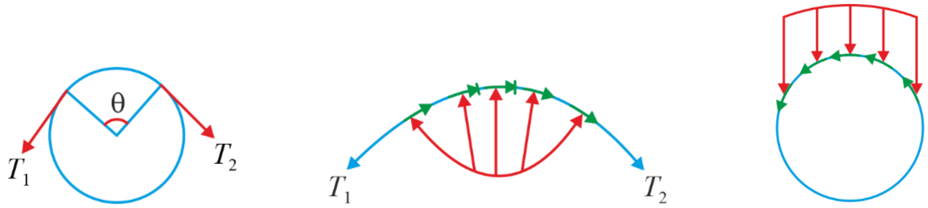

Fig. 3.10 Pressure on the pulley due to tension in belt

$$\boxed{\frac{T_1}{T_2} = e^{\mu\theta}}$$ (θ must be in radians)

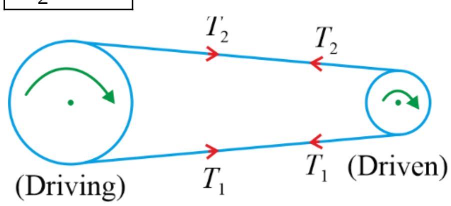

Fig. 3.11 Belt pulley system and tension in the belt

3.6 Rolling Friction

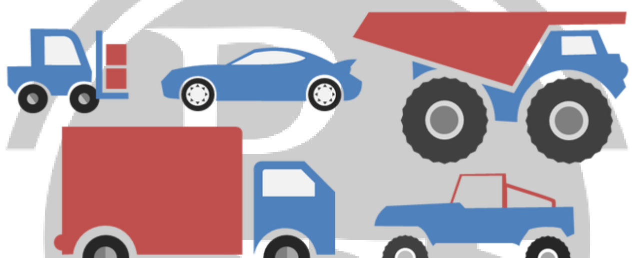

Fig. 3.12 Objects including rolling friction

Roller is at rest,

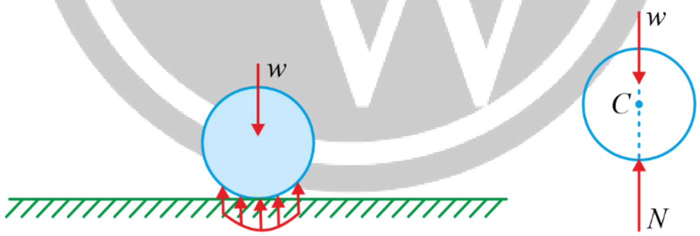

Fig. 3.13 FBD of roller at rest

A force P is applied to the roller to move the roller at constant velocity,

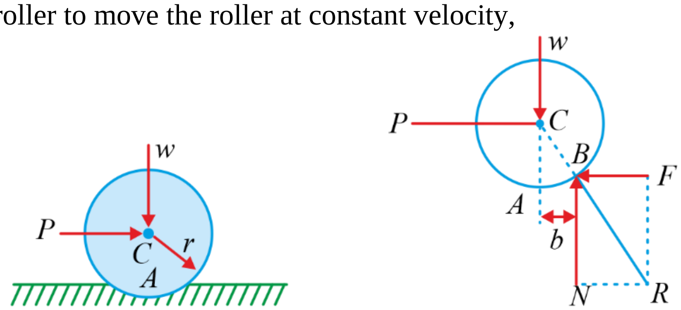

Fig. 3.14 FBD of roller under rolling friction and applied load

R = Resultant of normal force N and friction force F

b = Coefficient of rolling resistance (in mm)

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4 TRUSS

4.1 Truss

Truss is a structure which consists of several members connected together to support load.

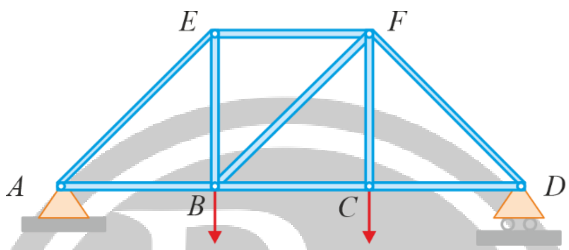

Fig. 4.1 A truss

4.2 Types of Trusses

Truss

- Plane Truss (all members are in a single plane)
- Space Truss (members are in different planes)

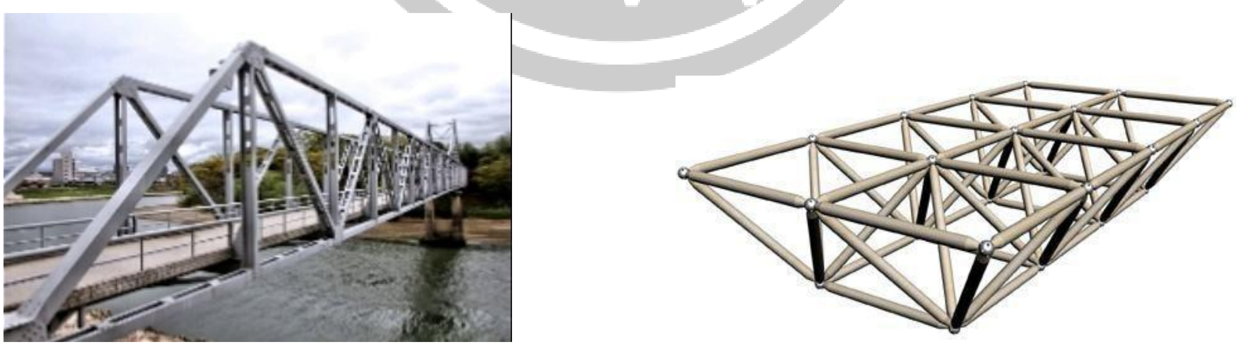

Fig. 4.2 Plane truss and space truss

4.3 Analysis of Plane Truss

- All members are connected only at the ends.
- All joints are frictionless pin joints.
- Loads are only applied at the joints
- All the members are subjected to only axial loads or in other words all the members are two force members.

- Weight of the members is assumed negligible.

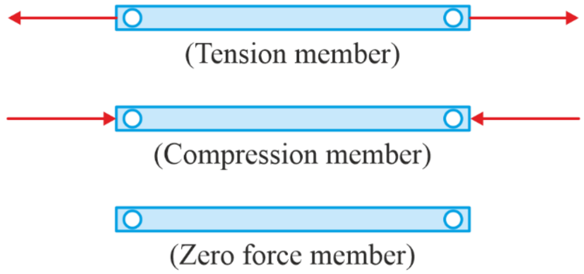

Fig. 4.3 Representation of forces on a member in truss

4.4 Perfect Truss

- A perfect truss is a truss which has just enough number of members to keep the truss in equilibrium.
- A perfect truss is statically determinate.

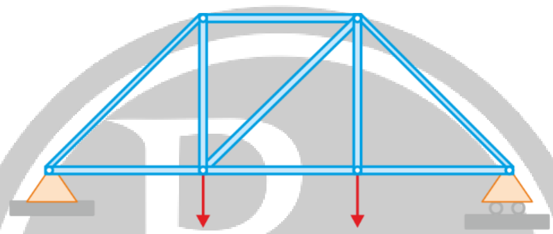

Fig. 4.4 Perfect truss

$m \rightarrow$ number of members

$j \rightarrow$ number of joints

Perfect trust must be statically determinate, so

Number of unknowns = Number of equilibrium equations

$$m + 3 = 2j$$

$$m = 2j - 3$$

- If $m = 2j - 3$ – Perfect truss (stable)
- If $m < 2j - 3$ – unstable truss
- If $m > 2j - 3$ – statically Indeterminate/ over rigid (stable)

4.5 Method of Joints

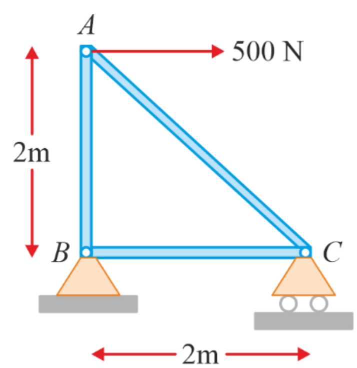

Fig. 4.5 A simple truss

FBD of joint A

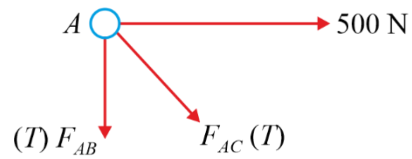

Fig. 4.6 Forces at point A in the truss

$$\Sigma F_x = 0$$

$$F_{AC} \cos 45 + 500 = 0$$

$$F_{AC} = -707.1 \text{ (C)}$$

$$\Sigma F_y = 0$$

$$F_{AB} - F_{AC} \sin 45° = O$$

$$F_{AB} = 500 \text{ N } (T)$$

FBD of joint C

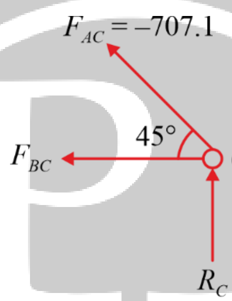

Fig. 4.7 Forces at point C in the truss

$$\Sigma F_x = 0$$

$$F_{BC} - 707.1 \cos 45° = 0$$

$$F_{BC} = 500 \text{ N } (T)$$

4.6 Method of Sections

Find the axial force in member BD.

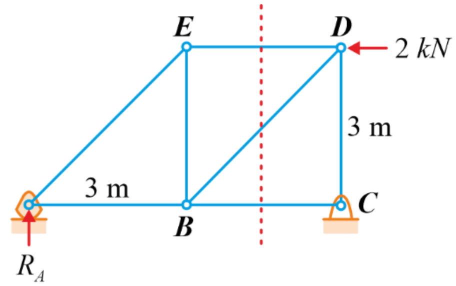

Fig. 4.8 A section taken in the truss

$$\Sigma M_c = 0$$

$$R_A \times 6 = 2 \times 3$$

$$R_A = 1 \text{ kN}$$

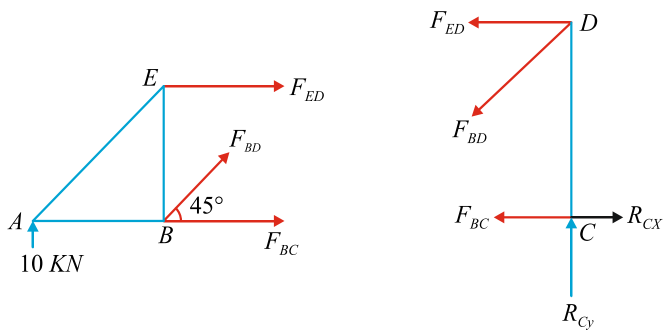

Fig. 4.9 Forces in the two parts of truss after taking section

$$\Sigma Fy = 0$$
$$F_{BD} \sin 45° + 1 = 0$$
$$F_{BD} = -1.414 \text{ kN}$$

4.7 Truss vs Frames

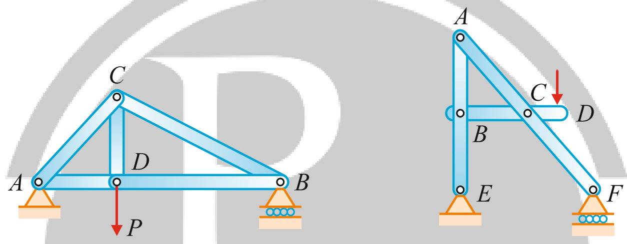

Fig. 4.10 Truss and frame

- In case of truss, all members are two force members, whereas there is at least one multi force member in frames.

4.8 Analysis of Plane Frames

- Frames are structures containing at least one multi force member.
- The members are connected at the ends or in between the ends.
- Loads act at the joints as well as on the members.

Two force members

Straight member
No force between the ends

Multi force members

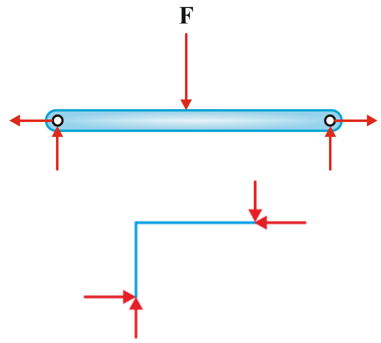

Fig. 4.11 Forces in the members

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5 CENTROID, COG, COM AND MOI

5.1 Centroid

It is the geometric centre of an object.

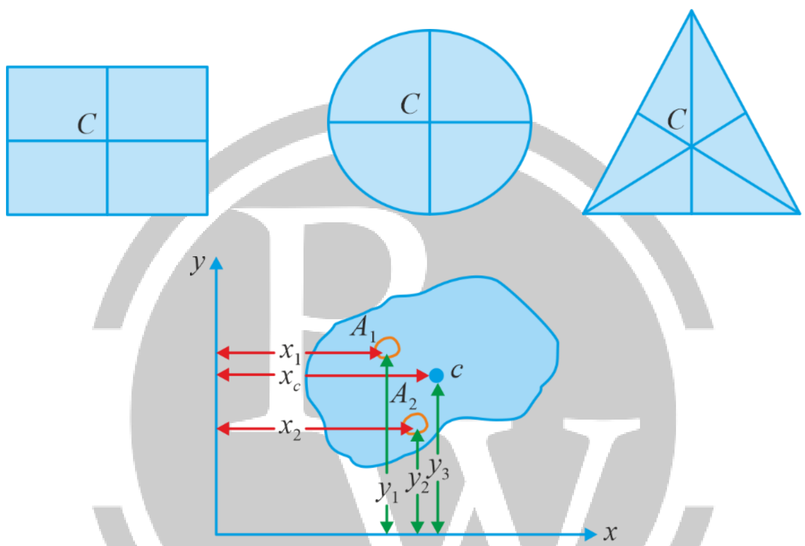


Fig. 5.1 Centroid of various shapes

$$x_c = \frac{A_1x_1 + A_2x_2 \ldots\ldots + A_nx_n}{A_1 + A_2 \ldots\ldots + A_n}$$

$$y_c = \frac{A_1y_1 + A_2y_2 \ldots\ldots + A_ny_n}{A_1 + A_2 \ldots\ldots + A_n}$$

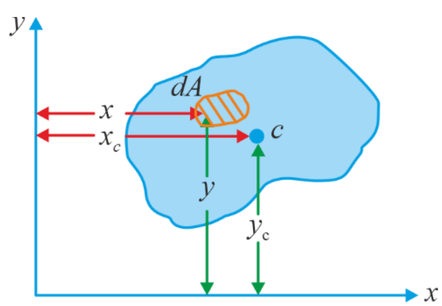


Fig. 5.2 Centroid of a lamina

$$x_c = \frac{\int x dA}{\int dA}$$

$$y_c = \frac{\int y\,dA}{\int dA}$$

5.2 Center of Mass

It is an imaginary point where the mass of the body can be assumed to be concentrated.

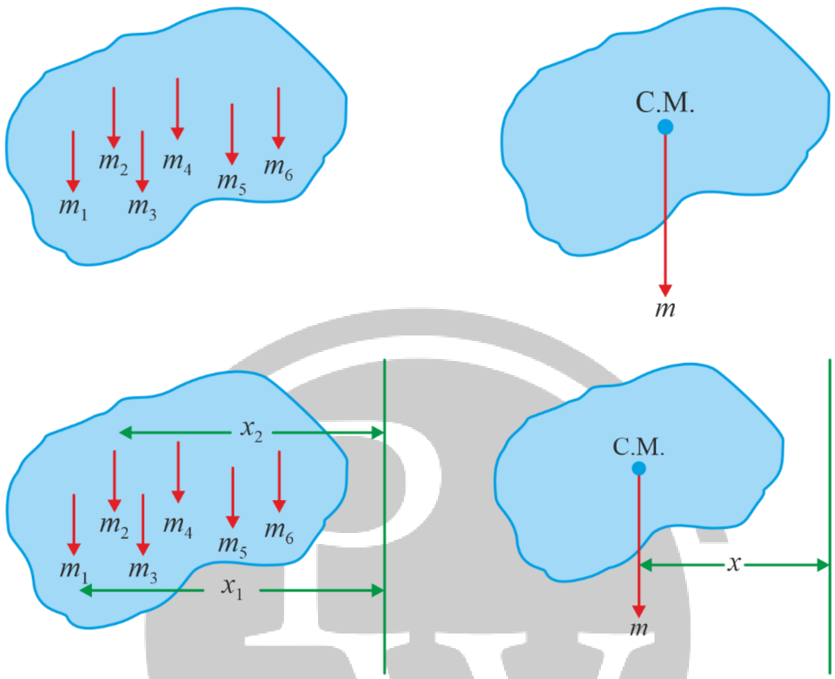


Fig. 5.3 Center of mass of an object

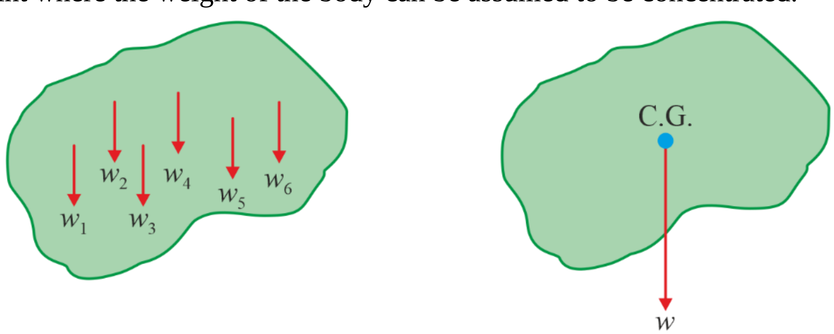
$$x = \frac{m_1x_1 + m_2x_2.........+m_nx_n}{m_1 + x_2.......+m_n}$$

Note:

If density of the body is same at every point, centroid and centre of mass will be same.

5.3 Center of Gravity

It is an imaginary point where the weight of the body can be assumed to be concentrated.



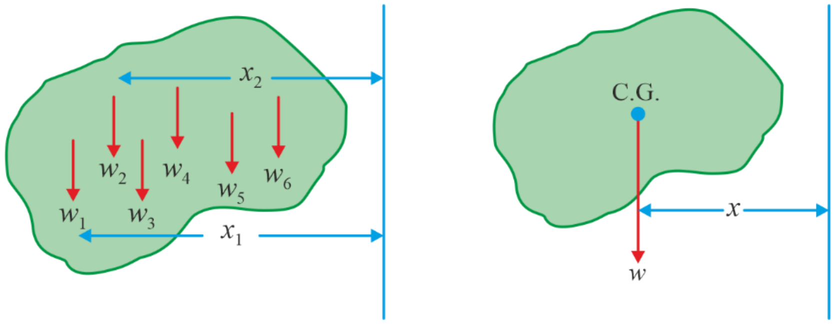


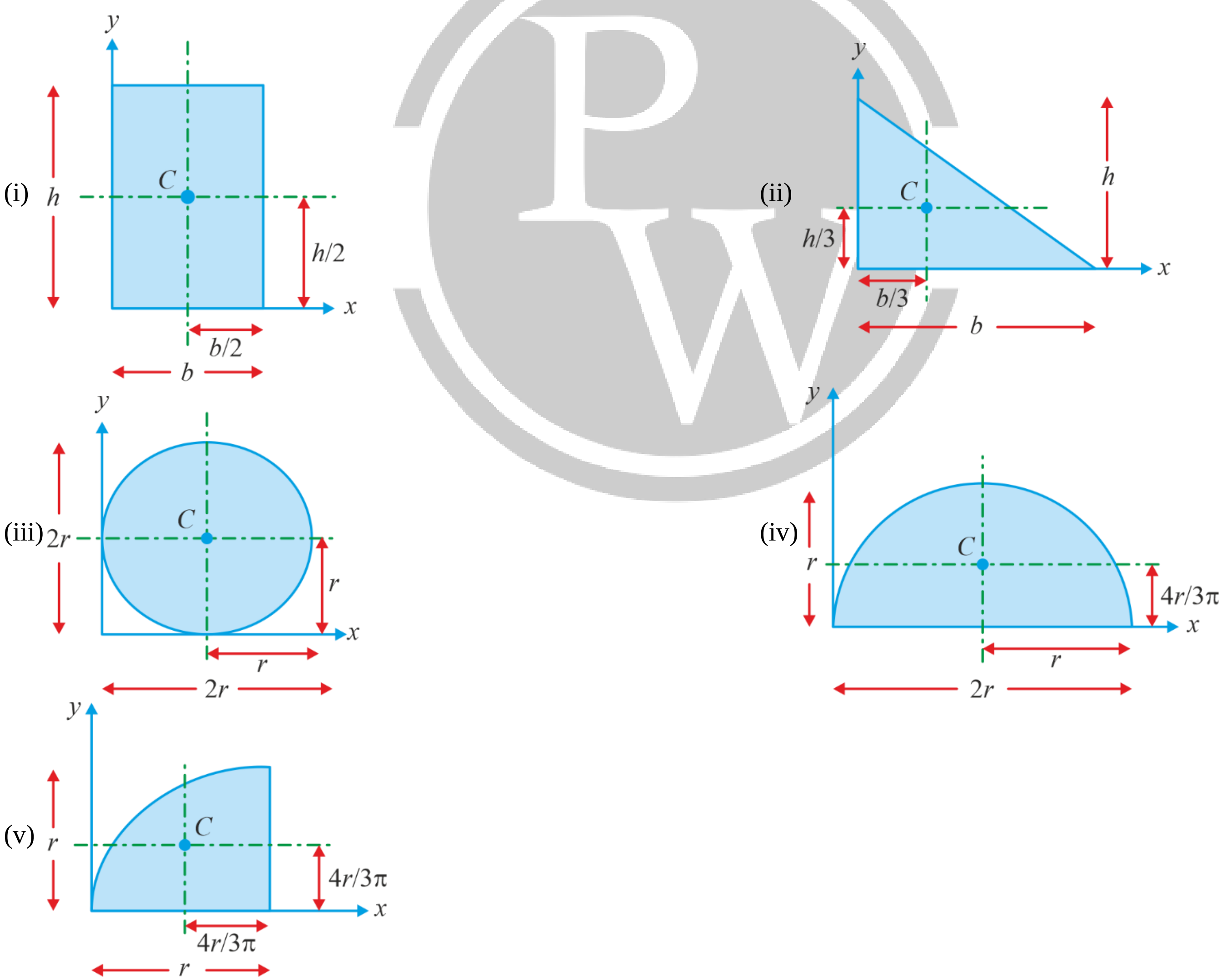
Fig. 5.4 Center of gravity of an object

$$x = \frac{W_1x_1 + W_2x_2 ....+ W_nx_n}{W_1 + W_2 ....+ W_n}$$

Note:

For small bodies g is constant, hence center of mass and center of gravity is same.

5.4 Centroid of some common areas



5.5 Moment of Inertia of Area (Second Moment of Area)

- It is a geometrical property of an area which indicates how its points are distributed about an axis.
- It signifies the resistance of an area against the applied moment (bending moment or twisting moment) about an axis.

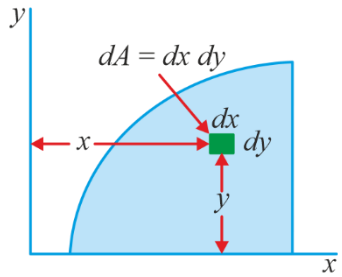


Fig. 5.5 MOI of an area

$$I_x = \int y^2 dA$$

$$I_y = \int x^2 dA$$

5.5.1 Polar moment of inertia

It is the moment of inertia of an area about the normal axis (z axis).

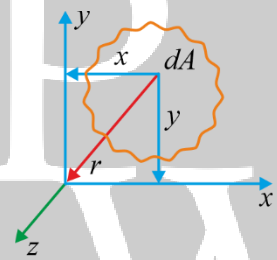


Fig. 5.6 Polar MOI of a section

$$r^2 = x^2 + y^2$$

$$I_z = \int r^2 dA = \int (x^2 + y^2) dA$$

$$I_z = \int x^2 dA + \int y^2 dA$$

$$I_z = I_x + I_y = J$$

5.5.2 Parallel Axis Theorem

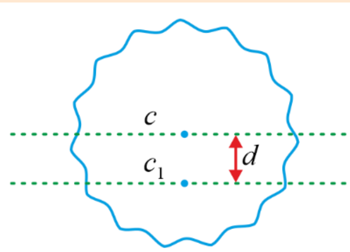


Fig. 5.7 An axis parallel to centroidal axis

$$I_{c_1} = I_c + Ad^2$$

5.5.3 Radius of Gyration

Consider an area A whose moment of inertia about an axis is I. Suppose the area is concentrated into a thin strip parallel of to that axis as shown in figure, such that its moment of inertia about the axis is same as I. Then the distance at which this strip is to be placed from the axis is called radius of gyration of the area about that axis.

$$I = Ak^2$$

$$k = \sqrt{\frac{I}{A}}$$

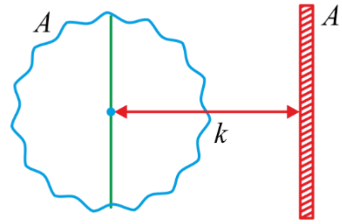


Fig. 5.8 Radius of gyration of a section

5.5.4 Moment of inertia of some common areas

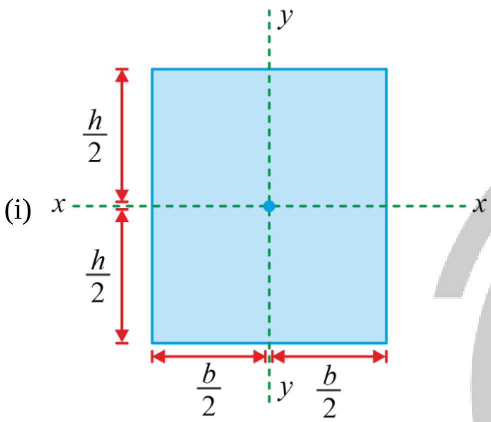


Fig. 5.9 Rectangular section

$$I_x = \frac{bh^3}{12}$$

$$I_y = \frac{hb^3}{12}$$

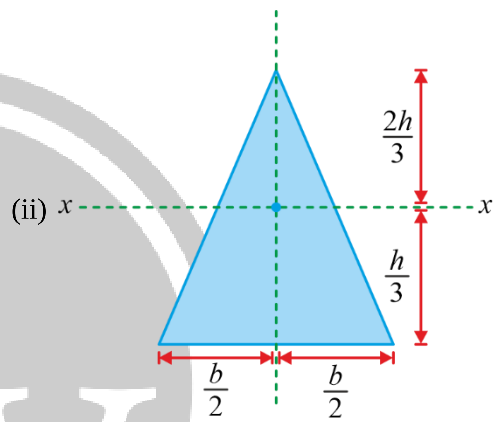


Fig. 5.10 Triangular section

$$I_x = \frac{bh^3}{36}$$

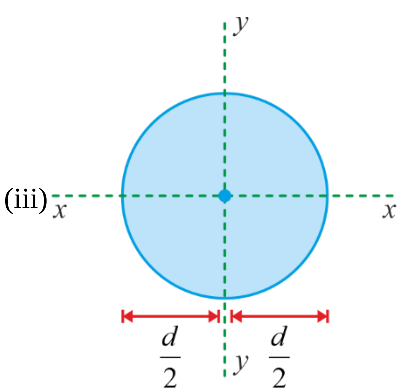


Fig. 5.11 Circular section

$$I_x = I_y = \frac{\pi}{64} d^4$$

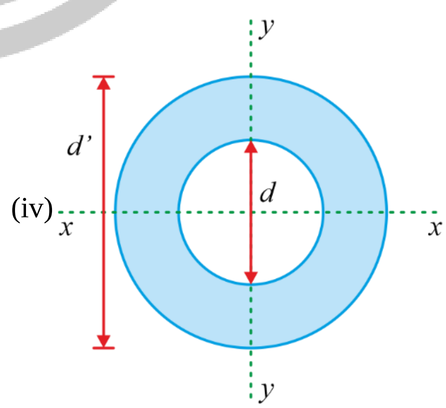


Fig. 5.12 Tube section

$$I_x = I_y = \frac{\pi}{64}\left(d_0^4 - d_i^4\right)$$

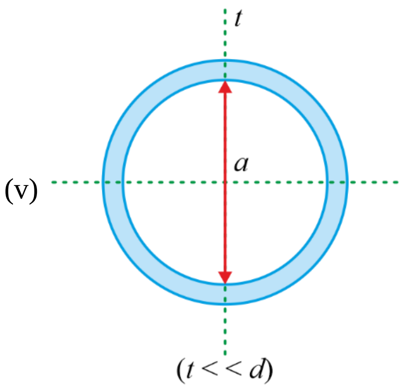


Fig. 5.13 Ring section

$$I_x = I_y = \frac{\pi d^3 t}{8}$$

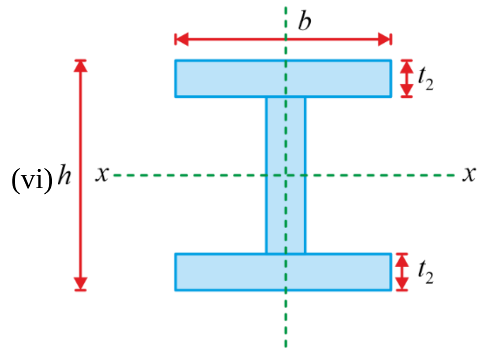


Fig. 5.14 I section

$$I_x = \frac{BH^3}{12} - \frac{bh^3}{12}$$

$b = B - t_1$

$h = H - 2t_2$

5.6 Moment of Inertia of Mass

It indicates how the mass of the body is distributed about an axis.

It signifies the resistance of a body against the applied torque about an axis.

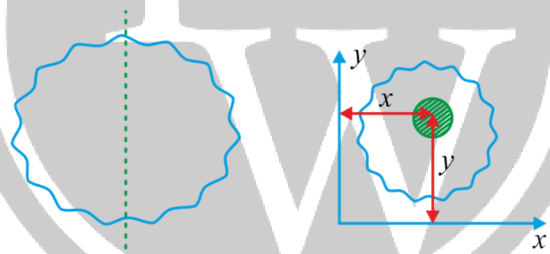


Fig. 5.15 Moment of Inertia of mass

$$I_x = \int y^2 . dm \text{ kg-m}^2$$

$$I_y = \int x^2 dm \text{ kg-m}^2$$

5.6.1 Parallel Axes Theorem

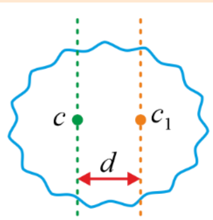


Fig. 5.16 An axis parallel to axis at COM of an object

$$I_{C_1} = I_C + md^2$$

5.6.2 Radius of Gyration

If the total mass of the body were concentrated to a point then the distance of that point at which it would have a moment of inertia the same as the body's actual distribution of mass, is called radius of gyration.

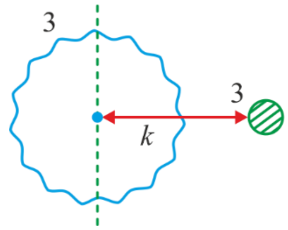


Fig. 5.17 Radius of gyration of an object

$$I = mk^2$$

$$\boxed{k = \sqrt{\frac{I}{m}}}$$

5.6.3 Moment of inertia of some common bodies

(i) Slender rod

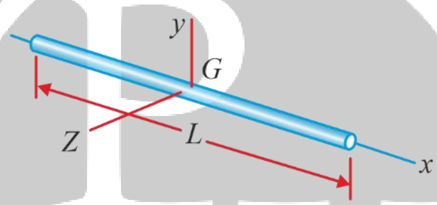


Fig. 5.18 Slender Rod

$$\boxed{I_y = I_z = \frac{mL^2}{12}}$$

$$I_z = I_z + m \times \left(\frac{L}{2}\right)^2$$

$$= \frac{mL^2}{12} + \frac{mL^2}{4}$$

$$\boxed{I_z = \frac{mL^2}{3}}$$

(ii) Thin Disc

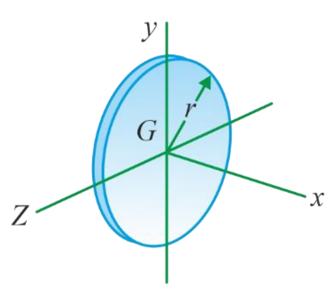


Fig. 5.19 Thin Disc

$$I_x = \frac{mr^2}{2}$$

(iii) Cylinder

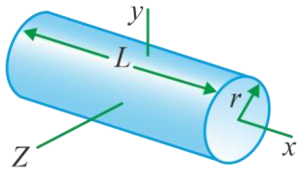


Fig. 5.20 Cylinder

$$I_x = \frac{mr^2}{2}$$

(iv) Sphere

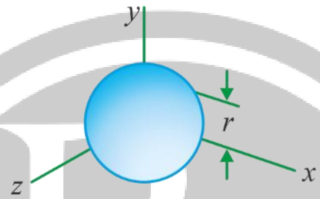


Fig. 5.21 Sphere

$$I_x = I_y = I_z = \frac{2}{5} mr^2$$

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6 KINEMATICS OF PARTICLES

6.1 Types of Motion in a plane

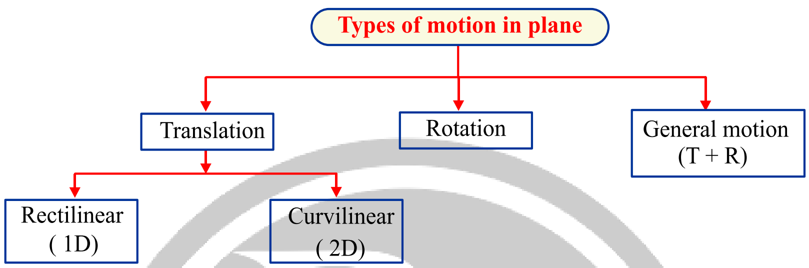


Fig. 6.1 Types of motion

6.1.1 Translational motion

- All the particles of the body move along identical parallel path.
- All the particles of the body have same displacement.

(i) Rectilinear Translational Motion: (1D)

- All the particles of the body move along **straight lines.**

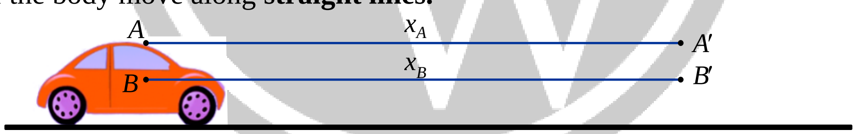
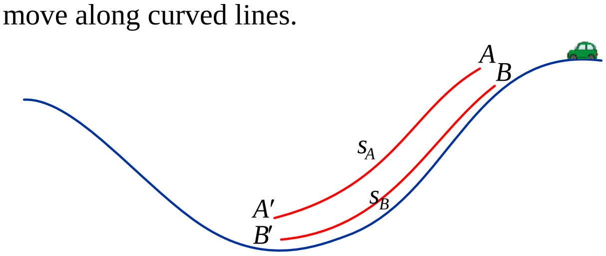


Fig. 6.2 A car moving in a straight line

$$x_A = x_B$$
$$v_A = v_B$$
$$a_A = a_B$$

(ii) Curvilinear Translational Motion: (2D)

- All the particles of the body move along curved lines.



$$s_A = s_B$$
$$v_A = v_B$$
$$a_A = a_B$$

Fig. 6.3 A car moving along a curved path

6.1.2 Rotational motion

- All the particles of the body move in circular paths.
- All the particles of the body have different displacement.

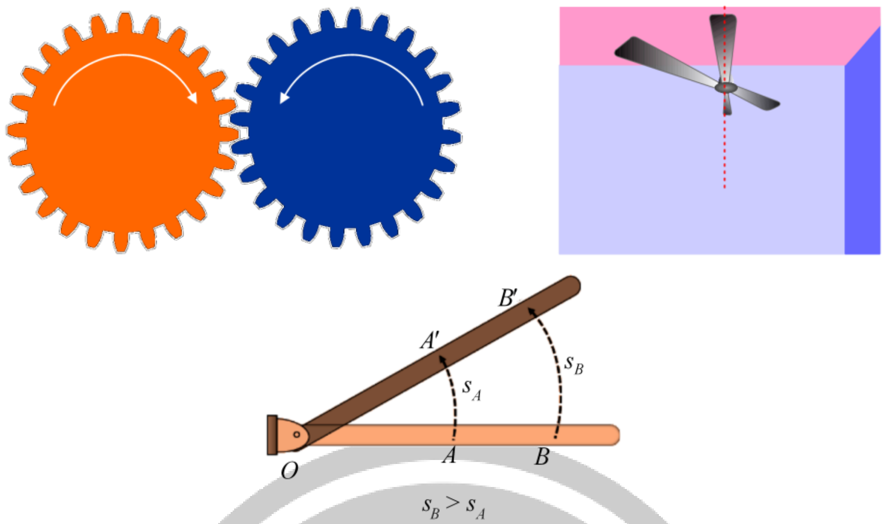


Fig. 6.4 Some examples of rotational motion

6.1.3 General Motion

- The body translates and rotates simultaneously.
- All the particles of the body move in different paths.
- All the particles of the body have different displacement.

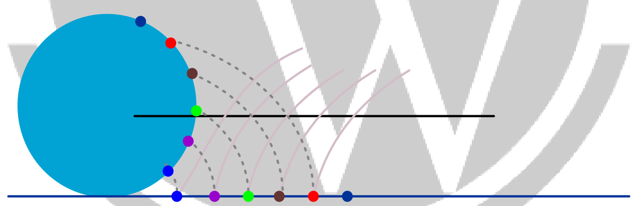


Fig. 6.5 A roller undergoing general motion

6.1.4 Dynamics of particles vs Rigid Bodies

- If shape and size of the body is not affecting the motion, we can consider the body to be a particle.
- By saying that the bodies are analysed as particles, we mean that only their motion as an entire unit will be considered; any rotation about their own mass center will be neglected.
- There are cases, however, when such a rotation is not negligible; the bodies cannot then be considered as particles. Such motions will be analysed in later chapters, dealing with the dynamics of rigid bodies

6.2 Translational Motion

- All the particles of the body move along identical parallel path.
- All the particles of the body have same displacement.

Rectilinear Translational Motion: (1D)

- All the particles of the body move along straight lines.

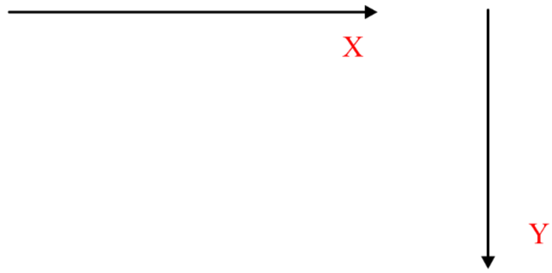


Fig. 6.6 Two perpendicular straight lines

6.2.1 Motion Variables

(i) Position
(ii) Displacement
(iii) Velocity
(iv) Acceleration

These are the variables which define the motion of the body.

(i) Position

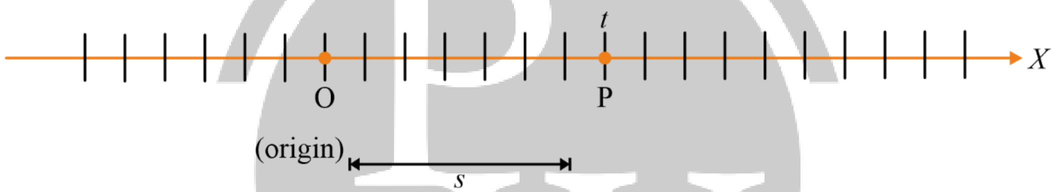


Fig. 6.7 Position of a particle at time t

(ii) Displacement

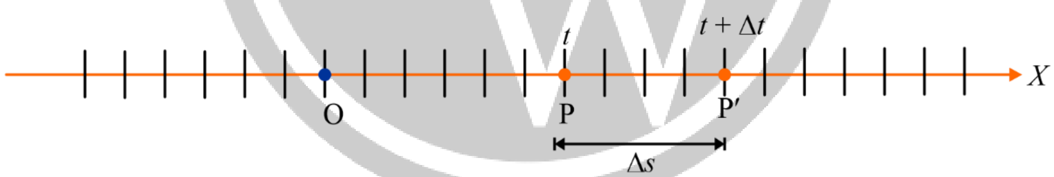


Fig. 6.8 Displacement of a particle at time t

$\Delta s \rightarrow$ displacement within time Δt.

(iii) Velocity

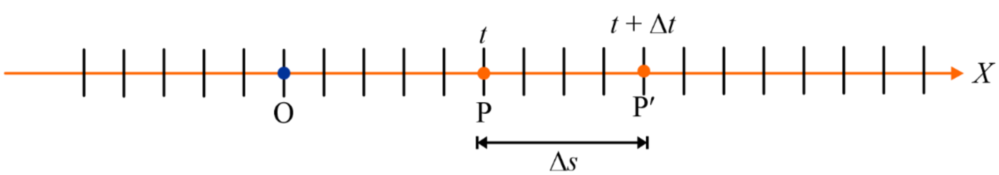


Fig. 6.9 Velocity of point at time t

$$\boxed{V_{avg} = \frac{\Delta s}{\Delta t}} \frac{m}{s}$$

Instantaneous velocity, $$\boxed{V = \frac{ds}{dt}}$$

(iv) Acceleration

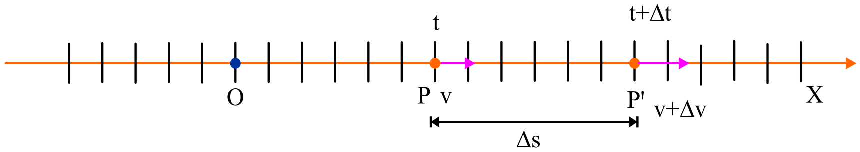


Fig. 6.10 Acceleration of a particle at time t

$$a_{avg} = \frac{\Delta v}{\Delta t} \quad \frac{m}{s^2}$$

Instantaneous acceleration

$$a = \frac{dv}{dt}$$

$$a = \frac{dv}{ds} \cdot \frac{ds}{dt} \Rightarrow v \cdot \frac{dv}{ds}$$

Note:
Negative acceleration (deceleration) means velocity is decreasing.

6.2.2 Graphical Interpretation

(1) Displacement – Time graph

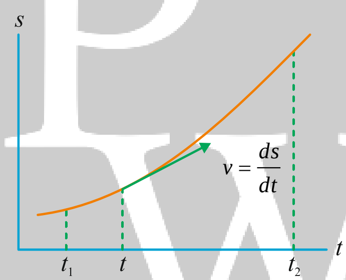


Fig. 6.11 Displacement-time graph

The slope of the displacement – time graph at any time is equal to the velocity at that time.

(2) Velocity – Time graph

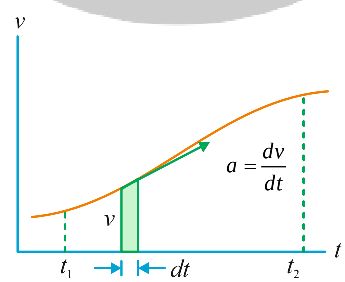


Fig. 6.12 Velocity-time graph

- The slope of the velocity – time graph at any time t is equal to the acceleration at that time.
- The net displacement of the particle during the interval from t_1 to t_2 is the corresponding area under the velocity – time graph, which is

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v\,dt \quad or \quad s_2 - s_1 = \text{(area under } \upsilon\text{-t curve)}$$

(3) Acceleration – Time graph

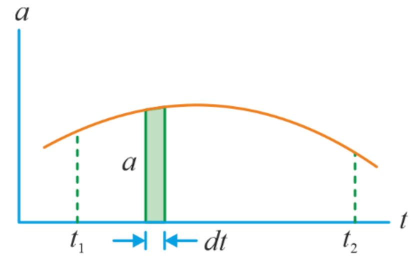


Fig. 6.13 Acceleration-time graph

- The net change in velocity of the particle during the interval from t_1 to t_2 is the corresponding area under the acceleration – time graph, which is

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt \;\; or \;\; v_2 - v_1 = \text{(area under a-t curve)}$$

6.2.3 Uniformly Accelerated Motion (a = constant)

(i) $a = \frac{dv}{dt}$

$$\int_u^v dv = \int_o^t a\,dt$$

$$v - u = at$$

$$\boxed{v = u + at}$$

(ii) $v = \frac{ds}{dt}$

$$\int_o^s ds = \int_o^t v\,dt$$

$$s = \int_o^t (u + at)\,dt$$

$$\boxed{s = ut + \frac{1}{2}at^2}$$

(iii) $a = \frac{vdv}{ds}$

$$\int_u^v v\,dv = \int_o^s a\,ds$$

$$\frac{v^2 - u^2}{2} = as$$

$$\boxed{v^2 - u^2 = 2as}$$

6.3 Curvilinear Translational Motion

- All the particles of the body move along curved path.

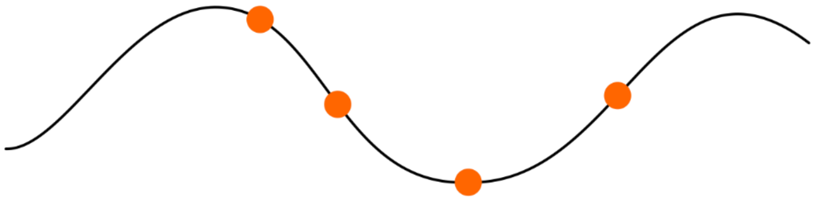


Fig. 6.14 A particle undergoing curvilinear motion

6.3.1 Motion Variables

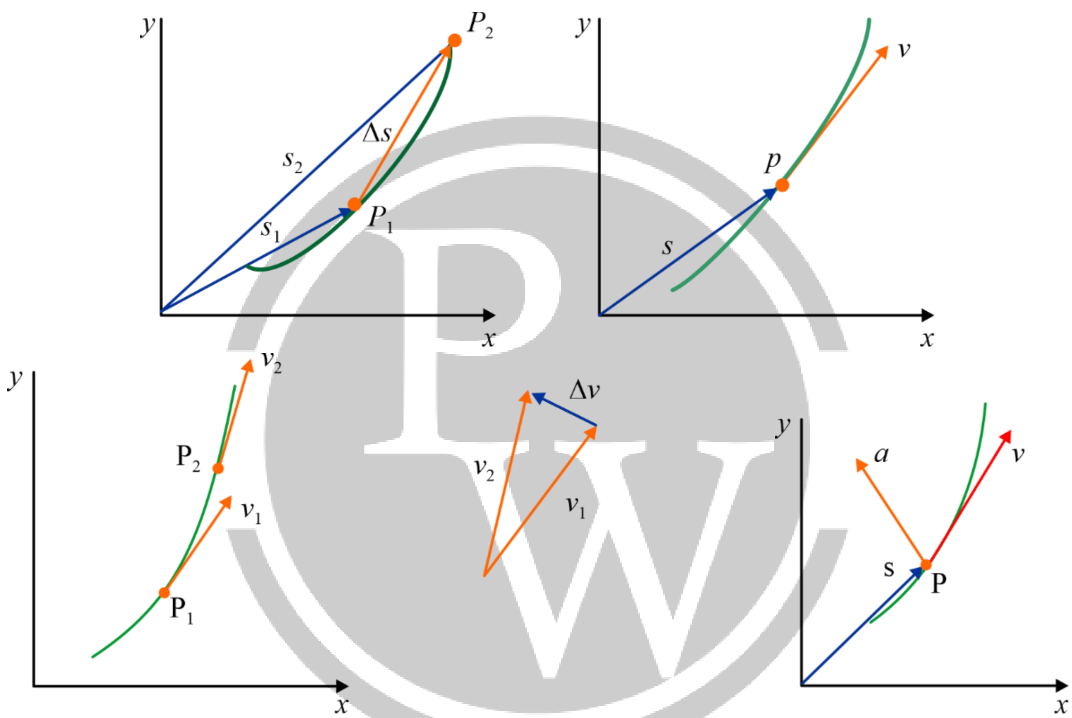


Fig. 6.15 Motion variables

$$\vec{s} = x\hat{i} + y\hat{j}$$

$$|s| = \sqrt{x^2 + y^2}$$

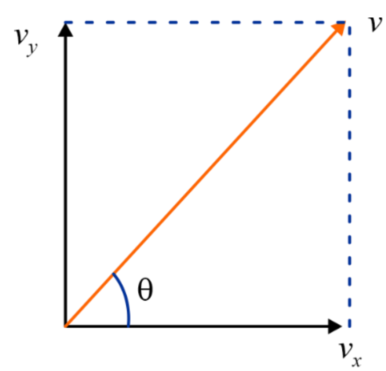


Fig. 6.16 Resultant velocity and its direction

$$\vec{v} = \frac{d\vec{s}}{dt} = \frac{d}{dt}\left(x\hat{i} + y\hat{j}\right)$$

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$|v| = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}\frac{v_y}{v_x}$$

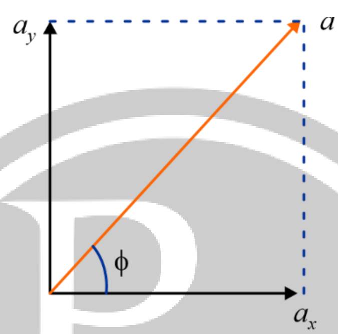


Fig. 6.17 Resultant acceleration and its direction

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v_x\hat{i} + v_y\hat{j}\right)$$

$$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

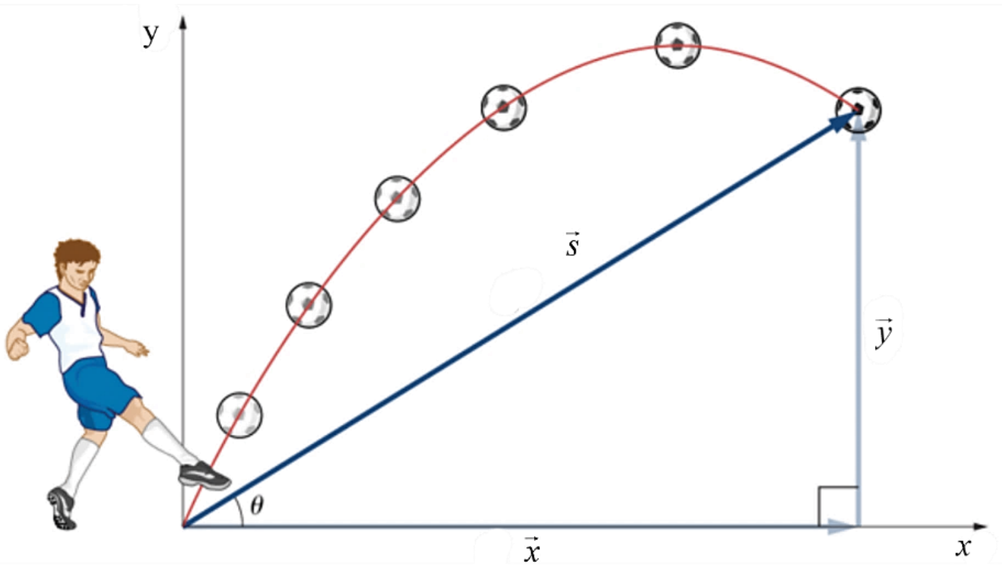
$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$

$$|a| = \sqrt{a_x^2 + a_y^2}$$

$$\phi = \tan^{-1}\frac{a_y}{a_x}$$

6.4 Projectile Motion

Projectile motion is experienced by an object or particle that is thrown near the Earth's surface and moves along a curved path under the action of gravity only.



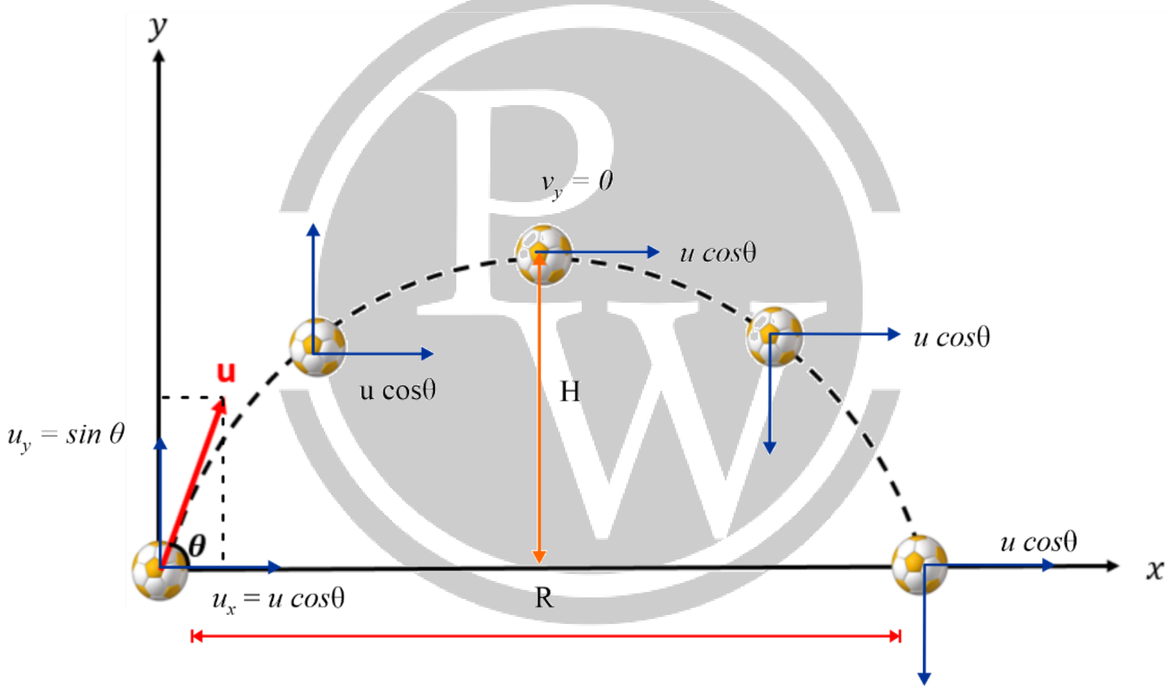


Fig. 6.18 Projectile motion

x	*y*
$a_x = 0$	$a_y = \pm g$
$v_x = u_x$	$v_y = u_y + a_y.t$
$s_x = u_x \times t$	$s_y = u_y t + 1/2 a_y t^2$

Important parameters to calculate:

1. Time of Flight (T) – total time of motion
2. Maximum Height (H) – maximum displacement in y
3. Rang (R) – maximum displacement in x

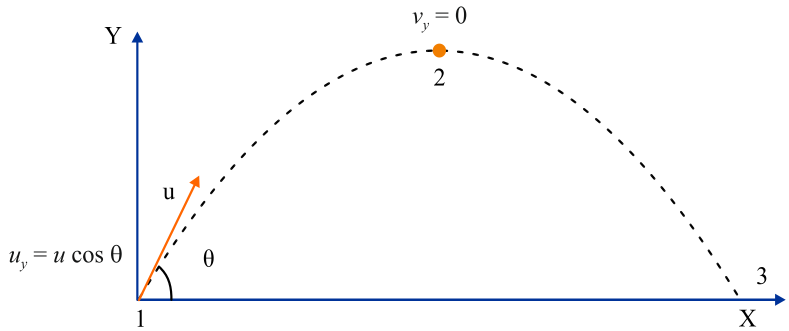


Fig. 6.19 Velocities of a particle in projectile motion

6.4.1 Time of flight (T)

Motion from 1 to 2 in y direction

$$v_y = u_y + a_y . t$$

$$0 = u \sin\theta - g.t$$

$$t = \frac{u \sin\theta}{g}$$

∵ Total time of flight, $T = 2.t$

$$\boxed{T = \frac{2u \sin\theta}{g}}$$

Alternatively:

Motion in y direction from 1 to 3

$$s_y = u_y.T + ½\, a_y.\, T^2$$

$$0 = u \sin\theta.\, T - ½\, g\, T^2$$

$$\boxed{T = \frac{2u \sin\theta}{g}}$$

6.4.2 Maximum Height (H)

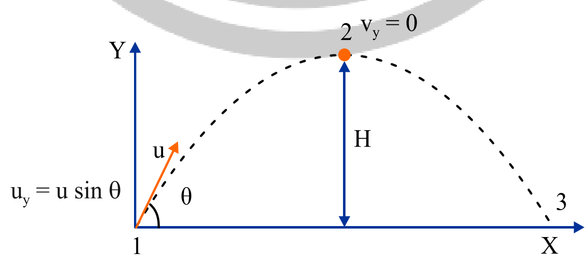


Fig. 6.20 Vertical velocities of a particle in projectile motion

Motion in Y from 1 to 2

$$v_y^2 - u_y^2 = 2a_y s_y$$

$$0 - (u \sin\theta)^2 = -2g.H$$

$$\boxed{H = \frac{u^2 \sin^2\theta}{2g}}$$

6.4.3 Range (R)

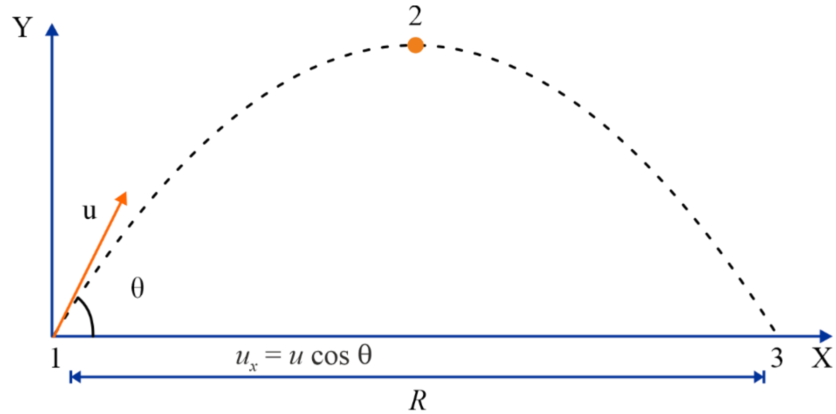


Fig. 6.21 Horizontal velocity of a particle in projectile motion

Motion in x from 1 to 3

$$s_x = u_x \cdot T + 1/2\, a_x \cdot T^2$$

$$R = u\cos\theta \times \frac{2u\sin\theta}{g} + 0$$

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}}$$

Maximum Range

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\sin 2\theta = 1 = \sin 90°$$

$$\theta = 45°$$

$$\boxed{R_{max} = \frac{u^2}{g}}$$

Range vs θ

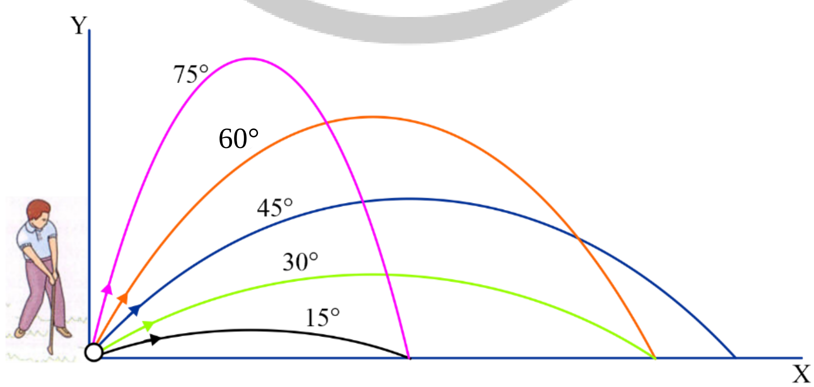


Fig. 6.22 Projectile motion at different angles

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7 KINEMATICS OF RIGID BODIES

7.1 Rotational Motion

- All the particles of the body move in circular paths.
- All the particles of the body have different displacement.

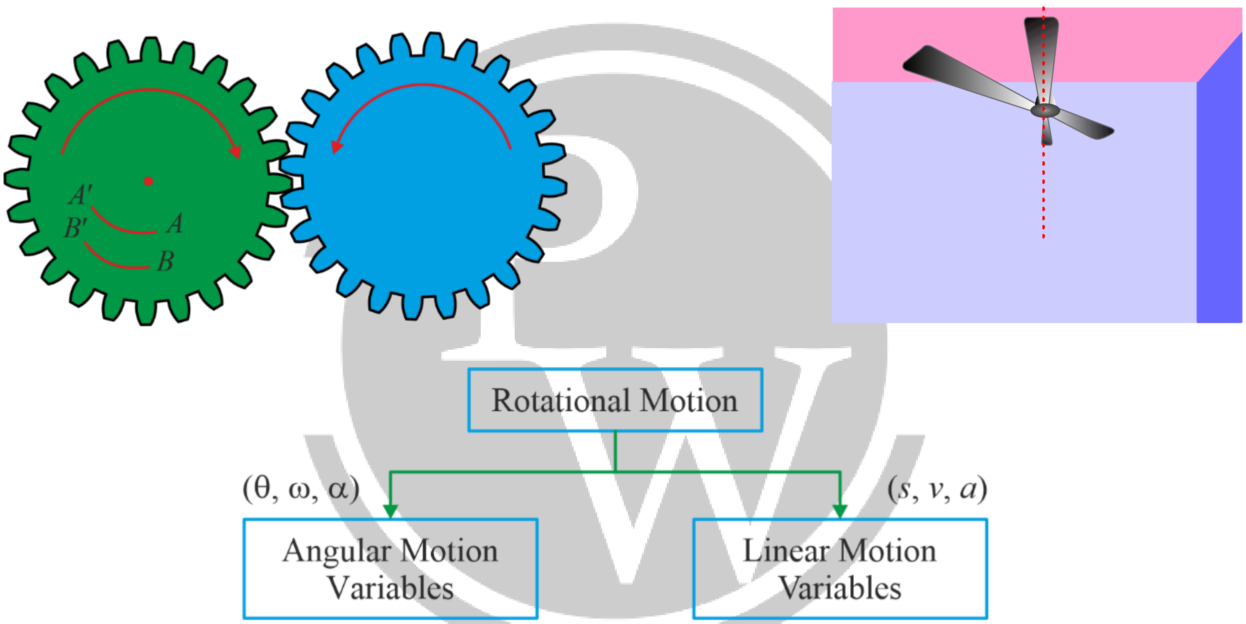

Fig. 7.1 Rotational motion and motion variables

7.1.1 Angular Motion Variables

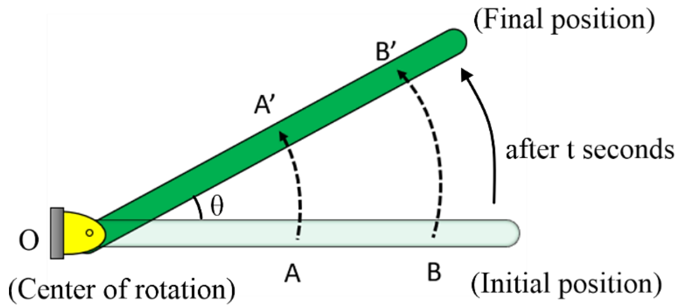

Fig. 7.2 A link rotating about a point

(1) Angular Displacement

$$\theta_A = \theta_B \qquad \text{(radians)}$$

- Angular displacement of every particle is same (θ).

(2) Angular velocity

$$\omega = \frac{dv}{dt} \qquad \text{(rad/s)}$$

ω of every particle is same.

$$\omega_A = \omega_B$$

(3) Angular acceleration

$$\alpha = \frac{d\omega}{dt} \qquad (\text{rad/s}^2)$$

$$\alpha = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt}$$

$$\alpha = \omega\cdot\frac{d\omega}{d\theta}$$

$$\alpha_A = \alpha_B$$

$$\theta \rightarrow S$$

$$\omega \rightarrow v$$

$$\alpha \rightarrow a$$

Constant acceleration	Continuous Motion
1. $\omega = \omega_0 + \alpha t$	1. $\omega = \frac{d\theta}{dt}$
2. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	2. $\alpha = \frac{d\omega}{dt}$
3. $\omega^2 - \omega_0^2 = 2\alpha\theta$	3. $\alpha = \omega\cdot\frac{d\omega}{d\theta}$

7.1.2 Linear Motion Variables

(1) Linear displacement

$$s = r\,\theta$$

where r is the distance of the point from the center of rotation

$$s_A \neq s_B$$

In rotational motion, as r changes linear displacement also changes.

(2) Linear velocity

$$v = r.\omega$$

$$v_A \neq v_B$$

(3) Linear acceleration

$$a = r.\alpha$$

$$a_A \neq a_B$$

7.2 General Motion (Translation + Rotation)

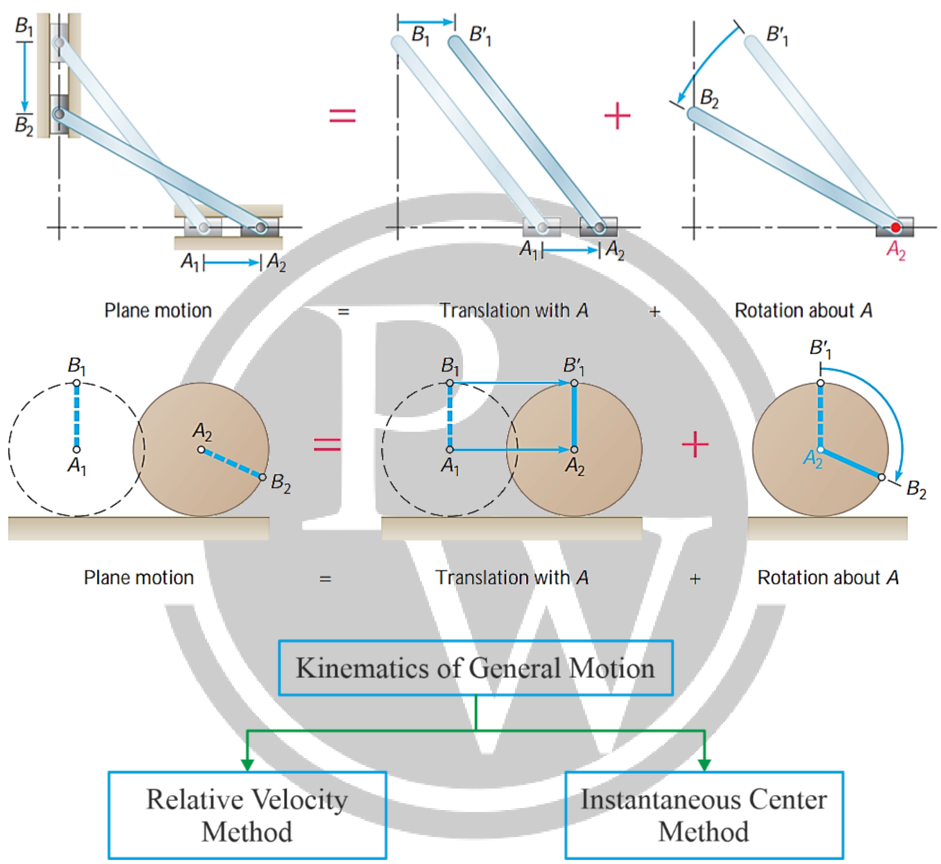

Fig. 7.3 General motion and methods to find motion variables

7.2.1 Relative Velocity Method

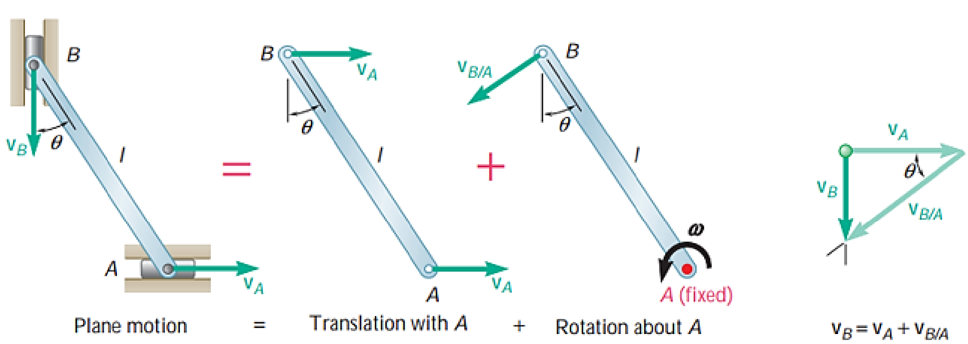

Fig. 7.4 A double slider mechanism and its velocity diagram

With A chosen as the reference point, the velocity of B is the vector sum of the translational portion V_A, plus the rotational portion $V_{B/A}$, which has the magnitude $V_{B/A} = \omega r$

Q. A rod of length 2 m is sliding in a corner as shown. At an instant when the rod makes an angle of 30 degrees with the horizontal plane, the velocity of point A on the rod is 8 m/s. Find

(i) Velocity of the end B (in m/s)

(ii) Velocity of the mid point (in m/s)

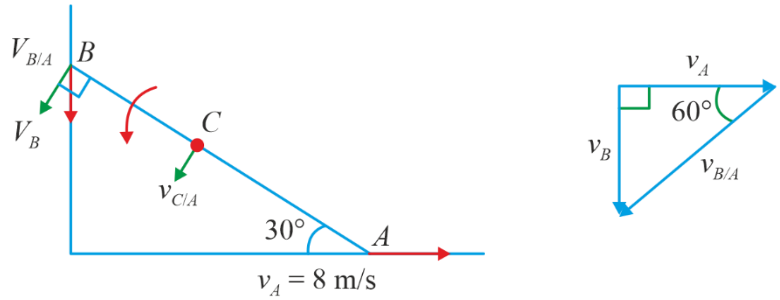

Fig. 7.5 Velocities of points in link AB and its velocity diagram

$$\vec{v}_B = v_A + \vec{v}_{B/A}$$

$$\tan 60° = \frac{v_B}{v_A}$$

$$v_B = 13.85 \text{ m/s}$$

$$v_{B/A} = \sqrt{v_A^2 + v_B^2}$$

$$\omega \times BA = \sqrt{8^2 + 13.85^2}$$

$$\omega \times 2 = 16$$

$$\omega = 8 \text{ rad/s}$$

$$\vec{v}_C = \vec{v}_A + \vec{v}_{C/A}$$

$$v_{C/A} = \omega \times CA$$

$$= 8 \times 1$$

$$= 8 \text{ m/s}$$

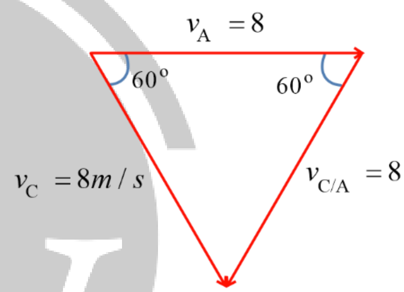

Fig. 7.6 Velocity diagram

From equilateral triangle

$$v_c = 8 \text{ m/s}$$

Alternatively:

Relative velocity of B w.r.t. A along AB = 0

$\therefore$ $\vec{v}_B$ along AB = $\vec{v}_A$ along AB

$$v_B \cos 60° = v_A \cos 30°$$

$$v_B = 8 \times \frac{\cos 30°}{\cos 60°}$$

$$v_B = 13.85 \text{ m/s}$$

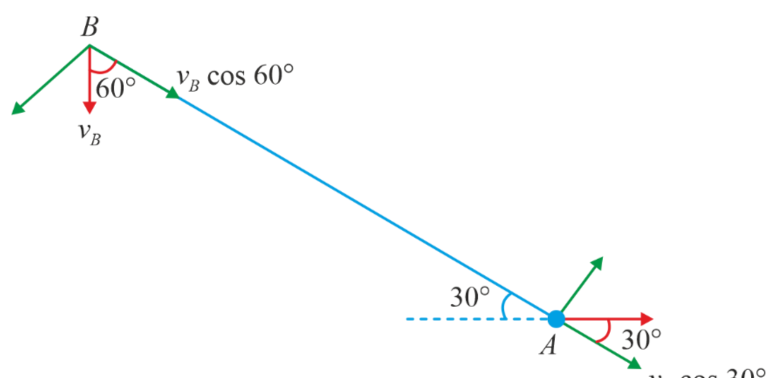

Fig. 7.7 Resolution of velocities at A &B

7.2.2 Instantaneous Center Method

- Combined motion of rotation and translation, may be assumed to be a motion of pure rotation about some imaginary center known as instantaneous center.
- As the position of body goes on changing, therefore the instantaneous center also goes on changing.

$$V_A = \omega \times IA$$

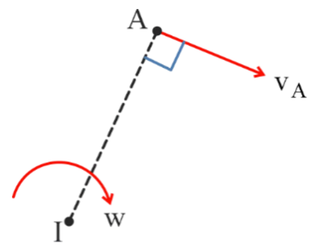

Fig. 7.8 Instantaneous center of A

Location of I Center

Case 1: When the direction of velocities of two particles A and B are known and V_A is not parallel to V_B.

I center lie at the intersection of two lines drawn at A and B, perpendicular to V_A and V_B respectively.

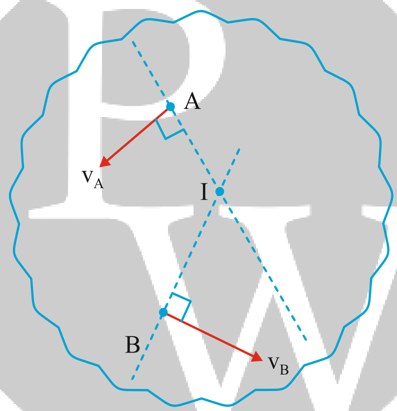

Fig. 7.9 I center when V_A is not parallel to V_B

Case 2: When the direction and magnitude of velocities of two particles A and B are known and V_A is parallel to V_B.

I center lie at the line connecting A and B.

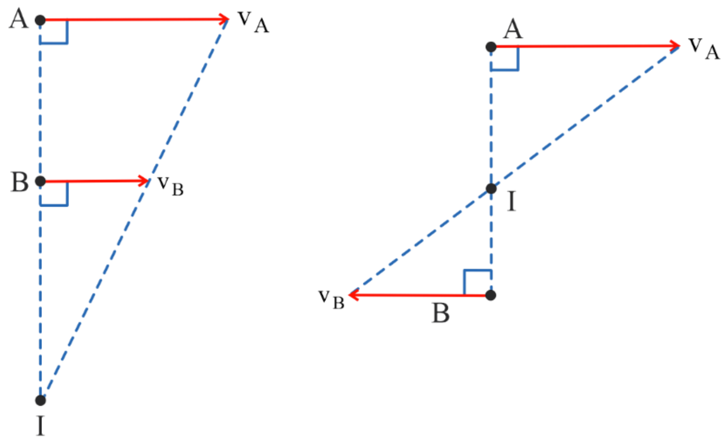

Fig. 7.10 I center when V_A is parallel to V_B

Case 3: Rolling without slipping on a fixed surface.

I center lie at the point of contact at that instant.

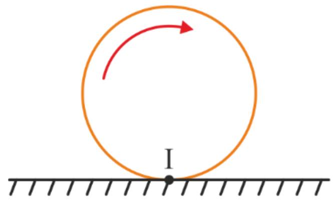

Fig. 7.11 I center of rolling object

Q. A rod of length 2 m is sliding in a corner as shown. At an instant when the rod makes an angle of 30 degrees with the horizontal plane, the velocity of point A on the rod is 8 m/s. Find

(i) Velocity of the end B (in m/s)

(ii) Velocity of the mid point (in m/s)

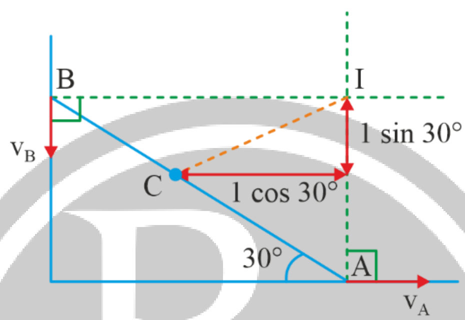

Fig. 7.12 I center of link AB in double slider 4 bar mechanism

$$v_A = \omega \times IA$$

$$8 = \omega \times 2 \sin 30°$$

$$\omega = 8 \text{ rad/s}$$

$$v_B = \omega \times IB$$

$$= 8 \times 2 \cos 30°$$

$$v_B = 13.85 \text{ m/s}$$

$$IC = \sqrt{(1\cos 30°)^2 + (1.\sin 30°)^2} = 1 \text{ m}$$

$$v_c = \omega \times IC$$

$$= 8 \times 1$$

$$v_c = 8 \text{ m/s}$$

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8 KINETICS OF PARTICLES

8.1 Kinetics of Particles

Relation between forces and motion variables (*s*, *v*, *a*)

- D'Alembert's method
- Work - Energy method
- Impulse - momentum method

8.1.1 D'Alembert's Principle

D'Alembert's Principle introduces a new force- Inertia Force. D'Alembert's principle is another form of Newton's second law of motion.

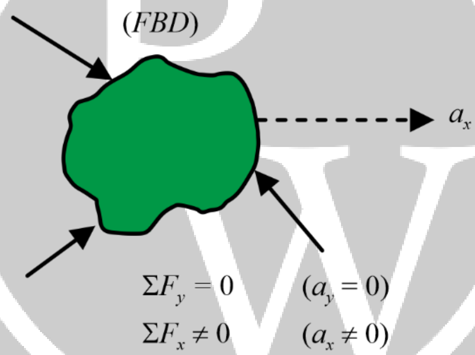


Fig. 8.1 An object with an acceleration in x-direction

Newton's law of motion:

$$\Sigma F_x = m \cdot a_x$$

$$\Sigma F_x - ma_x = 0$$

Total force in *x*

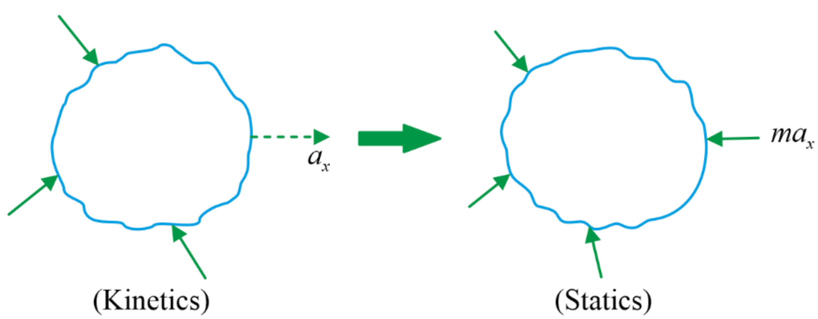


Fig. 8.2 Conversion of an object from kinetics to statics by introducing inertia force

1. An elevator of mass 3000 kg is moving vertically downwards with a constant acceleration. Starting from rest, it travels a distance of 40 m during an interval of 10 seconds. Find the cable tension (in N) during this time. Assume g = 10 m/s^2.

Sol.

$$\Sigma F_y = ma_y$$

$$30000 - T = 3000\ a$$

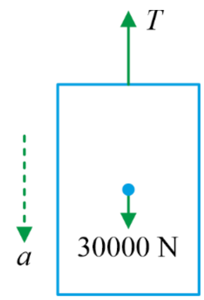
$$s = ut + \frac{1}{2}at^2$$

$$40 = 0 + \frac{1}{2}a \times 10^2$$

$$a = 0.8\ \text{m/s}^2$$

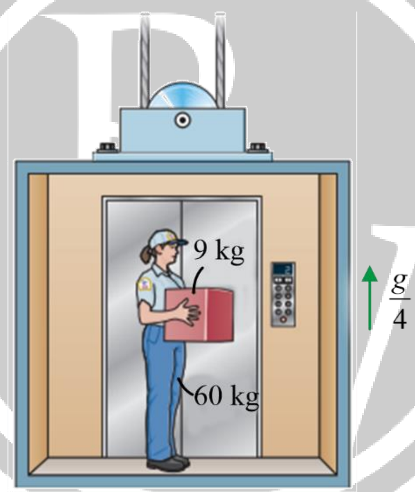
$$30000 - T = 3000 \times 0.8$$

$$T = 27{,}600\ \text{N}$$

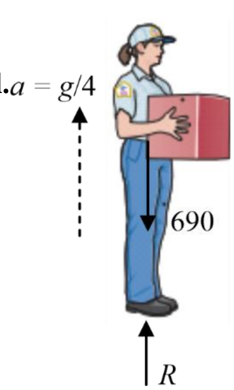


(FBD)

2. A 60-kg woman holds a 9-kg package as she stands within an elevator which briefly accelerates upward at a rate of $g/4$. Determine the force R which the elevator floor exerts on her feet during the acceleration interval. Assume g = 10 m/s^2.



Sol.

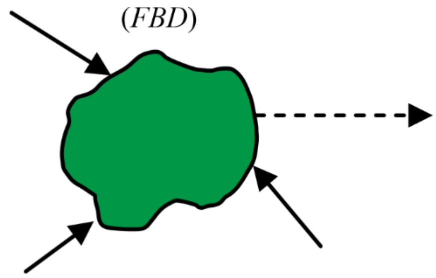


$$\Sigma F_y = ma_y$$

$$R - 690 = 69 \times \frac{g}{4} = 69 \times \frac{10}{4}$$

$$R = 862.5\ \text{N}$$

8.1.2 Work – Energy Method



ΣF = Net force in direction of motion

Fig. 8.3 FBD of an object

Work done on body = Charge in K.E. of body

$$\Sigma F \times s = \frac{1}{2} m (v^2 - u^2)$$

$$\Sigma F = \text{Net force in direction of motion.}$$

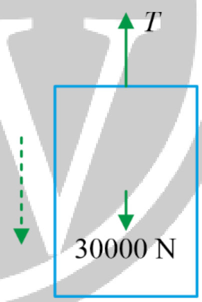
3. An elevator of mass 3000 kg is moving vertically downwards with a constant acceleration. Starting from rest, it travels a distance of 40 m during an interval of 10 seconds. Find the cable tension (in N) during this time.

Assume $g = 10$ m/s^2.

Sol.

$$\Sigma F \times s = \frac{1}{2} m (v^2 - u^2)$$

$$(30000 - T) \times 40 = \frac{1}{2} \times 3000 (v^2 - 0)$$



$$s = ut + \frac{1}{2} at^2$$

$$40 = 0 + \frac{1}{2} a \times 10^2$$

$$a = 0.8 \text{ m/s}^2$$

$$v = u + at$$

$$= 0 + 0.8 \times 10$$

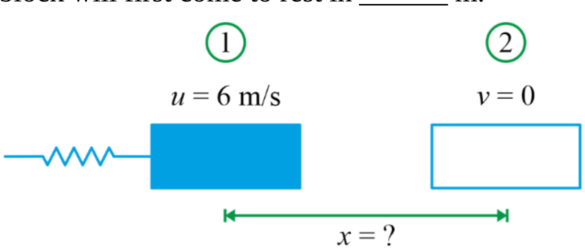
$$v = 8 \text{ m/s}$$

$$(30000 - T) \times 40 = \frac{1}{2} \times 3000 (8^2 - 0)$$

$$T = 27{,}600 \text{ N}$$

4. The 10 kg block is moving to the right at 6 m/s when the 360 N/m spring is unstretched. If the horizontal plane is frictionless, the block will first come to rest in _______ m.

Sol.



Conservation of Energy

$$\underset{(1)}{(\text{Total energy})} = \underset{(2)}{(\text{Total energy})}$$

$$\frac{1}{2}mu^2 = \frac{1}{2}kx^2$$

$$10 \times 6^2 = 360 \times x^2$$

$$x = 1 \text{ m}$$

Work – Energy

$$\text{Work done} = \text{Change in K.E.}$$

$$0 = \frac{1}{2}m\,(0 - u^2) + \frac{1}{2}kx^2$$

$$\frac{1}{2}mu^2 = \frac{1}{2}kx^2$$

- Conservation of energy is only applicable if there is no friction.
- If there is friction, apply work-energy.

8.1.3 Conservation of Momentum

When the resultant of the external forces acting on a system of particles is zero, the linear momentum of the system is conserved.

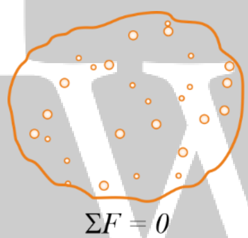


Fig. 8.4 A system of particles with net external force zero

$$\Sigma F \cdot t = m.(v - u)$$

$$0 = m.(v - u)$$

$$u = v$$

8.2 Collision (Impact) of Elastic Bodies

Impact occurs when two bodies collide with each other during a very short period of time.

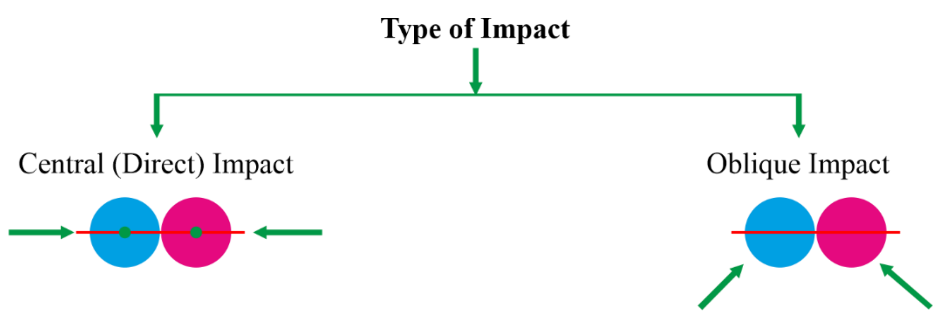


Fig. 8.5 Types of impact

8.2.1 Central (Direct) Impact

Fig. 8.6 Before and after velocities in Direct impact

Let $(u_1 > u_2)$ $v_1 = ?$ Let $(v_2 > v_1)$

$v_2 = ?$

Principle of Conservation of Momentum:

Initial momentum of system = Final momentum of system

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad \text{. . . (i)}$$

$$\begin{pmatrix} \vec{v} = +ve \\ \overleftarrow{v} = -ve \end{pmatrix}$$

Newton's Law of Restitution

- For a given pair of bodies in collision, the ratio of relative velocity after collision and relative velocity before collision is always same.
- This ratio is known as coefficient of restitution (e). Its value depends on the material.

$$e = 0 \text{ to } 1$$

$$e = 1 \text{ for perfectly elastic bodies}$$

$$e = 0 \text{ for perfectly plastic bodies}$$

$$e = \frac{v_2 - v_1}{u_1 - u_2} \quad \text{. . . (ii)}$$

- For perfectly elastic collision, kinetic energy of the system is conserved.

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

- If $e < 1$, some kinetic energy is lost.

$$\text{Loss in K.E.} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

Special Cases

Case 1: If there is perfectly plastic collision between two bodies, the bodies stick together and move as single body after collision.

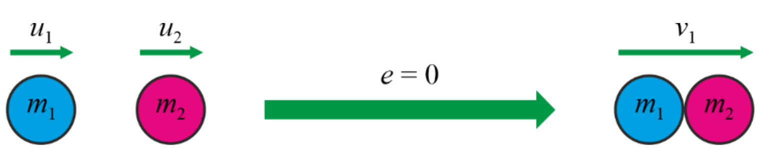


Fig. 8.7 Direct impact of perfectly plastic bodies

$$e = \frac{v_2 - v_1}{u_1 - u_2} = 0$$

$$v_2 = v_1$$

Case 2: If there is perfectly elastic collision between two bodies, of equal masses, the velocities of the bodies interchange after collision.

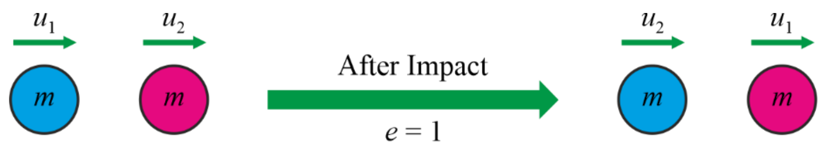


Fig. 8.8 Direct impact of perfectly elastic bodies

$$e = 1 = \frac{v_2 - v_1}{u_1 - u_2}$$

$$v_2 - v_1 = u_1 - u_2 \quad \text{. . . (i)}$$

$$mu_1 + mu_2 = mv_1 + mv_2$$

$$u_1 + u_2 = v_1 + v_2 \quad \text{. . .(ii)}$$

$$v_1 = u_2$$

$$v_2 = u_1$$

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9 KINETICS OF RIGID BODIES

9.1 Kinetics of Rigid Bodies

Relation between forces and motion variables. (s, v, a)

- D'Alembert's method
- Work - Energy method
- Impulse - momentum method

Rotation + General motion

9.1.1 D'Alembert's Method

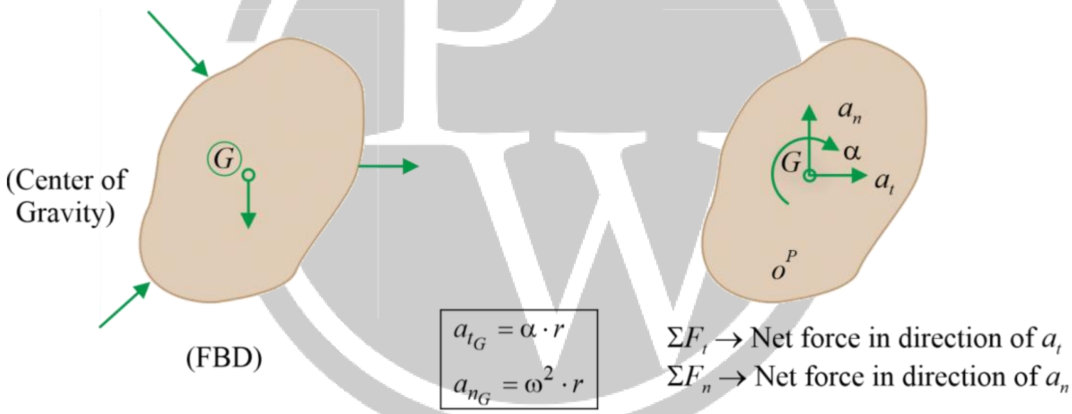


Fig. 9.1 An object under application of external loads and its accelerations

$$\Sigma F_t = m \cdot a_t$$

$$\Sigma F_n = m \cdot a_n$$

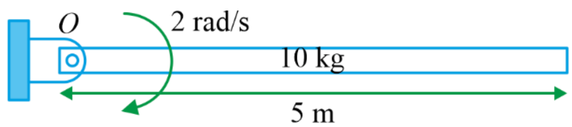
$$\Sigma M_G = I_G \cdot \alpha$$

OR

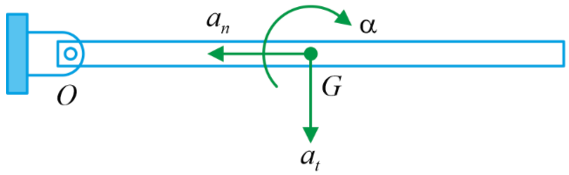
$$\Sigma M_P = I_P \cdot \alpha$$

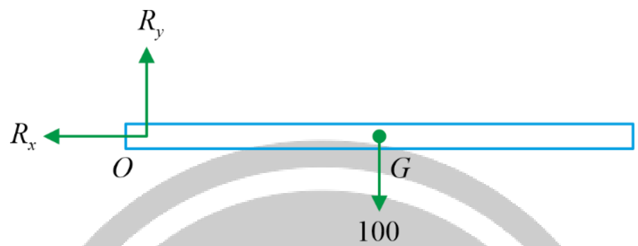
I_G = Mass MOI of body about G

1. A slender rod of mass 10 kg and length 5 m is rotating with an angular velocity of 2 rad/s at the instant as shown in figure. Find the angular acceleration of the rod and the horizontal and vertical reaction at the hinge.



Sol.





$$\Sigma F_n = m \cdot a_n = m \cdot \omega^2 r = m\omega^2(OG)$$

$$R_X = 10 \times 2^2 \times 2.5$$

$$R_X = 100 \text{ N}$$

$$\Sigma F_t = m \cdot a_t = m \cdot \alpha \cdot r = m \cdot \alpha\ (OG)$$

$$(100 - R_y) = 10 \times \alpha \times 2.5 \quad \ldots \text{(i)}$$

$$\Sigma M_G = I_G \cdot \alpha = \left(\frac{ml^2}{12}\right) \cdot \alpha$$

$$R_y \times 2.5 = \left(\frac{10 \times 5^2}{12}\right) \cdot \alpha \quad \ldots \text{(ii)}$$

By solving equation (i) and (ii)

$$R_y = 25 \text{ N}$$

$$\alpha = 3 \text{ rad/s}^2$$

Alternatively:

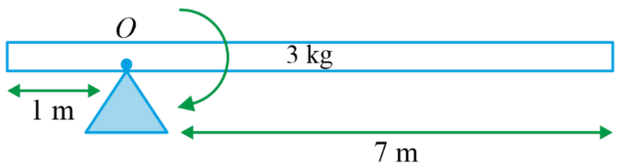
$$\Sigma M_0 = I_0 \cdot \alpha = \left(\frac{ml^2}{3}\right) \cdot \alpha$$

$$100 \times 2.5 = \left(\frac{10 \times 5^2}{3}\right) \times \alpha$$

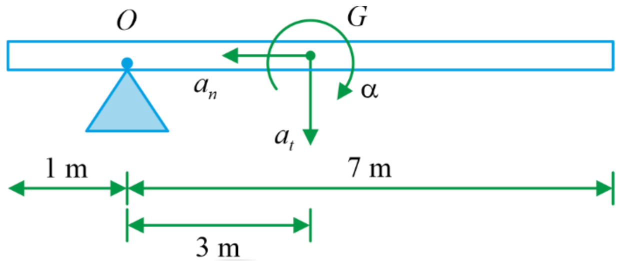
$$\alpha = 3 \text{ rad/s}^2$$

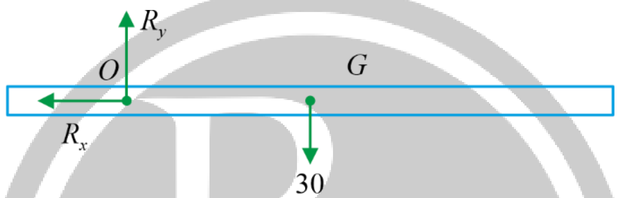
2. A uniform slender rod (8 m length and 3 kg mass) rotates in a vertical plane about a horizontal axis 1 m from its end as shown in the figure. The magnitude of the angular acceleration (in rad/s^2) of the rod at the position shown is ______.

(g = 10 m/s^2) **[GATE-ME-14:2M]**



Sol.





$$I_0 = I_G + m \cdot (OG)^2$$

$$= \frac{3\times8^2}{12} + 3\times3^2$$

$$I_0 = 43\ kg - m^2$$

$$\Sigma M_0 = I_0\, \alpha$$

$$30 \times 3 = 43 \cdot \alpha$$

$$\alpha = 2.093\ \text{rad/s}^2$$

9.1.2 Work – Energy Method

Work done on the body = Change in K.E. of the body.

Kinetic Energy

(1) Pure Rotation:

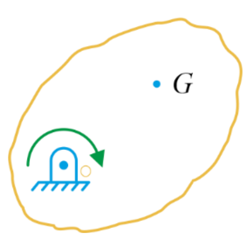


Fig. 9.2 An object under pure rotation

$$\text{K.E.} = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$$

OR

$$= \frac{1}{2}[I_G + m \cdot r^2]\,\omega^2$$

$$\text{K.E.} = \frac{1}{2}I_o\omega^2$$

(2) General Motion (Rotation + Translation):

$$\text{K.E.} = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$$

OR

$$\text{K.E.} = \frac{1}{2}I_I \cdot \omega^2$$

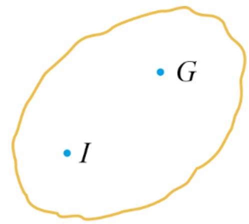


Fig. 9.3 Center of gravity and I center of an object

Work Done

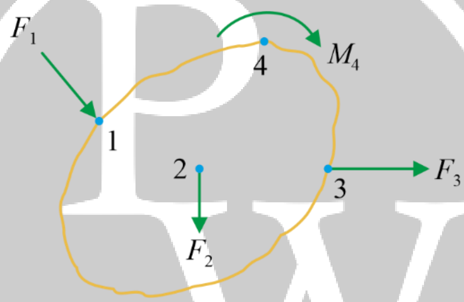
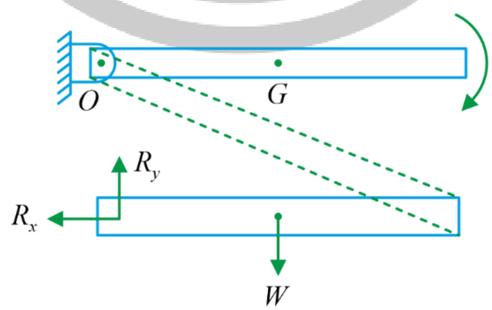


Fig. 9.4 FBD of an object under application of loads and moment

$$\text{W.D.} = \pm F_1s_1 \pm F_2s_2 \pm F_3s_3 \pm M_4\theta_4$$

Forces with zero work done

(1)



W.D. by R_x and $R_y = 0$

as displacement of O = 0

(2) Rolling without Slipping:

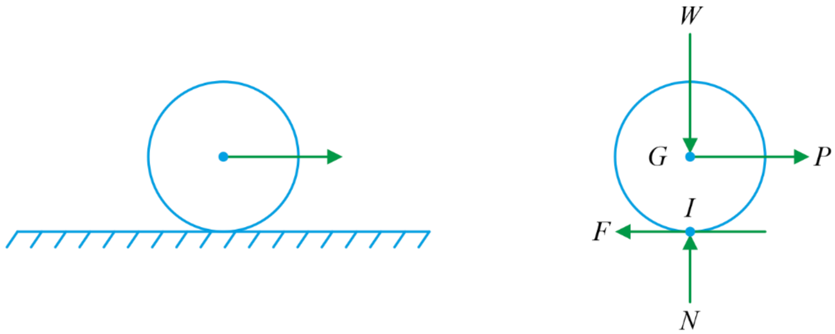
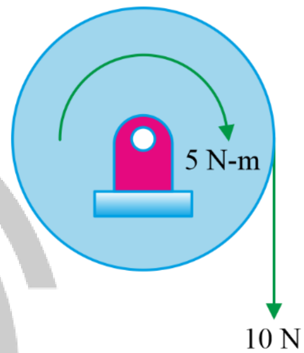


Fig. 9.5 A roller rolling without slipping and its FBD

W.D. of $F = 0$

as $V_I = 0$

3. The 30-kg disk of radius 0.2 m shown in figure is pin supported at its center. Determine the number of revolutions it must make to attain an angular velocity of 20 rad/s starting from rest. It is acted upon by a constant force of 10 N, which is applied to a cord wrapped around its periphery, and a constant couple moment of 5 N-m. Neglect the mass of the cord in the calculation.



Sol.

$$S_P = \theta \times r$$

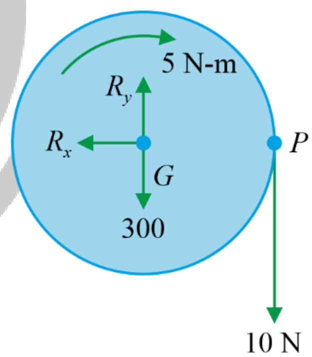
$$S_P = \theta \times 0.2$$

$$10 \times S_P + 5 \times \theta = \frac{1}{2}m(v_G^2 - u_G^2) + \frac{1}{2}I_G(\omega_2 - \omega_0^2)$$

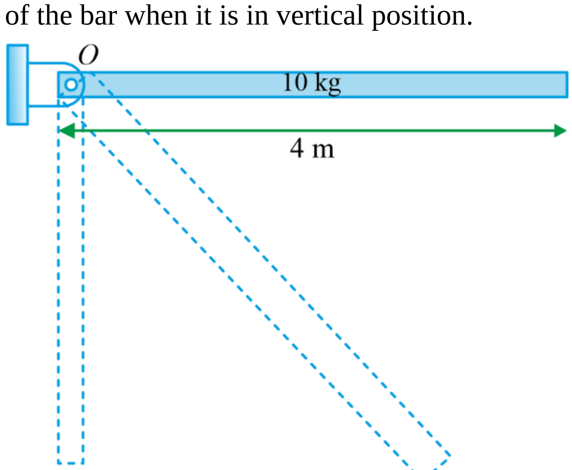
$$10 \times 0.2 \times \theta + 5 \times \theta = \frac{1}{2}m(0-0) + \frac{1}{2} \times \frac{30 \times 0.2^2}{2}(20^2 - 0)$$

$$\theta = 17.14 \text{ radius}$$

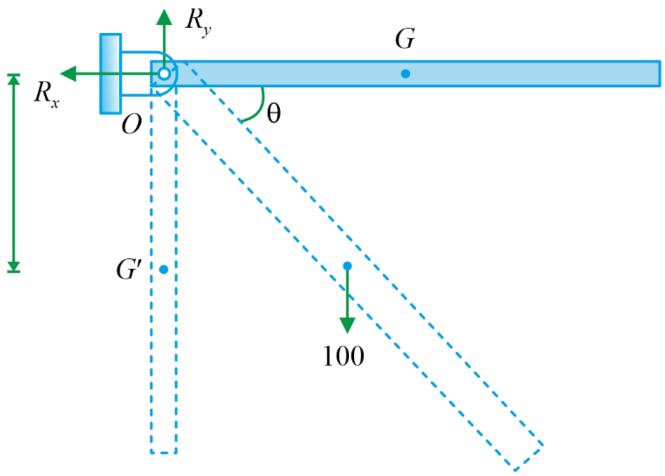
$$\text{No. of revelations} = \frac{\theta}{2\pi} = \frac{17.14}{2\pi} = 2.73$$



4. A uniform bar of mass 10 kg and length 4 m hangs from a frictionless hinge. It is released from rest in the horizontal position. Find the angular velocity of the bar when it is in vertical position.



Sol. $S_G = \frac{4}{2} = 2\text{m}$



$$100 \times S_G = \frac{1}{2}m(v_G^2 - u_G^2) + \frac{1}{2}I_G(\omega^2 - \omega_0^2)$$

OR

$$100 \times S_G = \frac{1}{2}I_0(\omega^2 - \omega_0^2)$$

$$100 \times 2 = \frac{1}{2}\left(\frac{10 \times 4^2}{3}\right)(\omega^2 - 0)$$

$$\boxed{\omega = 2.74} \text{ rad/s}$$

9.1.3 Impulse – Momentum Method

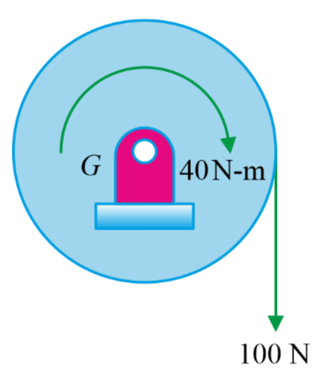
Fig. 9.6 An object under the application of load and its velocities

$$\Sigma F_x \times t = m(v_{Gx} - u_{Gx})$$

$$\Sigma F_y \times t = m(v_{Gy} - u_{Gy})$$

$$\Sigma M_G \times t = I_G(\omega - \omega_0)$$

5. The 20 kg disk of radius 0.75 m shown in figure is pin supported at its center. It is acted upon by a constant force of 100 N, which is applied to a cord wrapped around its periphery, and a constant couple moment of 40 N-m. Find the angular velocity of the disc and horizontal and vertical reactions at the hinge, 2 sec after starting from ret.



Sol.

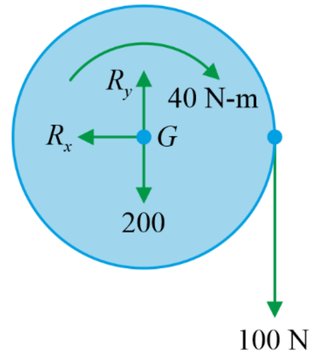
$$v_{Gx} = 0$$

$$u_{Gx} = 0$$

$$v_{Gy} = 0$$

$$u_{Gy} = 0$$

$$\Sigma F_x \times t = m(v_{Gx} - u_{Gx})$$
$$R_x \times 2 = 0$$
$$R_x = 0$$
$$\Sigma F_y \times t = m(v_{Gy} - u_{Gy})$$
$$(R_y - 200 - 100) \times 2 = 0$$
$$R_y = 300 \text{ N}$$
$$\Sigma M_G \times t = I_G(\omega - \omega_0)$$
$$(100 \times 0.75 + 40) \times 2 = \frac{20 \times 0.75^2}{2}(\omega - 0)$$
$$\omega = 40.88 \text{ rad/s.}$$



9.2 Lagrange's Equation of Motion

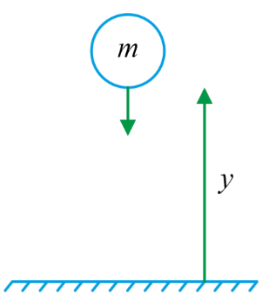
$$\boxed{L = T - U}$$
(Lagrangian) (K.E.) (P.E.)

q = Generalised coordinate.

Equation of Motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$
$$\left(\dot{q} = \frac{dq}{dt}\right)$$

6.



$$q = y, \ \dot{q} = \dot{y}$$
$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2$$
$$U = mgh = mgy$$
$$L = T - U$$
$$L = \frac{1}{2}m\dot{y}^2 - mgy$$
$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial y} = -mg$$

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = m\ddot{y}$$

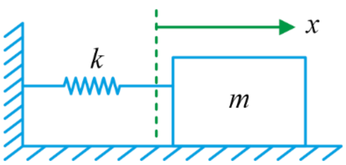
Equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$m\ddot{y} - (-mg) = 0$$

$$\ddot{y} + g = 0$$

$$\ddot{y} = -g \text{ (–ve sign indicates } a \text{ is } \downarrow)$$

7. 

$$q = x, \; \dot{q} = \dot{x}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$U = \frac{1}{2}kx^2$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

Equation of motion:

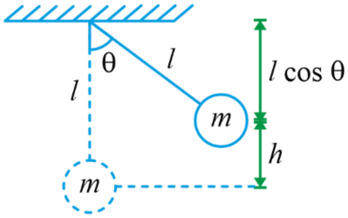
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} - (-kx) = 0$$

$$m\ddot{x} + kx = 0$$

$$\boxed{\ddot{x} + \left(\frac{k}{m}\right)x = 0} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

8.



(String is massless)

$$q = \theta,\ \dot{q} = \dot{\theta}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega l)^2$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

$$U = mgh = mg\,(l - l\cos\theta)$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl + mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

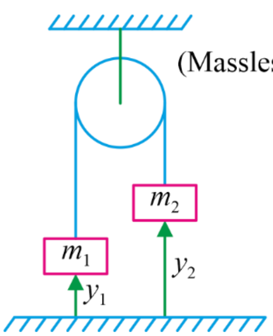
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$ml^2\ddot{\theta} - (-mgl\sin\theta) = 0$$

$$l^2\ddot{\theta} + gl\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0} \quad (\sin\theta \approx \theta)\ \ \omega_n = \sqrt{\frac{g}{l}}$$

9.



(Massless and frictionless pulley)

$$y_1 + y_2 = c$$

$$y_2 = c - y_1$$

$$\dot{y}_2 = -\dot{y}_1$$

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2$$

$$U = m_1gy_1 + m_2gy_2 = m_1gy_1 + m_2g\,(c - y_1)$$

$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - m_1gy_1 - m_2gc + m_2gy_1$$

$$\frac{\partial L}{\partial y_1} = -m_1g + m_2g = (m_2 - m_1)g$$

$$\frac{\partial L}{\partial \dot{y}_1} = (m_1 + m_2)\dot{y}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = (m_1 + m_2)\ddot{y}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) - \frac{\partial L}{\partial y_1} = 0$$

$$(m_1 + m_2)\ddot{y}_1 - (m_2 - m_1)g = 0$$

$$\ddot{y}_1 = \frac{(m_2 - m_1)g}{(m_1 + m_2)}$$

❑❑❑