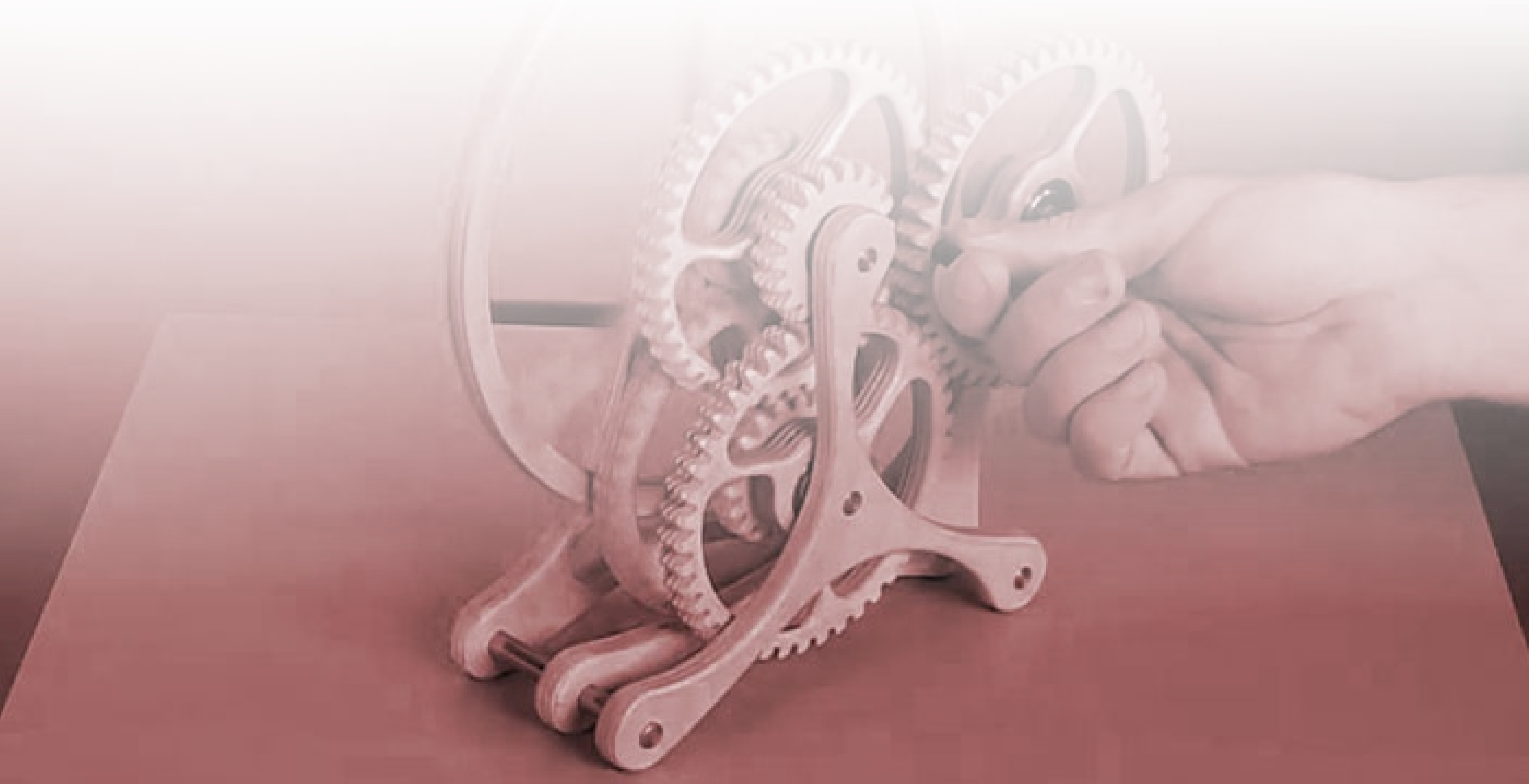
Theory of Machines



Published By:

Physics Wallah

ISBN: 978-93-94342-39-2

Mobile App: Physics Wallah (Available on Play Store)

Website: www.pw.live

Email: support@pw.live

Theory of Machines

INDEX

1 SIMPLE MECHANISMS

1.1 Types of Constrained Motion

- **Completely Constrained:** Motion in definite direction of force applied.
- **Successfully Constrained:** Motion is possible in more than one direction but it is made in only one direction with the help of third link.
- **Incompletely (unconstrained) Motion:** Motion is possible in more than one direction depending on direction of force.

1.2 Resistant Body

- A body/link which can transfer motion/power at least in one direction

1.3 Kinematic Link

- **Rigid link:** Crank, Connecting rod, Piston, etc.
- **Flexible link:** Belt, Rope, Chain drives, etc.
- **Fluid link:** Hydraulic brakes, Ram, Jack, etc.

Links may also be classified as

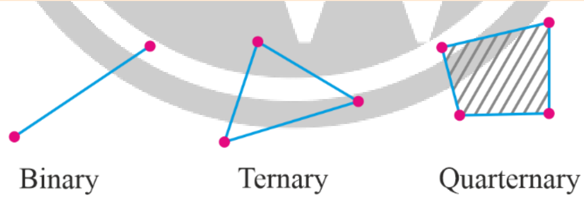


Fig. 1.1 Links

1.4 Kinematic Pair

- Joint of two links having constrained relative motion between them.

(a) According to nature of Relative Motion:

- Turning pair (Revolute/Pin Joint)
- Sliding Pair (Prismatic Pair)
- Screw Pair (Helical Pair)
- Rolling Pair
- Spherical Pair

(b) According to nature of contact:

- **Lower Pair:** Surface/Area contact
- **Higher Pair:** Line/Point contact.

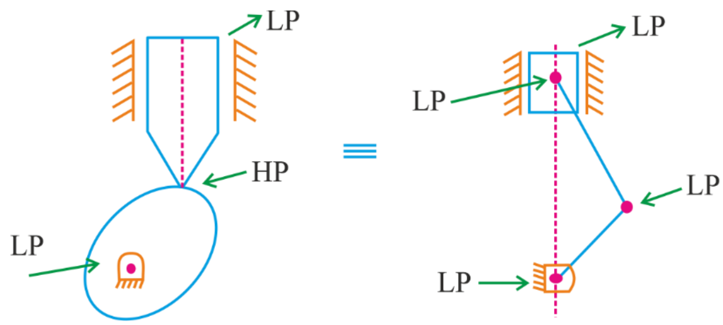


Fig. 1.2 Cam and Follower

$$\boxed{1\,\text{Higher Pair} \cong 2\,\text{Lower Pairs}}$$

(c) According to nature of mechanical constraint:

- **Self-closed pair:** Permanent contact between links.
- **Forced closed pair:** Links are in contact either by gravity action or spring action.

1.5 Types of Joint

- Binary Joints →
- Ternary Joint →
- Quaternary Joint →

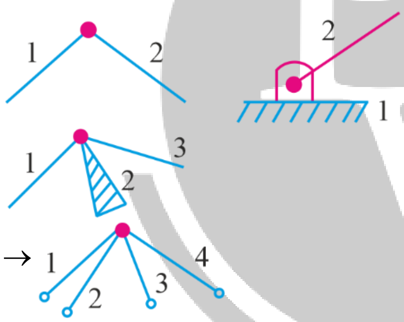


Fig. 1.3 Joints

> **Note:**
> If n links are connected at joint ≡ (n-1) Binary Joints

1.6 Kinematic Chain

- Closed chain having constrained relative motion.
- If one link of kinematic chain is fixed it becomes mechanism.

1.7 Degree of Freedom (Mobility)

- Minimum number of independent variables required to define position/motion of the system.
- For rigid body in space:

$$\boxed{\text{DOF} = 6 - \text{Number of restraints}}$$

- Restraint represents those motions which are not possible.
- For 2-D planar mechanisms:

$$\boxed{F = 3(l-1) - 2j - h}$$

l = number of links

h = number of higher pairs.

j = number of binary joints.

1.8 Case of Redundant Link

- Link which don't produce any extra constraint. Such link should not be counted while finding DOF.

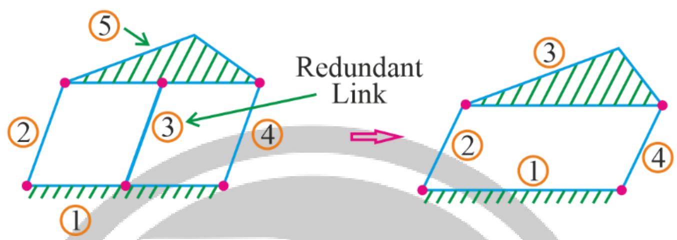


Fig. 1.4 Mechanism with Redundant link

1.9 Redundant Degree of Freedom

- Sometimes one or more links of a mechanism can be moved without causing any motion to rest of links of mechanism. such link is said to have redundant degree of freedom (F_r)

$$\boxed{F = 3(l-1) - 2j - h - F_r}$$

F_r = number of dummy motion (number of redundant DOF)

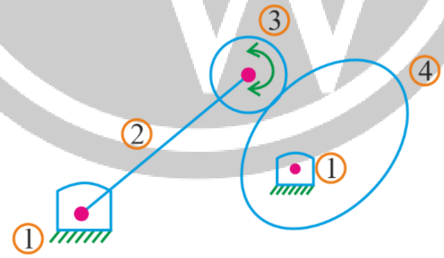


Fig. 1.5 Mechanism with Redundant DOF

Link 3 $\rightarrow$ have redundant DOF

Physical significance of DOF of Mechanism

F = 3 (4 –1) – 2 (3) – 1–1 = 1

- If F = 1 $\rightarrow$ Kinematic Chain
- If F = 0 $\rightarrow$ Frame /structure
- If F < 1 $\rightarrow$ Super structure/Indeterminate Structure.
- If F > 1 $\rightarrow$ Unconstrained Chain

- DOF of open chain:

$\boxed{F = 3l - 2j - h} \rightarrow$ No link is fixed.

1.10 Grubler's Criteria

- Origin of first kinematic chain,

$$F = 1,\ h = 0$$
$$F = 3(l-1) - 2j - h$$
$$1 = 3l - 3 - 2j - 0 \quad \Rightarrow \quad \boxed{3l - 2j - 4 = 0}$$

$l_{min} = 4$ to make a kinematic chain with all lower pairs.

- Four bar mechanism
- Single slider crank mechanism
- Double slider crank mechanism

1.11 Four bar Mechanism

- Four links with four turning pairs
- Input and output links $\rightarrow$ rotates only
- Complete rotation $\rightarrow$ Crank
- Partial rotation $\rightarrow$ Rocker

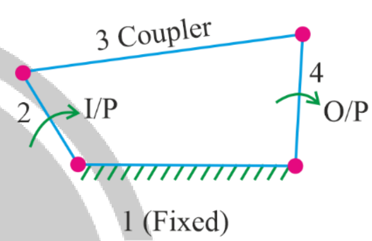


Fig. 1.6 Four bar chain mechanism

1.12 Grashoff's Law

- For continuous relative motion:

$\boxed{(s+l) \leq (p+q)} \rightarrow$ Class-I mechanisms

s = length of shortest link
l = length of longest link
p, q = length of other two links.

Case: I $(s + l) < (p + q)$: Law satisfies [class I]

- Shortest link fixed: Double crank mechanism.
- Shortest link is coupler: Double rocker mechanism
- Link adjacent to shortest link is fixed: Crank –Rocker Mechanism

Case: II $(s + l) = (p + q)$: Law satisfies [class I]

E.g.: $s = 20cm$, $l = 50cm$, $p = 40cm$, $q = 30cm$

Parallelogram Linkage

l – Fixed: Double crank
s – Fixed: Double crank

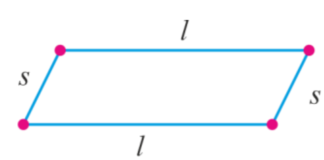


Fig. 1.7 Parallelogram Linkage

Deltoid Linkage

s – Fixed: Double crank
l – Fixed: Crank-rocker mechanism

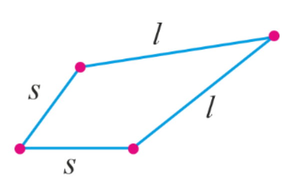


Fig. 1.8 Deltoid Linkage

Case: III (*s* + *l*) > (*p* + *q*): Law does not satisfy

- Class-II mechanism
- Double rocker mechanism

1.13 Transmission Angle for 4-bar Mechanism

Angle between coupler link and output link

= transmission angle.

$$\mu \begin{cases} \text{minimum;} & \theta = 0^\circ \\ \text{maximum;} & \theta = 180^\circ \end{cases}$$

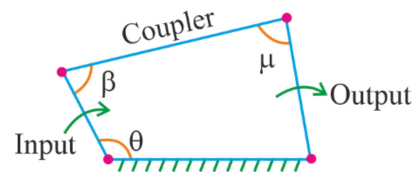


Fig. 1.9 Transmission angle

1.14 Single Slider Crank Chain

- In 4-bar chain, one turning pair replaced by sliding pair.

(a) First Inversion of single slider-crank mechanism:

Cylinder is fixed

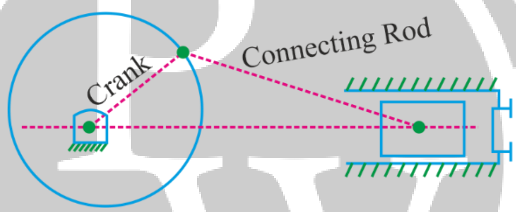


Fig. 1.10 Single slider Crank Chain

Rotary motion $\longleftrightarrow$ Reciprocating motion

$$\left\{ \begin{array}{l} \text{Application:} \\ \bullet \text{ Reciprocating Engine} \\ \bullet \text{ Reciprocating compressor} \end{array} \right\}$$

(b) Second inversion of slider crank mechanism:

Crank Fixed

Application:

- Whitworth quick return mechanism
- Rotary engine (Gnome engine)

Whitworth Quick Return Mechanism:

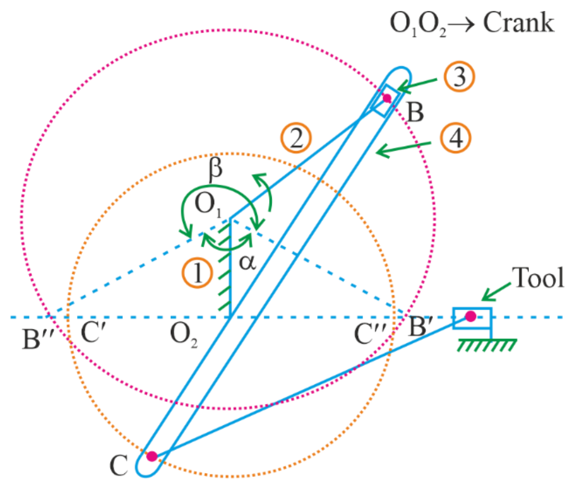


Fig. 1.11 Whitworth Quick Return Mechanism

Forward stroke: (B' B B")

$$\text{Quick Return Ratio} = \frac{\text{time of cutting}}{\text{time of return}} = \frac{\beta}{\alpha} > 1$$

(c) Third inversion of slider Crank Mechanism:

Connecting Rod is fixed.

Application:

- Oscillating cylinder Engine.
- Crank & slotted lever mechanism.

  Used in shaping and Slotting machine

  Velocity is max at the mid stroke.

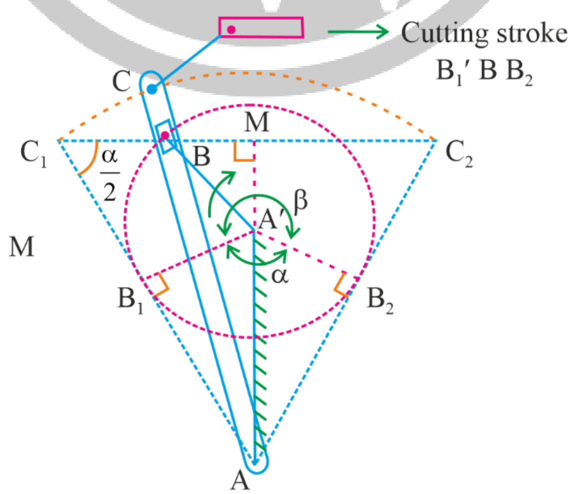


Fig. 1.12 Crank and slotted lever mechanism.

CA = length of slotted Bar.

A′B = length of crank

$$QRR = \frac{(\text{time})_{\text{cutting}}}{(\text{time})_{\text{return}}} = \frac{\beta}{\alpha} > 1$$

AA′ = length of connecting rod.

Stroke length = $2(C_1A) \times \frac{A'B_1}{AA'}$

(d) Fourth inversion of slider crank mechanism:

Slider is fixed

Application:

- Hand Pump / Bull Engine.

1.15 Double Slider Crank Chain

- Four bar chain with two turning pairs; two sliding pairs.

(a) **First Inversion:** Slotted bar is fixed.

Application: Elliptical Trammel used to draw ellipse

(b) **Second Inversion:** Any one slider is fixed.

Application: Scotch Yoke Mechanism [To convert reciprocating motion into rotation and vice versa]

(c) **Third Inversion:** Link connecting sliders is fixed.

Application: Oldham coupling [used to connect shafts having lateral misalignment.]

$$\omega_{driver} : \omega_{driven} = 1.1$$

1.16 Mechanical Advantage

- Ratio of output (Force/torque) to the input (force/torque), at any instant.

$$MA = \frac{F_o}{F_I} = \frac{T_o}{T_I} = \frac{\text{Load}}{\text{Effort}}$$

- Relation between *MA* and efficiency

$$\eta = \frac{p_o}{p_i} = \frac{F_o \cdot V_0}{F_i V_i} = \frac{T_o \cdot \omega_o}{T_i \cdot \omega_i}$$

$$MA = \eta \cdot \frac{V_i}{V_o} = \eta \frac{\omega_i}{\omega_o}$$

1.17 Toggle Positions in 4-bar Chain

It represents extreme position of output link as a rocker in 4-bar chain.

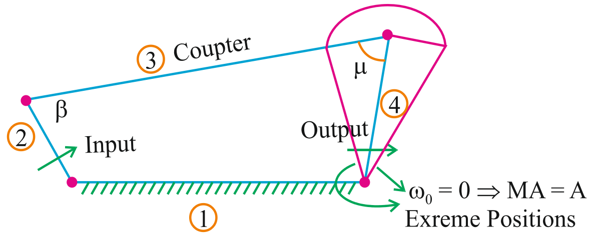


Fig. 1.13 Toggle Positions

1.18 Intermittent Motion Mechanism

- Motion is repeated after specific time internal while input is given continuously.
- Geneva mechanism → Indexing
- Ratchet Mechanism → Clocks

1.19 Straight Line Motion Mechanism

Exact Straight-line motion mechanism

- Hart's mechanism
- Scott Russell Mechanism
- Modified Scott Russell mechanism.

Approx. straight-line motion mechanism

- Grass Hooper's Mechanism
- Robert's mechanism
- Tchebichoff's mechanism
- Watt indicator mechanisms
- Pantograph

❒❒❒

2 Velocity and Acceleration Analysis

2.1 Velocity of a Point on Link

- Relative velocity of one point of a link w.r.t. other point is always perpendicular to the link.
- For a link undergoing pure translation, relative velocity between points is zero.

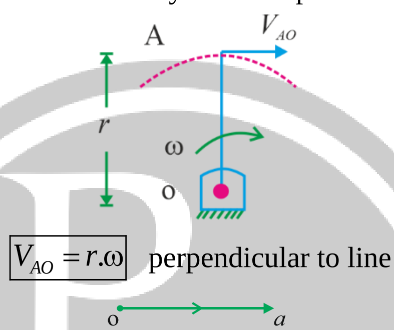


Fig. 2.1 Vector

2.2 Instantaneous Centre (I-centre)

- Any link at any instant can be assumed in pure rotation about imaginary point in space known as I-centre.
- I-centre may keep on changing as links move.
- For two bodies having velocity motion between them, there is an I-centre.
- In a mechanism, if l = number of links. Then

  Number of I-Centres = $\frac{l(l-1)}{2}$

- Basic I-centres:

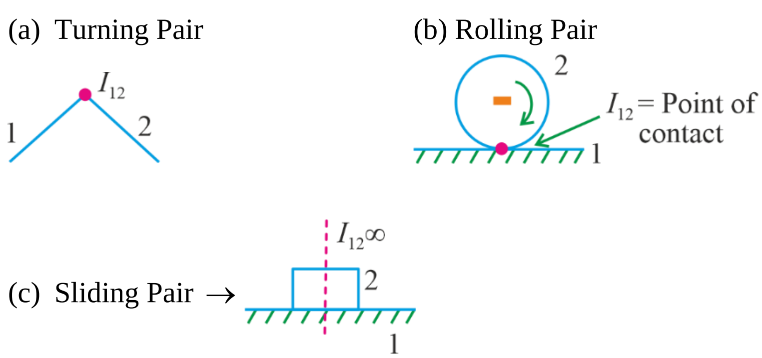


Fig. 2.2 I-centre

- **Centrode:** Locus of I-centres for relative motion between links.
- **Axode:** The line passing through 1-centre and perpendicular to the plane of motion is known as instantaneous axis. The locus of instantaneous axis for a link during the whole motion is known as Axode. It is surface

Motion	Centrode	Axode
General	Curve	Curved surface
Rectilinear	Straight Line	Plane surface
Pure Rotation	Point	Straight Line

2.3 Kennedy's Theorem

- For example. For link 2, 3, 5 I_{23}, I_{35} I_{25} will be in a line, for any three links having relative motion among them, their three I-centres must lie in a straight line.

2.4 Angular Velocity Ratio Theorem

$$\omega_m (I_{mn} I_{1m}) = \omega_n (I_{mn} I_{1n})$$

If I_{1m} and I_{1n} lies at same side of I_{mn} then sense of and ω_m & ω_n will be same

2.5 Maximum Velocity During Cutting Stroke and Return Stroke in Crank-Slotted Bar

- During cutting stroke:

  $\omega \rightarrow$ Uniform angular velocity of crank

  $r \rightarrow$ crank radius

  $l_1 \rightarrow$ length of slotted bar

  $l_e \rightarrow$ length of connecting rod (fixed link)

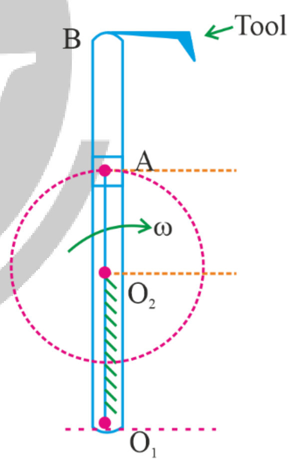


Fig. 2.3 Crank and Slotted lever mechanism

$$\omega_{S.L.\text{max cutting stroke}} = \frac{r\omega}{l+r}$$

$$\omega_{S.L.\text{max return stroke}} = \frac{r\omega}{l-r}$$

Note:

$\omega_{S.L.}$ & Speed of cutter will be max at mid stroke when slotted lever becomes vertical

- For any rotating link, we have two components of accelerations

  (i) Radial/centripetal acceleration

  (ii) Tangential acceleration

2.5.1 Acceleration Analysis

$$(a_{BA})_{\text{radial}} = \frac{V_{BA}^2}{(BA)} = (AB)\cdot\omega^2$$

$$(a_{BA})_{\tan g.} = (AB)\cdot\alpha \rightarrow \perp^r \text{ perpendicular to link}$$

$$|a_{BA}| = \sqrt{(a_{BA})_{\text{radial}}^2 + (a_{BA})_{\text{tang.}}^2}$$

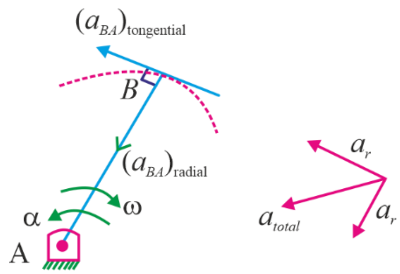


Fig. 2.4 Acceleration Components

2.6 Coriolis Acceleration

- This acceleration is associated with the slider when it is sliding on a rotating body.

$$a_c = 2 \times V \times \omega$$

V = sliding velocity of slider

ω = angular velocity of body on which slider is sliding.

Direction of:

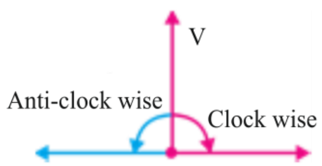
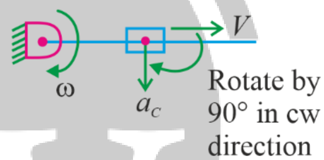
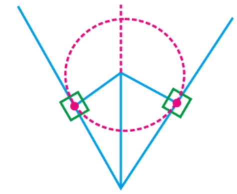
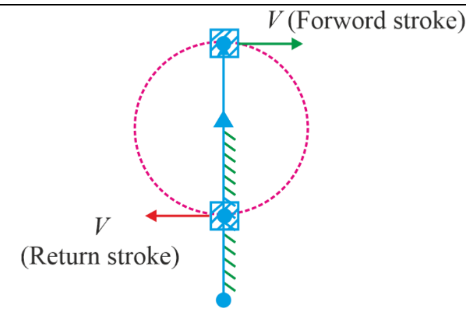
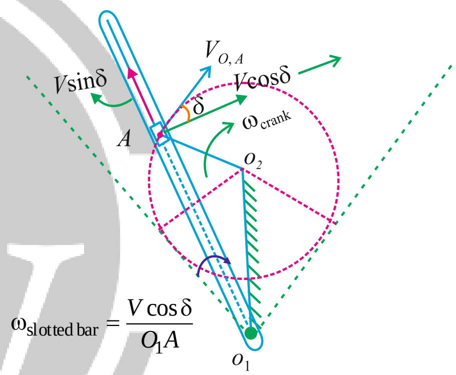
1. Take the sense of ω 2. Rotate V in that sense by 90º 	For Example: 

Fig. 2.5 Coriolis Acceleration

Note:
- Coriolis Acceleration exists in Crank and slotted bar QRMM.
- At four positions, a^c becomes zero.

At the extreme positions. At extreme ends, $\omega_{\text{slotted bar}} = 0 \Rightarrow a_C = 0$	 **Fig. 2.6 Coriolis Acceleration**
At the mid-stroke At mid-stroke, slider isn't sliding ($V_s = 0$) $\Rightarrow a_C = 0$.	 **Fig. 2.7 Coriolis Acceleration**
• Between extreme position a mid-stroke. $a^C = 2 \times V_{\text{sliding}} \times \omega_{S.L.}$ $\Rightarrow \left[a^C = 2(V \sin\delta)\right] \cdot \left(\frac{V\cos\delta}{O_1A}\right)$	 **Fig. 2.8 Crank and Slotted Lever Mechanism**

2.7 Rubbing Velocity

- If r = radius of pin at joint O,

  V_r = (Rubbing velocity) = $r(\omega_1 \pm \omega_2)$

  (+) = when links move in opposite direction

  (–) = when links move in same direction.

❒❒❒

3 CAM AND FOLLOWERS

3.1 Introduction

- A cam is mechanical member used to impart desired motion to a follower by direct contact.
- The cam may be rotating or reciprocating whereas the follower may be rotating, reciprocating or oscillating.
- It is used in automatic machine, IC engine, machine tools, printing control mechanism.

Element of Cam & Follower

- A driver member known as the cam.
- A driven member called the follower.
- A frame which supports the cam and guides the follower.

Point to remember:

A cam and follower combination belong to the category of higher pair.

3.2 Types of Cam

3.2.1 According to the shape:

- **Wedge and flat cams**

A wedge cam has a translational motion. The follower can either translate or oscillate.

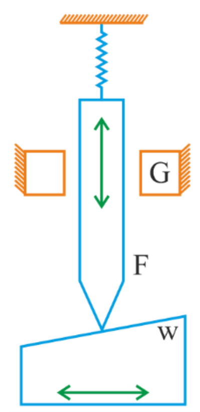


Fig. 3.1 Wedge and flat cams

- **Radial or Disc Cams**

A cam in which the follower moves radially from the centre of rotation of the cam is known as a radial or disc cam.

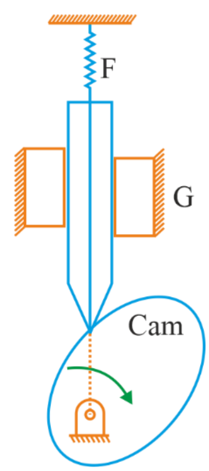


Fig. 3.2 Radial or Disc Cams

- **Spiral cams**
  A spiral cam is a face cam in which a groove is cut in the form of a spiral. It is used in computer.

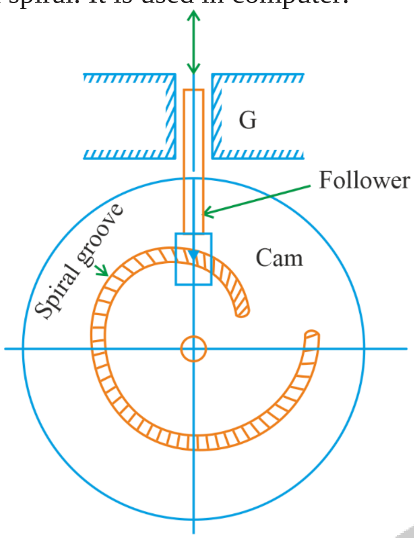


Fig. 3.3 Spiral cams

- **Cylindrical cams**
  In a cylindrical cam, a cylinder which has a circumferential contour cut in surface, rotates about its axis. It is also known as barrel or drum cams.

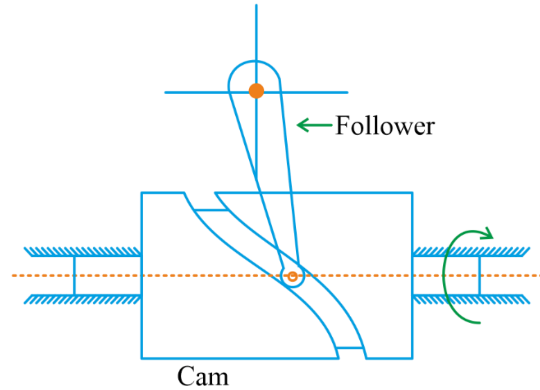


Fig. 3.4 Cylindrical cams

- **Conjugate cams**
  It is a double disc cam and preferred when the requirement is low wear, low noise, better control of the follower, high speed, high dynamic loads etc.

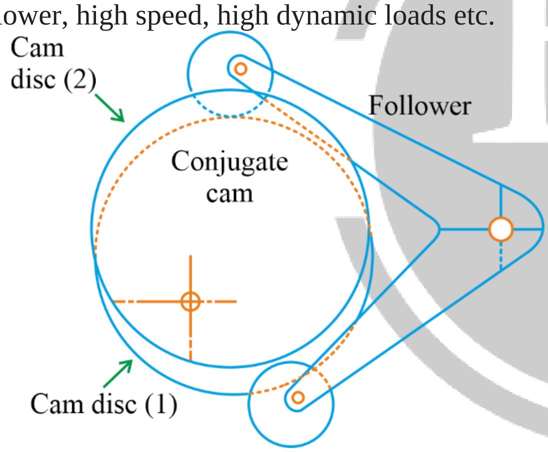


Fig. 3.5 Conjugate cams

- **Globoidal cams**
  It has two types of surfaces i.e. convex or concave. It is used when moderate speed and the angle of oscillation of the follower is large.

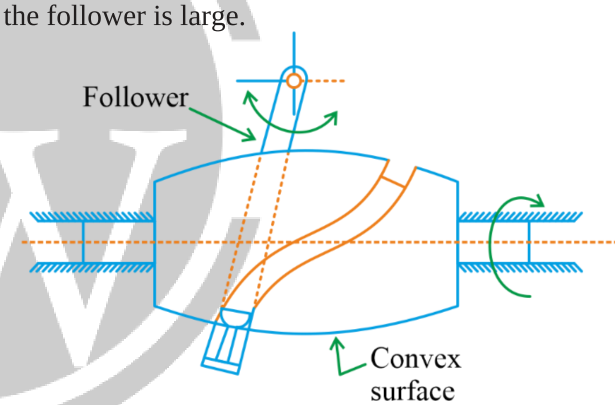


Fig. 3.6 Globoidal cams

3.2.2 According to the follower movement:

- **Rise-Return-Rise (RRR)**
  In this, there is alternate rise and return of the follower with no period of dwells. The follower has a linear or an angular displacement.

- **Dwell-Rise-Return Dwell (D-R-R-D)**
  In this cam, there is rise and return of the follower after a dwell.

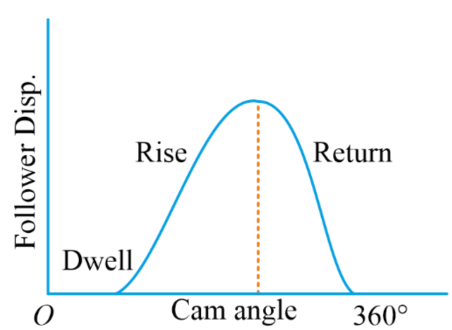


Fig. 3.8

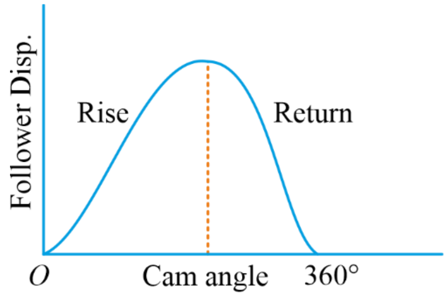
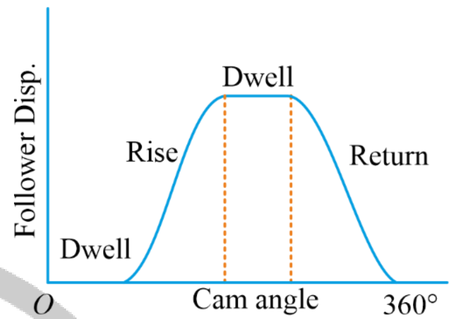


Fig. 3.7

- **Dwell-Rise-Dwell-Return-Dwell (D-R-D-R-D)**

  The dwelling of the cam is followed by rise and dwell and subsequently by return and dwell. In case the return of the follower is by a fall, the motion may be known as Dwell-Rise-Dwell (DRD).



D-R-D-R-D

Fig. 3.9

3.3 Type of followers

3.3.1 According to the Shape

- **Knife-edge follower:**
  Its use is limited as it produces a great wear of the surface at the point of contact

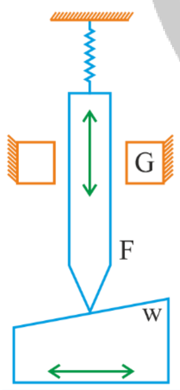


Fig. 3.10 Knife-edge follower

- **Roller follower:**
  At low speeds, the follower has a pure rolling action, but at high speeds, some sliding also occurs.

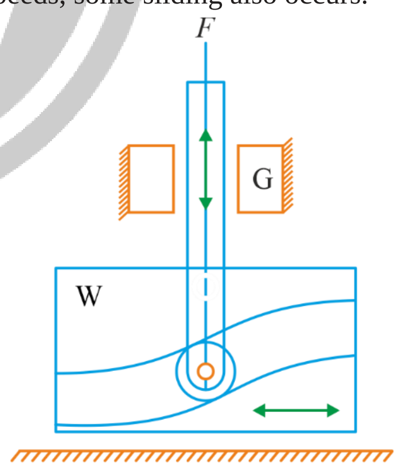


Fig. 3.11 Roller follower

- **Mushroom follower:**
  It does not pose the problem of jamming the cam

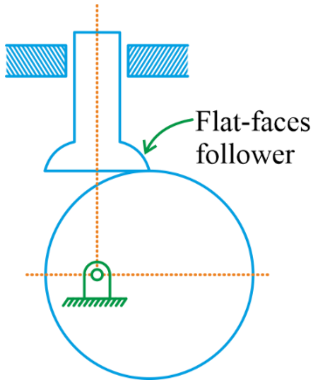


Fig. 3.12 Mushroom follower

3.3.2 According to the Movement

- **Reciprocating follower:**
  In this type, as the cam rotates the follower reciprocates or translates in the guides.

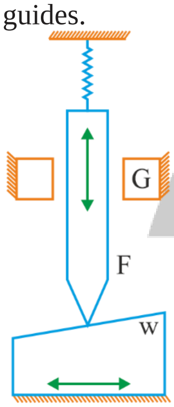


Fig. 3.13 Reciprocating follower

- **Oscillating follower:**
  The follower is pivoted at a suitable point on the frame and oscillates as the cam makes the rotary motion.

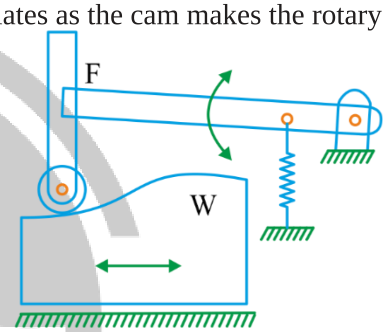


Fig. 3.14 Oscillating follower

3.3.3 According to the Location

- **Radial follower:**
  The follower is known as a radial follower if the line of movement of the follower passes through the centre of rotation of the cam

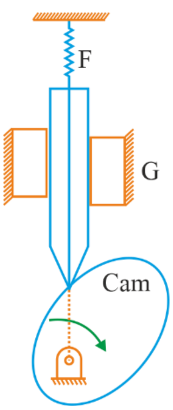


Fig. 3.15 Radial follower

- **Offset follower:**
  If the line of movement of the roller follower is offset from the centre of rotation of the cam, the follower is known as an offset follower.

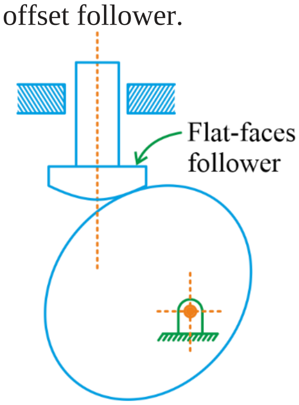


Fig. 3.16 Offset follower

3.4 Terminology and Force Exerted by Cam

3.4.1 Terminology of Cam

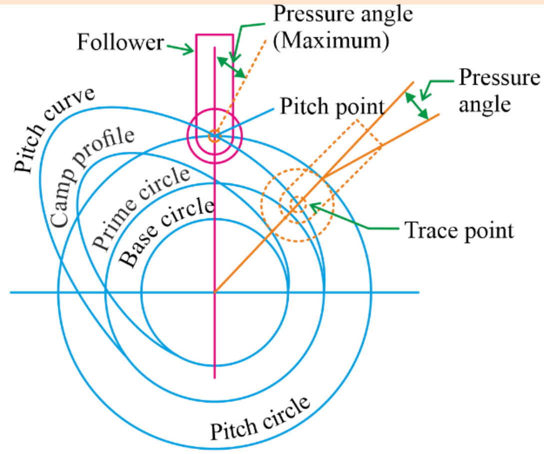


Fig. 3.17 Terminology of cam

- **Base circle**
  It is the **smallest** circle **tangent** to the cam profile drawn from the centre of rotation of a radial cam.
- **Pressure angle:** The pressure angle, representing the steepness of the cam profile, is the angle between the normal to the pitch curve at a point and the direction of the follower motion. It varies in magnitude at all instants of the follower motion.
- **Pitch Point:** It is the point on pitch curve at which the pressure angle is **maximum**.
- **Prime circle:** The **smallest** circle drown **tangent** to the pitch curve is known as prime circle.
- **Angle of Ascent (ϕ_a):** It is the angle through which the cam turns during the time the follower rise.
- **Angle of dwell (δ):** Angle of dwell is the angle through which the cam turns while the follower **remains stationary** at the highest or **lowest position.**
- **Angle of Decent (ϕ_d):** It is the angle through which the cam turns during the time the follower returns to the initial position.
- **Angle of Action:** It is the total angle moved by cam during the time, between the beginning of rise and the end of return of the follower.
- **Point to remember:**
- The dynamic effects of acceleration (jerks) usual speed of the cams.

3.4.2 Force Exerted by Cam

The force exerted by a cam on the follower is always normal to the surface of the cam at the point of contact. The vertical component (F cosα) lifts the follower whereas the horizontal component (F sin α) exerts lateral pressure on the bearing.

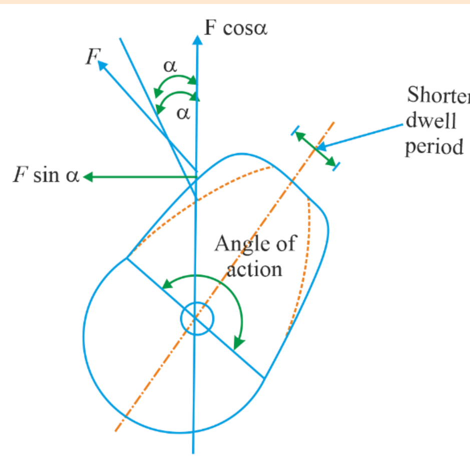


Fig. 3.18 Force Component on cam

3.5 Standard follower Motion Analysis

s = follower displacement (instantaneous)

h = maximum follower displacement

V = velocity of the follower

f = acceleration of the follower

θ = cam rotation angle (instantaneous)

φ = cam rotation angle for the maximum follower displacement

β = angle on the harmonic circle

3.5.1 Simple Harmonic Motion (SHM)

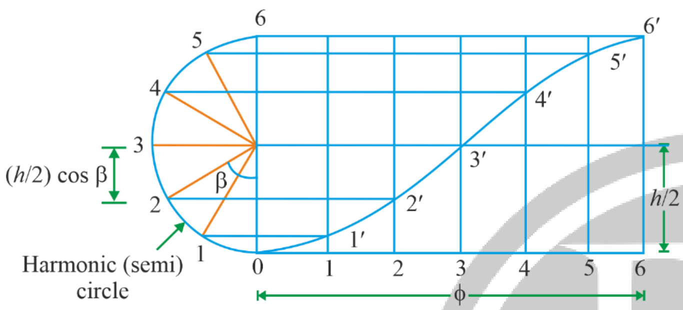


Fig. 3.19 Simple Harmonic Motion (SHM)

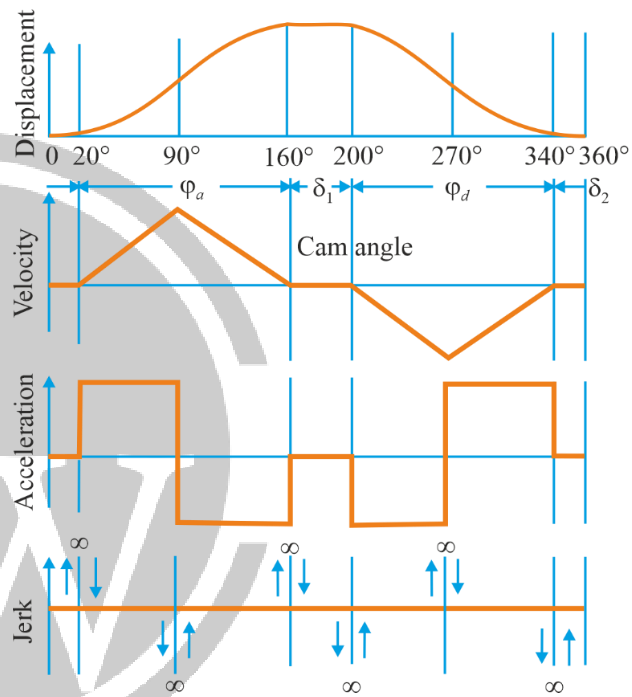


Fig. 3.20 Simple Harmonic Motion (SHM)

- **Displacement of follower**

$$s = \frac{h}{2}\left(1 - \cos\frac{\pi\omega t}{\varphi}\right)$$

- **Maximum velocity**

$$v_{max} = \frac{h}{2}\frac{\pi\omega}{\varphi} \text{ at } \theta = \frac{\varphi}{2}$$

- **Maximum acceleration**

$$f_{max} = \frac{h}{2}\left(\frac{\pi\omega}{\varphi}\right) \text{at } \theta = 0$$

3.5.2 Constant Acceleration and Deacceleration (Parabolic)

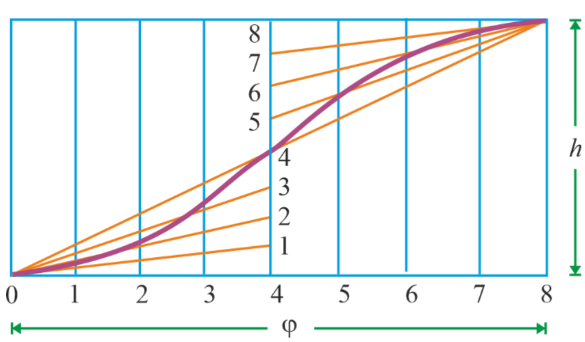


Fig. 3.21 Constant Acceleration and Deacceleration (Parabolic)

- **Displacement of follower**

$$s = \frac{h}{2} \text{ and } t = \frac{\varphi/2}{\omega}$$

- **Maximum velocity of follower**

$$v_{max} = \frac{4h\omega}{\varphi^2}\frac{\varphi}{2} = \frac{2h\omega}{\varphi}$$

at $\theta = \frac{\varphi}{2}$.

- **Acceleration of follower**

$$f = \frac{2h/2}{\varphi^2/4\omega^2} = \frac{4h\omega^2}{\varphi^2}$$

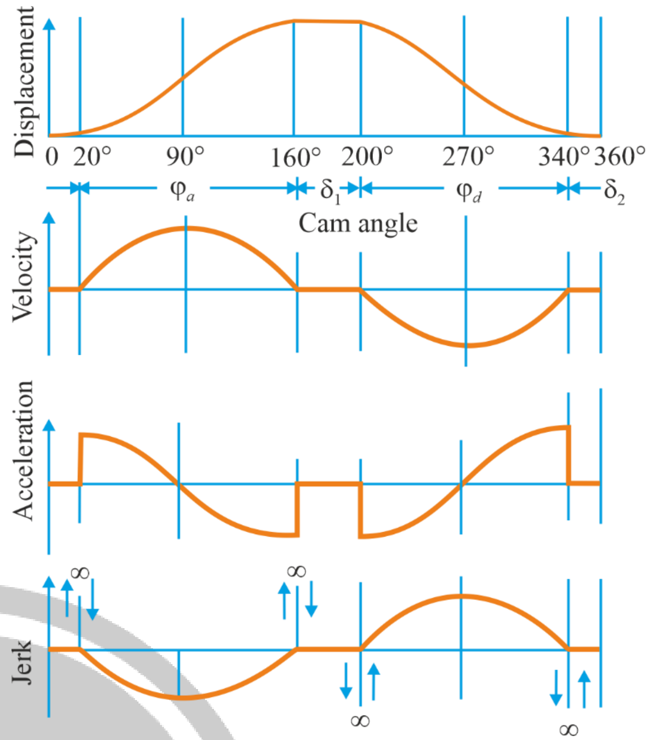


Fig. 3.22 Constant Acceleration and Deacceleration (Parabolic)

3.5.3 Constant Velocity

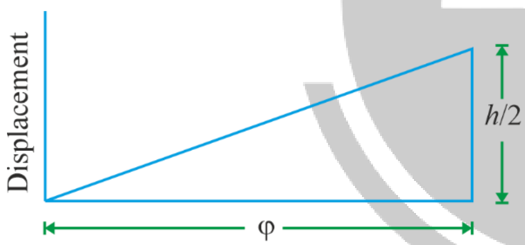


Fig. 3.23 Constant Velocity

- **Displacement of followers**

$$s = h\frac{\theta}{\varphi} = h\frac{\omega t}{\varphi}$$

- **Velocity of followers**

$$v = \frac{ds}{dt} = \frac{h\omega}{\varphi} \text{ constant}$$

- **Acceleration of follower**

$$f = \frac{dv}{dt} = 0$$

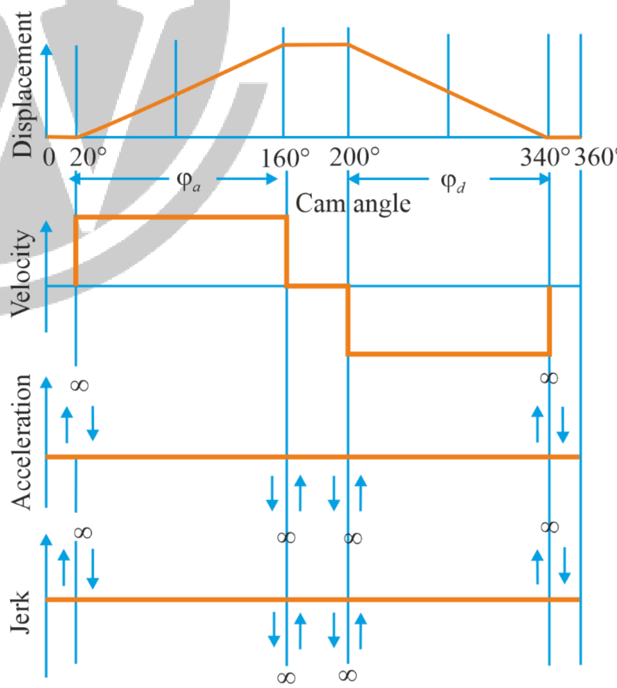


Fig. 3.24 Constant Velocity

3.5.4 Cycloidal

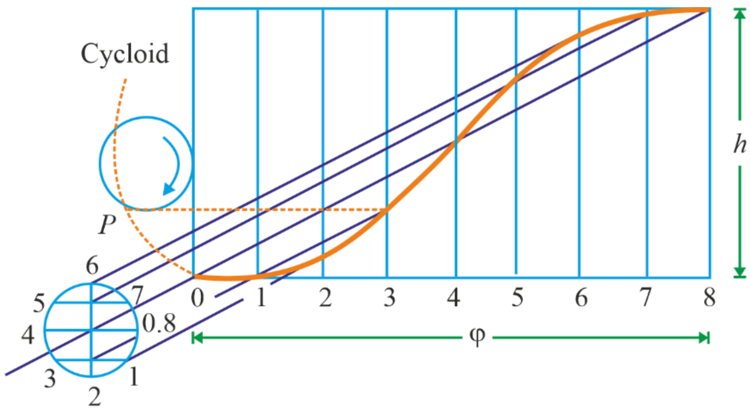


Fig. 3.25 Cycloidal

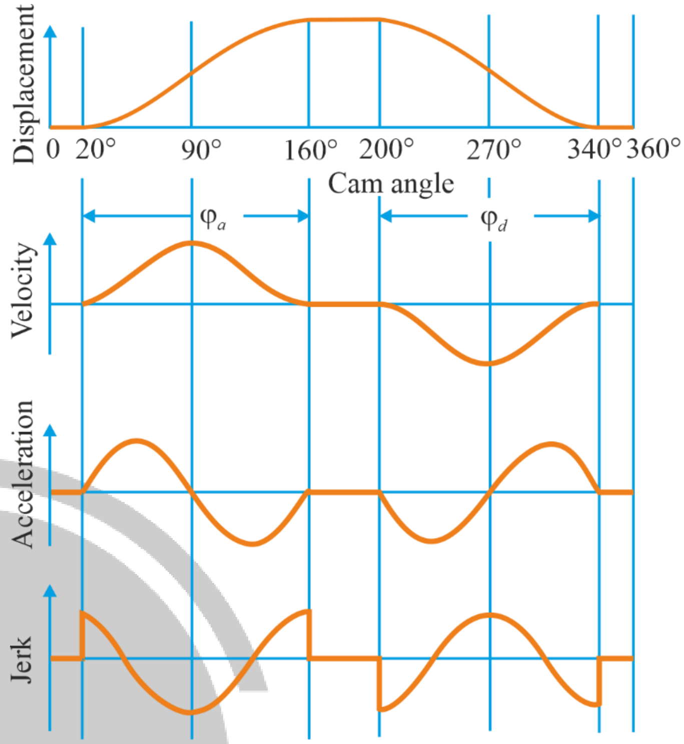


Fig. 3.26 Cycloidal

- **Displacement of follower**

$$s = \frac{h}{\pi}\left(\frac{\pi\theta}{\varphi} - \frac{1}{2}\sin\frac{2\pi\theta}{\varphi}\right)$$

- **Maximum velocity of follower**

$$v_{\max} = \frac{2h\omega}{\varphi} \text{ and } \theta = \frac{\varphi}{2}$$

- **Maximum acceleration of followers**

$$f_{\max} = \frac{2h\pi\omega^2}{\varphi^2} \text{ and } \theta = \frac{\varphi}{4}$$

❒❒❒

4 GEARS AND GEAR TRAIN

4.1 Gears & Its Classification of Gears

- Gears are used to transfer power/motion from one shaft to anther in such away than ratio $\frac{\omega_1}{\omega_2}$ remains constant during entire motion.
- According to relative position of their shaft axis:

4.1.1 Parallel Shaft

1. **Spur gear:**
   - Straight tooth parallel to axis of gear.
   - No axial thrust
   - High impact stress on tooth
2. **Helical Gear:**
   - Straight tooth inclined to axis of rotation.
   - Low impact stress due to gradual load application.
   - Problem of axial thrust.
3. **Double Helical (Herring bone) Gear:**
   - Axial thrust problem eliminated.

4.1.2 Intersecting Shaft

1. **Straight Bevel Gear:**
   - Connects shaft at an angle running at low speed.
   - If size of both gear is same: Mitre gears
   - Subjected to impact stress
2. **Spiral Bevel/Helical Bevel:**
   - Low impact stress
   - Problem of axial thrust

4.1.3 Skew shaft (non-parallel and non-intersecting)

- Pure rolling motion isn't possible.

1. **Crossed Helical Gears:**
   - Two shafts can be set at any angle by choosing suitable helix angle.
2. **Worm Gear:**
   - Very large speed reduction ratio

   Worm (Driver) $\rightarrow$ Very less dia. & very large spiral angle.

   Worm (Wheel) $\rightarrow$ Very large dia. & very less spiral angle.

Note:
- In power transmission, smaller bodies are made drivers.
- Hence generally gears are also known as speed reduction device.

4.2 Terminology of Gear:

(a) Circular Pitch (P_c):

It is distance from one point of tooth to the same point of adjacent tooth measured along pitch circle.

$$p_c = \frac{\pi D}{T}$$

(b) Diametral Pitch (P_d):

It is defined as ratio of number of teeth to diameter of pitch circle

$$p_d = \frac{T}{D}$$

(c) Module (m):

It is ratio of diameter of pitch circle to number of teeth

$$m = \frac{D}{T}$$

(d) Gear Ratio:

It is the ratio of number of teeth on gear to the number of teeth on pinion

$$G = \frac{T_G}{T_p}$$

(e) Velocity Ratio:

It is ratio of ω_{driver} to ω_{driven} gear

D $\rightarrow$ Pitch circle diameter; T $\rightarrow$ tooth.

- For two mating gears, circular pitch **and** module must be same.

4.3 Meshing of Involute Gears

- Normal drawn at any point on involute curve will become tangent to its base circle.

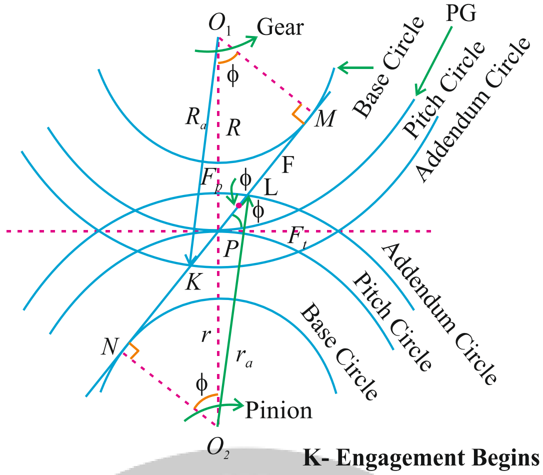


Fig. 4.1 Involute Teeth

4.3.1 Line of Action [NM]

- Driving gear tooth, exerts force on driven gear tooth at contact point Q along line of action MN.
- Pitch point (P) and Point of contact (Q) lies along a line of action.

4.3.2 Pressure Angle (ϕ)

- Angle between line of action and common tangent to pitch circle
- Line of action remains fixed (in case of involute gear) while point of contact changes continuously along line of action between K and L as gear tooth engage and disengage
- Time interval in which point of contact (Q) is travelling from start of engagement to end of engagement is termed as one engagement period.

4.3.3 Path of Contact/Contact Length

- Contact length = Approach length + Recess length

i.e., $KL = KP + PL$

Path of approach – (KP):

$$KP = \sqrt{R_a^2 - R^2\cos^2\phi} - R\sin\phi$$

Path of Recess – (PL):

$$PL = \sqrt{r_a^2 - r^2\cos^2\phi} - r\sin\phi$$

So, contact length = $KP + PL$ = $\sqrt{R_a^2 - R^2\cos^2\phi} + \sqrt{r_a^2 - r^2\cos^2\phi} - (R+r)\sin\phi$

4.3.4 Arc of Contact

- Length travelled by pinion/gear along their pitch circle in one engagement period.

Arc of contact = Arc of approach + Arc of Recess

$$\text{Arc of Contact} = \frac{\text{Path of contact}}{\cos\phi}$$

Note:

In one engagement period:

$$\left.\begin{aligned}(\text{Angle Turned})_{\text{By pinion}} &= \frac{\text{Arc of contact}}{r} \\ (\text{Angle Turned})_{\text{By gear}} &= \frac{\text{Arc of contact}}{R}\end{aligned}\right\}$$

4.3.5 Contact Ratio

- It gives number of pairs of teeth engaged in one engagement period.

$$\text{Contact Ratio} = \frac{\text{Arc of contact}}{\text{Circular Pitch}}$$

4.3.6 Path of Contact in Rack and Pinion Arrangement

Path of contact = $KP + PL$

$$PL = \sqrt{r_a^2 - r^2\cos^2\phi} - r\sin\phi$$

$$KP = \frac{\text{Addendum of Rack}}{\sin\phi}$$

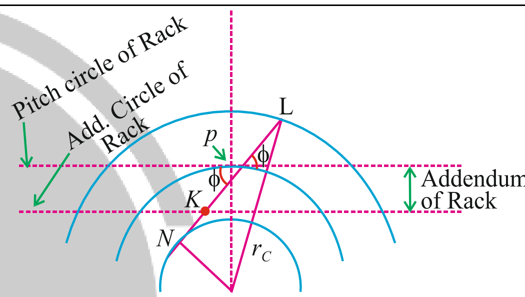


Fig. 4.2 Rack and Pinion in Mesh

4.4 Power Transmission in Gears

- F = normal thrust between teeth

⇒ $F_t = F\cos\phi$ → Tangential component (Power Transmission)

⇒ $F_R = F\sin\phi$ → Radial component (Thrust action on bearing.)

⇒ Torque = F_t × pitch circle radius.

⇒ Power Transmitted, $P = F_a \cdot \omega \times \text{Torque}$

$$p = \frac{2\pi NT}{60} \rightarrow watt$$

N → rpm

T → N-m

⇒ Efficiency of gear drive:

$$\eta = \frac{p_o}{p_i} = \frac{T_o \cdot \omega_o}{T_i \cdot \omega_i} = \frac{F_o \cdot V_o}{F_i \cdot V_i}$$

4.5 Velocity of Sliding

$V_{\text{sliding}} = (\omega_p + \omega_g)$ × distance between pitch point and point of contact

$V_{\text{sliding}} = (0)$ motion is pure rolling at pitch point = Zero

4.6 Interference in Involute Gears

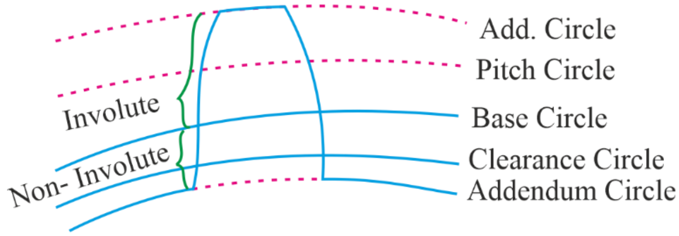


Fig. 4.3 Gear tooth

- The profile of gear is involute outside the base circle and inside base circle it is non-involute.
- Mating of involute profile of one gear with non-involute profile of other results into interference.

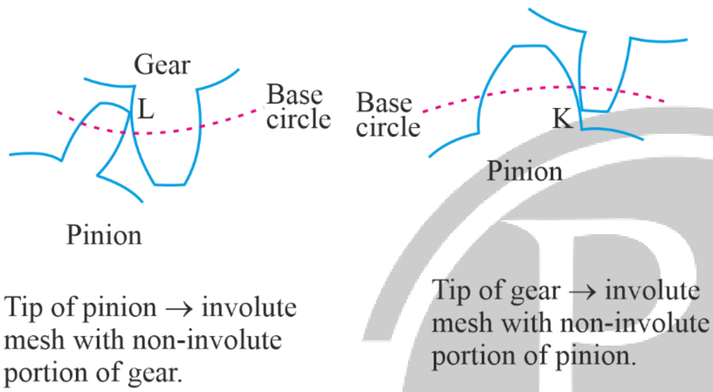


Fig. 4.4 Path of Contact

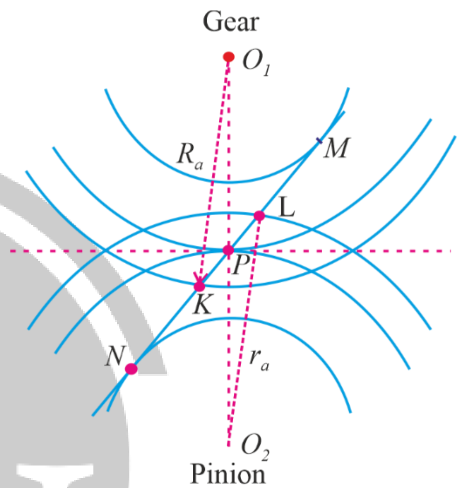


Fig. 4.5 Path of Contact

Interference occurs if:

= $R_a > O_1N$ or $r_a > O_2M$

So, Last safety point of K is N.

Last safety point of L is M.

4.7 Methods to Prevent Interference

1. **Undercutting of Gear:** Removal of material of non-involute portion below base circle.

   Limitation: Strength of tooth decreases at the base, so used only in low power transmission.

2. **Increase Pressure Angle:**

   ϕ can be increased by reducing base circle radius.

   $R_B \downarrow \Rightarrow$ non-involute portion $\downarrow \Rightarrow$ interference $\downarrow$

   Limitation: $\phi_{max} \approx 20^o$ to 25^o

3. **Stubbing the tooth:** Portion of tip of tooth of gear is removed, thus preventing that portion to contact non-involute portion.

4. **Limitation:** Contact ratio decreases

5. Increasing Number of teeth

By increasing number of teeth on same pitch circle, tooth size decreases.

$R_A \downarrow$

Interference $\downarrow$

4.8 Minimum Tooth Required to Prevent Interference

- **For pinion:**

$$t_{\min} = \frac{2A_p}{\sqrt{1+G(G+2)\sin^2\phi}-1}$$

- **For gear:**

$$T_{\min} = \frac{2A_g}{\sqrt{1+\frac{1}{G}\left(\frac{1}{G}+2\right)\sin^2\phi}-1}$$

Where, A_p, A_g = Fractional addendum (Addendum of pinion and gear for one module (m))

$$\left[G = \frac{T}{t}\right]$$

- Full depth involute: Addendum = module
- Stub involute: Addendum < module

Minimum tooth required on pinion to prevent interference in rack and pinion Arrangement:

$$t_{\min} = \frac{2A_r}{\sin^2\phi}$$

$A_r \rightarrow$ Fractional addendum on rack

Note:

(a) In case, if fractional addendum on gear and pinion is same i.e., $A_p = A_g$, then gear will do interference first. Make gear safe by providing $T_{\min}$.

(b) Say addendum of on gear $A_g = 0.8 \Rightarrow$ 20% portion of gear is stubbed.

4.9 Cycloidal Tooth Profile

- Phenomena of interference is absent.
- Tooth have spreading flank and thus stronger.

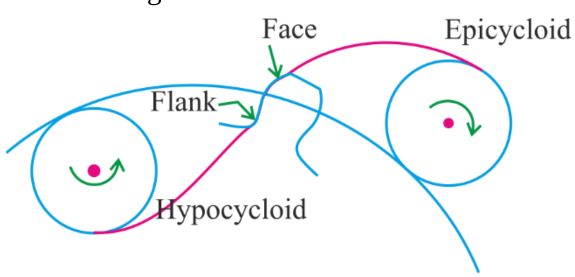


Fig. 4.6 Cycloidal Tooth Profile

- $\left.\begin{array}{l}\text{At engagement}: \phi = \phi_{max} \\ \text{At pitch point}: \phi = 0^{o} \\ \text{At engagement}: \phi = \phi_{max}\end{array}\right\} \rightarrow \text{variable } \phi, \text{less smooth running}$
- $\left.\begin{array}{l}\text{Flank} \rightarrow \text{Hypocycloid} \\ \text{Face} \rightarrow \text{Epicycloid}\end{array}\right\} \rightarrow \text{complicated design, difficult to menufecture}$

4.10 Gear Trains

When more than two gears are made to mesh with each other to transmit power from one shaft to another, such a combination is called gear train

$$\text{Speed Ratio} = \frac{\omega_{\text{driver}}}{\omega_{\text{driven}}} = \frac{1}{\text{Train Value}}$$

4.10.1 Simple Gear Train

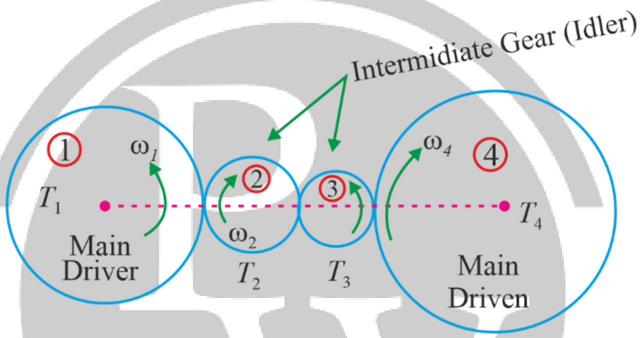


Fig. 4.7 Gear Train

$$\frac{\omega_1}{\omega_2} = \frac{T_2}{T_1}; \quad \frac{\omega_2}{\omega_3} = \frac{T_3}{T_2}; \quad \frac{\omega_3}{\omega_4} = \frac{T_4}{T_3}$$

$$\left[\frac{\omega_1}{\omega_4} = \frac{T_4}{T_1} = \text{Speed Ratio}\right]$$

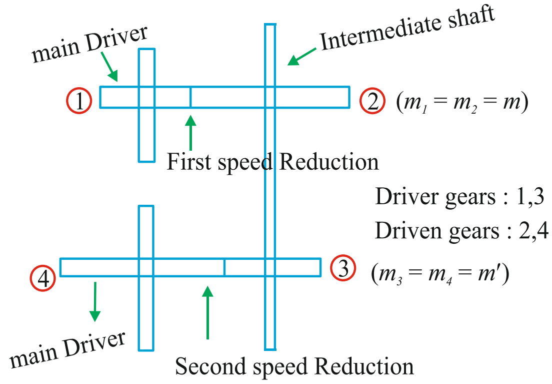
- Intermediate gears don't affect speed ratio (Idler gears)
- If number of idlers are odd than direction of driver and driven are same
- If number of idlers are even than direction of driver and driven are same are opposite.

4.10.2 Compound Gear Train

- At least one of the intermediate shafts have more than one gear in use.

4.10.3 Reverted Gear Train

- Form of compound gear train which connects two co-axial shafts.

$\frac{\omega_1}{\omega_2} = \frac{T_2}{T_1}; \quad \frac{\omega_3}{\omega_4} = \frac{T_4}{T_3}$ $\quad (\therefore \omega_2 = \omega_3)$ $\Rightarrow \left[\frac{\omega_1}{\omega_4} = \frac{T_2 \cdot T_4}{T_1 \cdot T_3} = \text{Speed Ratio}\right]$ Also, $r_1 + r_2 = r_3 + r_4$ $\Rightarrow m(t_1 + t_2) = m'(t_3 + t_4)$	 **Fig. 4.8 Reverted Gear Train**

4.10.4 Epicyclic Gear Train or Planetary

- If apart from motion of gear, any of the gear axis is also rotating about same axis, it is known as epicyclic gear train.
- DOF of epicyclic gear train is 2
- To rotate axis of gear, a link is used known as arm/carrier.
- One gear is fixed either sun or ring.

4.10.5 Fixing Torque/Holding Torque:

$$\Rightarrow \boxed{(T_{O/P} \cdot \omega_{O/P}) + \eta(-T_{I/P} \cdot \omega_{I/P}) = 0} \quad \text{...(I)}$$

η = efficiency of gear train.

For equilibrium, net torque on gear train = 0

$$\boxed{T_{O/P'} + T_{I/P} + T_{\text{fixing}} = 0} \quad \text{... (II)}$$

- For given input torque $T_{I/P}$ we can find output torque from equations (I) and then fixing torque from equation (II).

❒❒❒

5 Analysis of Single Slider Crank Mechanism

5.1. Analysis of Single Slider Crank Mechanism

If crank is rotating with constant angular speed ω in clockwise direction.

r = crank radius

l = length of connecting rod

θ = Angle turned by crank from line of stroke or from TDC/BDC.

$n = \frac{l}{r}$ = obliquity ratio

ω = crank speed of rotation.

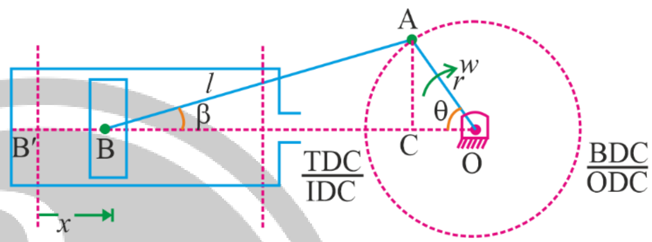


Fig. 5.1 Single Slider Crank Mechanism

5.1.1 Piston Displacement

$x = OB' - OB = (l + r) - (l\cos\beta + r\cos\theta)$

From figure $AC = l\sin\beta = r\sin\theta$

$$\sin\beta = \frac{\sin\theta}{n} \Rightarrow \cos\beta \frac{\sqrt{n^2 - \sin^2\theta}}{n}$$

$$x = (nr + r) - \left[nr \cdot \frac{\sqrt{n^2 - \sin^2\theta}}{n} + r\cos\theta \right]$$

$$x = r(1 - \cos\theta) + r\left(n - \sqrt{n^2 - \sin^2\theta}\right)$$

$$\boxed{x = r\left[(1 - \cos\theta) + \left(n - \sqrt{n^2 - \sin^2\theta}\right)\right]}$$

5.1.2 Piston Velocity

$$V = \frac{dx}{dt} = r\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right] \cdot \frac{d\theta}{dt}$$

$$V = r\omega\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right] \rightarrow \text{exact.}$$

If n is large, $n^2 - \sin^2\theta \approx n^2$

$$\boxed{V = r\omega\left[\sin\theta + \frac{\sin 2\theta}{2n}\right]} \rightarrow \text{Approximate.}$$

5.1.3 Piston Acceleration

$$a = \frac{dV}{dt} = \frac{dV}{d\theta}\cdot\frac{d\theta}{dt}$$

$$a = r\omega^2\left[\cos\theta + \frac{\cos 2\theta}{n}\right] \rightarrow \text{Approx.}$$

5.1.4 Angular Velocity of Connecting Rod

$$\sin\beta = \frac{\sin\theta}{n} \Rightarrow \cos\beta\cdot\frac{d\beta}{dt} = \frac{\cos\theta}{n}\cdot\frac{d\theta}{dt}$$

$$\frac{\sqrt{n^2 - \sin^2\theta}}{n}\cdot(\omega_{Cr}) = \frac{\cos\theta}{n}\times\omega_{\text{crank}}$$

$$\boxed{\omega_{CR} = \omega_{\text{crank}}\cdot\frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}}$$

If $\quad n >>> \sin\theta$

$$\Rightarrow \quad \boxed{\omega_{CR} = (\omega_{\text{crank}})\cdot\frac{\cos\theta}{n}}$$

5.1.5 Angular Acceleration of Connecting Rod

$$\alpha_{CR} = \frac{\omega_{CR}}{dt} \quad \Rightarrow \boxed{\alpha_{CR} = -\omega_{\text{crank}}^2\cdot\frac{\sin\theta}{n}}$$

5.2 Force Analysis of Slider – Crank Mechanism

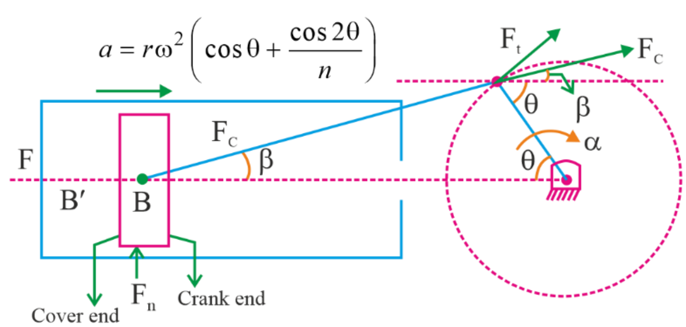


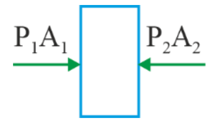
Fig. 5.2 Force component on Single Slider Crank Mechanism

m = mass of reciprocating part

Assumption – mass/Inertia Effect of connecting rod is neglected.

5.2.1 Fluid Pressure Force

P_1 = Fluid pressure at cover end.
P_2 = Fluid pressure at crank end.
A_1 = Area of piston exposed to fluid pressure at cover end.
A_2 = Area of piston exposed to fluid pressure at crank end.



$$A_1 = \frac{\pi}{4}D^2 \quad ; \quad D \rightarrow \text{Bore size}$$

$$A_2 = \frac{\pi}{4}\left(D^2 - d^2\right) \quad ; \quad d \rightarrow \text{Piston rod diameter}$$

$$\boxed{F_P = P_1A_1 - P_2A_2}$$

5.2.2 Piston Effort/ Effective Driving Force on Piston

$$\boxed{F = F_P - m \times a - F \pm mg}$$

F $\rightarrow$ Friction
mg $\rightarrow$ In case of vertical engines
ma $\rightarrow$ Inertia force (F_i)

5.2.3 Force Transmitted along Connecting Rod

$$F_C \cos\beta = F \Rightarrow \boxed{F_C = \frac{F}{\cos\beta}}$$

5.2.4 Normal Thrust against Cylinder Walls

$$F_n = F_C \sin\beta = F \tan\beta$$

$$\Rightarrow \boxed{F_n = F \tan\beta}$$

5.2.5 Radial Thrust on Crankshaft Bearing

$$F_R = F_C \cos(\theta + \beta)$$

$$\Rightarrow \boxed{F_R = \frac{F}{\text{Cos}\beta} \cdot \cos(\theta + \beta)}$$

5.2.6 Crank Effort (F_t)

$$F_t = F_C \sin(\theta + \beta)$$

$$\Rightarrow \boxed{F_t = \frac{F}{\cos\beta} \cdot \sin(\theta + \beta)}$$

5.2.7 Torque/Turning Moment on Crankshaft

$$T = F_t \cdot r$$

$\Rightarrow \boxed{T = F_r \cdot \frac{\sin(\theta+\beta)}{\cos\beta}}$

Or, $\boxed{T = F_r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right)}$ → from this expression we say that Torque, $\tau = f(\theta)$

Therefore, T ≠ Constant

Note:

If it is required to find speed at which gudgeon pin load reverse its direction. Then F_C will become first zero than reverse its direction i.e., $F_C = \frac{F}{\cos\beta} = 0 \quad \Rightarrow \boxed{F = 0}$

❒❒❒

6 FLYWHEEL

6.1 Introduction

- A flywheel is a rotating disc or rim that is used to store rotational kinetic energy $\left(\frac{1}{2}I\omega^2\right)$.
- Flywheels have a significant mass moment of inertia and thus resist changes in rotational speed.
- Energy is transferred to a flywheel by applying torque to it, thereby increasing its rotational speed, and hence it stores the energy.
- Conversely, a flywheel releases stored energy by applying torque to a mechanical load, thereby its rotational speed decreases.
- For example, a flywheel is used to maintain constant angular velocity of the crankshaft in a reciprocating engine. In this case, the flywheel which is mounted on the crankshaft stores energy when torque is exerted on it by a firing piston and it releases its energy to mechanical loads when piston is not exerting torque on it.

6.1.1 Turning Moments Diagrams

- A plot of T vs. θ is known as the turning moment diagram. The inertia effect of the connecting rod is usually ignored while drawing these diagrams. The turning moment diagrams for different types of engines are being given below:

(1) Single cylinder - Double acting steam engine -

Fig shows the turning - moment diagram for a single cylinder double – acting engine.

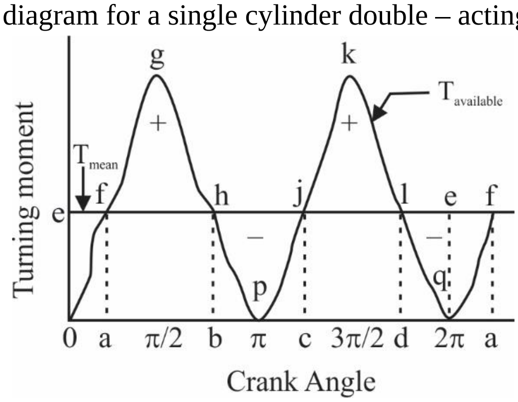


Fig. 6.1 Turning moment diagram

> **Note:**
> Generally, engines are connected against a constant load. In such case $T_{resisting}$ or T_{load} will be equal to T_{mean}.

- It can be observed that during the outstroke (ogp) the turning moment is maximum when the crank angle is little less than 90° and zero when the crank angle is zero and 180°. A somewhat similar turning moment diagram is obtained during the instroke (pkq).

- The area of the turning-moment diagram is proportional to the work done per revolution as the work is the product of the turning moment & the angle turned.
- The mean torque against which the engine works is given by

$$T_{mean} = \frac{\text{work done in one cycle}}{\text{Time period}}$$

$$oe = \frac{\text{Area ogpkq}}{2\pi}$$

- Where oe is the mean torque and is mean height of the turning moment diagram.

- The maximum engine speed is at b & d and minimum speed is at c and a. the greatest speed is the greater of the two maximum speeds and the least speed is the lesser of the two minimum speeds.
- The difference between the greatest and the least speeds of the engine over one revolution is known as the fluctuation of speed.

(2) Multi – Cylinder Engines –

- Fig. 6.2 shows the turning – moment diagram for a multi–cylinder engine

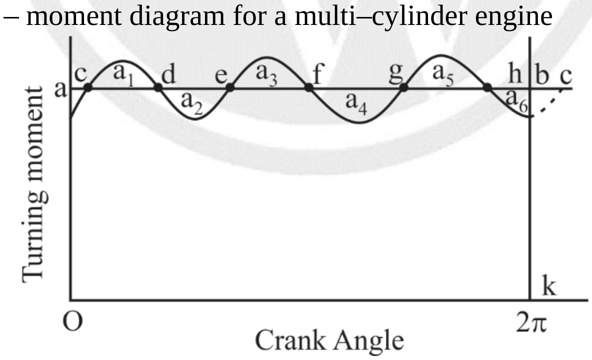


Fig. 6.2 Turning moment diagram

- The mean torque line ab intersects the turning moment curve at c, d, e, f, g and h. The area under the wavy curve is equal to the area oabk.
- The speed of the engine will be maximum when the crank positions correspond to d, f and h, and minimum corresponding to c, e and g.

6.2 FLUCTUATION OF ENERGY

- Let a_1, a_3 and a_5 be the areas in work units of the portions above the mean torque ab of the turning – moment diag. (fig (c)). These areas represent quantities of energies added to the flywheel. Similarly, areas a_2, a_4 and a_6 ab represent quantities of energies taken from the flywheel.
- The energies of the flywheel corresponding to position of the crank are as follows:

Crank position	Flywheel energy
c	E
d	$E + a_1$
e	$E + a_1 - a_2$
f	$E + a_1 - a_2 + a_3$
g	$E + a_1 - a_2 + a_3 - a_4$
h	$E + a_1 - a_2 + a_3 - a_4 + a_5$
c	$E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

- The greatest of these energies is the maximum kinematic energy of the flywheel and for the corresponding crank position, the speed is maximum.

- The least of these energies is the least kinetic energy of the flywheel and for the corresponding crank position, the speed is minimum.

(1) Coefficient of fluctuation of energy:

- It is the ratio of maximum fluctuation of energy to the total work done in a cycle.

$$C_E = \frac{\Delta KE_{max}}{\text{WD per cycle}}$$

- Maximum fluctuation of speed

$$\Delta\omega_{max} = (\omega_1 - \omega_2)$$

(2) Coefficient of fluctuation of speed:

- The difference between the greatest speed and the least speed is known as the maximum fluctuation of speed & the ratio of the maximum fluctuation of speed to the mean speed is the coefficient of fluctuation of speed.

$$C_s = \frac{\omega_1 - \omega_1}{\omega_{mean}}$$

Where

$$\omega_{mean} = \frac{\omega_1 + \omega_2}{2}$$

- Maximum fluctuation of energy

$$(\Delta KE)_{max} = \Delta E = \frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_2^2$$

$$= I\left(\frac{\omega_1 + \omega_2}{2}\right)(\omega_1 - \omega_2)$$

$$\Delta E = I\omega \frac{(\omega_1 - \omega_2)}{\omega} \times \omega$$

$$\boxed{\Delta KE_{max} = I\omega^2 C_s}$$

also known as fundamental equation of flywheel

Where I = moment of inertia of the fly wheel

ω_1 = maximum speed

ω_2 = minimum speed

ω = mean speed = $\frac{\omega_1 + \omega_2}{2}$

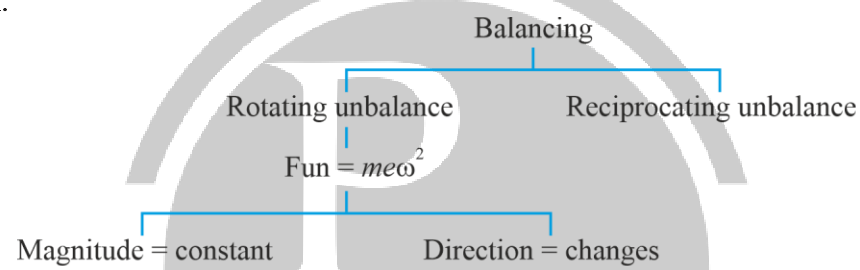
C_s = coefficient of fluctuation of speed

❑❑❑

7 BALANCING

7.1 Types of Balancing

- Balancing is the process of designing and modify machinery so that the unbalance is reduced to an acceptable level and if possible is eliminated entire.
- Generally, it is done by redistributing the mass which may be accomplished by addition or removal of mass from various mess members.
- There are two basic types of unbalances - rotating unbalance and reciprocating unbalance - which may occur separately or in combination.



7.1.1 Static Balancing

- A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation.
- Single transverse plane
- The net dynamic force acting on the shaft is equal to zero.

$$\Sigma\vec{F} = 0$$

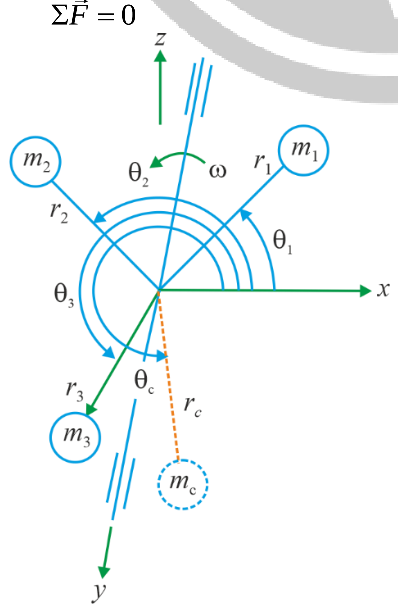


Fig. 7.1 Single Plane Unbalance System

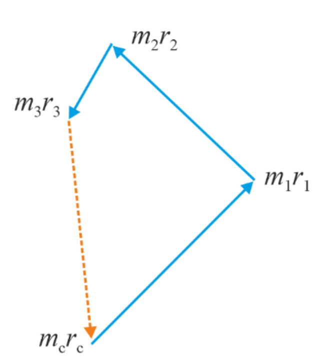


Fig. 7.2 Force Polygon

7.1.2 Dynamic Balancing

- When several masses rotate in different plane, the centrifugal forces, in addition to being out of balance, also form couples.
- A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.
- More than one transverse plane

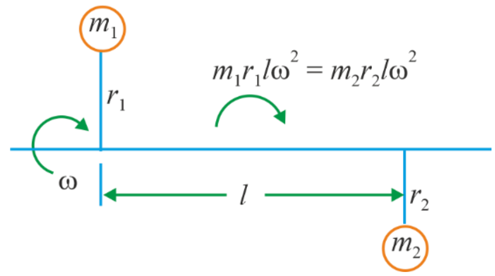


Fig. 7.3 Multi transverse Plane Unbalance System

- The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero.

  $\Sigma\vec{F} = 0$
- $\Sigma\vec{M} = 0$

7.2 Balancing of Reciprocating Mass

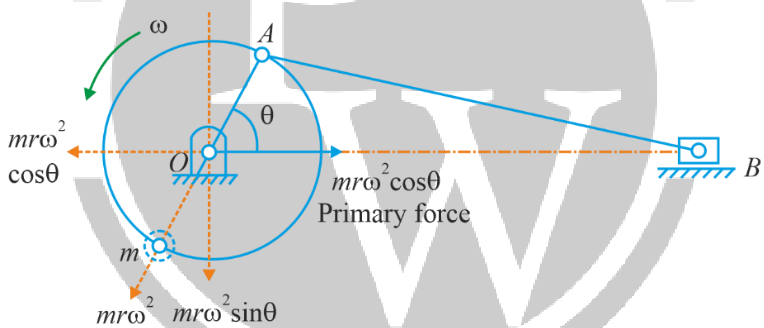


Fig. 7.4 Single Slider Crank Mechanism

- Total unbalance force due to reciprocating mass (m):

$$F = mr\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$= mr\omega^2\cos\theta + mr\omega^2\frac{\cos 2\theta}{n}$$

- Primary force = $mr\omega^2\cos\theta$. The secondary force = $mr\omega^2\left(\frac{\cos 2\theta}{n}\right)$
- Maximum value of the primary force = $mr\omega^2$
- Maximum value of the secondary force = $\frac{mr\omega^2}{n}$

7.2.1 By Balancing Fraction (c) of the Reciprocating Mass

- Primary force balanced by the mass = $cmr\omega^2 \cos\theta$
- Primary force unbalanced by the mass = $(1 - c)\, mr\omega^2 \cos\theta$
- Vertical component of centrifugal force which remains unbalanced
- $= cmr\omega^2 \sin\theta$
- Resultant unbalanced force at any instant

$$= \sqrt{[(1-c)mr\omega^2 \cos\theta]^2 + [cmr\omega^2 \sin\theta]^2}$$

- The resultant unbalanced force is minimum when $c = \frac{1}{2}$
- If m_P is the mass at the crankpin and c is the fraction of the reciprocating mass m to be balanced, the mass the crankpin may be considered as $(cm + m_P)$ which is to be completely balanced.

7.3 Parameters Related to the Partial Balancing in Locomotives

7.3.1 Hammer Blow

- Hammer-blow is the maximum vertical unbalanced force caused by the mass provided to balance the reciprocating masses.
- Its value is $mr\omega^2$. Thus, it varies as a square of the speed.
- At high speeds, the force of the hammer-blow could exceed the static load on the wheels and the wheels can be lifted off the rail when the direction of the hammer-blow will be vertically upwards.

7.3.2 Variation of Tractive Force

- Total unbalanced primary force or the variation in the tractive force

$$= -(1-c)mr\omega^2(\cos\theta - \sin\theta)$$

- For its maximum value

$$\theta = 135° \text{ or } 315°$$
$$\theta = 135°$$

- Maximum variation of Tractive force:

$$= \pm\sqrt{2}(1-c)mr\omega^2$$

7.3.3 Swaying Couple

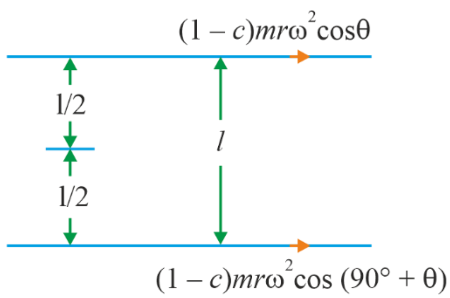


Fig. 7.5 Swaying Couple

- Swaying couple = moments of forces about the engine centre line

$$= [(1-c)mr\omega^2\cos\theta]\frac{1}{2} - [(1-c)mr\omega^2\cos(90°+\theta)]\frac{1}{2}$$

$$= (1-c)mr\omega^2(\cos\theta+\sin\theta)\frac{1}{2}$$

- For its maximum value:

θ = 45° or 225°

- Maximum swaying couple = $\pm\frac{1}{\sqrt{2}}(1-c)mr\omega^2 l$

7.3.4 Secondary Balancing

- Secondary force = $mr\omega^2\frac{\cos 2\theta}{n}$
- Its frequency is twice that of the primary force and the magnitude $\frac{1}{n}$ times the magnitude of the primary force.

Parameter	Actual	Imaginary
Angular velocity	ω	2ω
Length of crank	r	$\frac{r}{4n}$
Mass at the crank pin	m	m

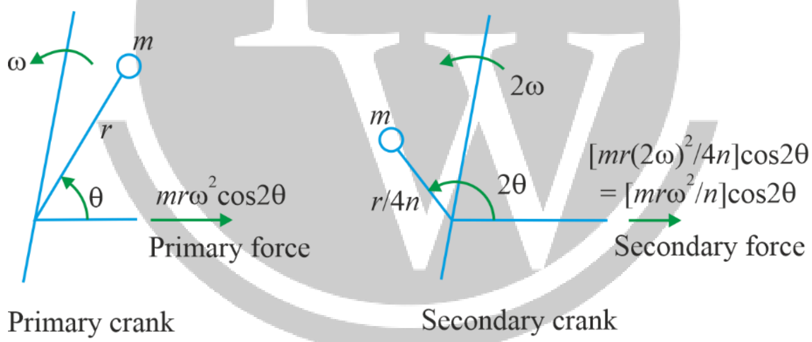


Fig. 7.6 Secondary Balancing

- Centrifugal force induced in the imaginary crank = $\frac{mr(2\omega)^2}{4n}$
- Component of this force along line stroke = $\frac{mr(2\omega)^2}{4n}\cos 2\theta$

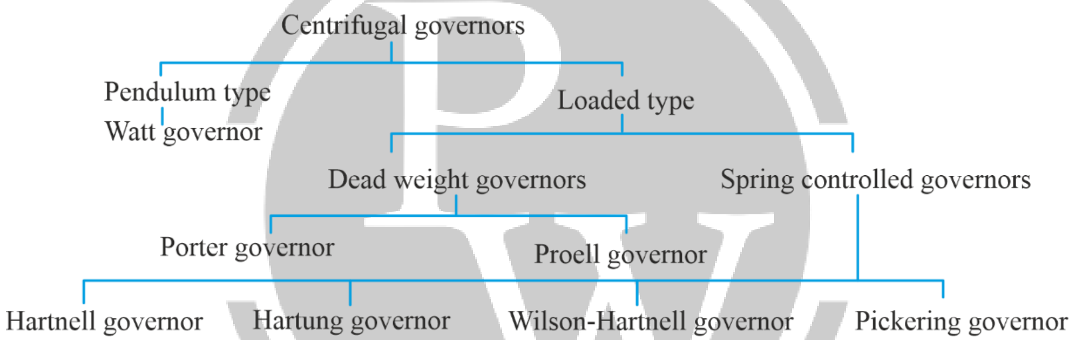
❒❒❒

8 GOVERNOR

8.1 Introduction

- Governor is to regulate the mean speed of an engine.
- Automatically controls the supply of working fluid (fuel) to the engine with the varying load conditions and keeps the mean speed within certain limits.

8.2 Types of Governor



8.3 Terminology

- **Height of a governor:** It is vertical distance from the centre of the ball to a point where the axes of the arms (or arms produced) intersect on the spindle axis.
- **Equilibrium speed:** It is speed at which the governor balls, arms etc., are in complete equilibrium and the sleeve do not tend to move upwards or downwards.
- **Mean equilibrium speed.** It is speed at the mean position of the balls or the sleeve.
- **Maximum and minimum equilibrium speeds:** The speeds at the maximum and minimum radius of rotation of the balls, without tending to move either way is known as maximum and minimum equilibrium speeds respectively
- **Sleeve lift.** It is the vertical distance which the sleeve travels due to change in equilibrium speed.

8.4 Watt Governor

m = Mass of the ball in kg,

w = Weight of the ball in newtons = $m.g$,

T = Tension in the arm in newtons,

ω = Angular velocity of the arm and ball about the spindle axis in rad/s,

r = Radius of the path of rotation of the ball i.e., horizontal distance from the centre of the ball to the spindle axis in metres,

F_C = Centrifugal force acting on the ball in newtons = $m.\omega^2.r$, and

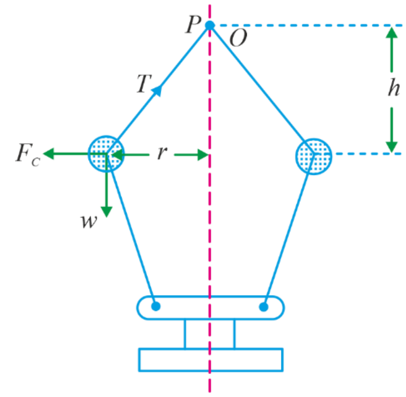


Fig. 8.1 Watt governor

8.4.1 Height of the Governor

$$h = \frac{9.81}{(2\pi N/60)^2} = \frac{895}{N^2} \text{ metres}$$

Note:

The height of a governor h, is inversely proportional to N^2. Therefore, at high speeds, the value of h is small. This governor only works satisfactorily at relatively low speeds *i.e.*, from 60 to 80 r.p.m.

8.4 Porter Governor

m = Mass of each ball in kg,

w = Weight of each ball in newtons = $m.g$,

M = Mass of the central load in kg,

W = Weight of the central load in newtons = $M.g$,

r = Radius of rotation in metres,

α = Angle of inclination of the arm (or upper link) to the vertical, and

β = Angle of inclination of the line (or lower link) tot the vertical.

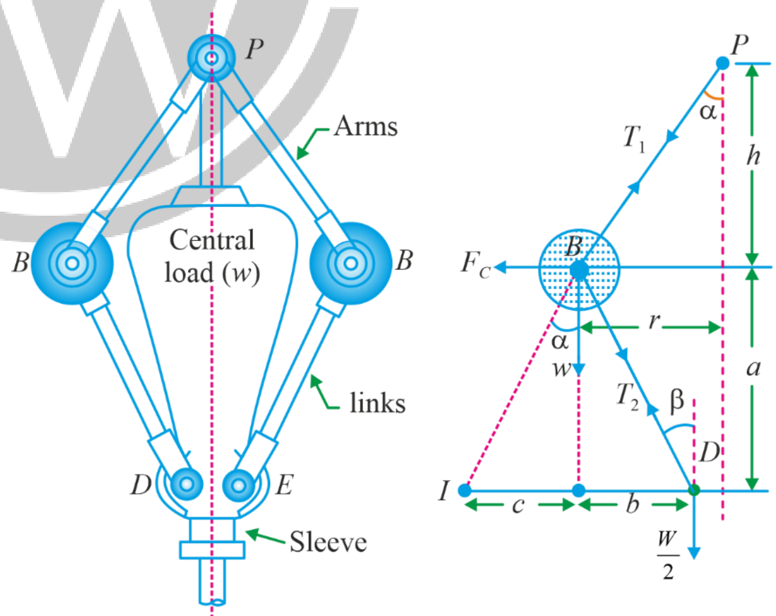


Fig. 8.2 Porter governor

8.4.1 Equilibrium Equation About I-centre

$$mr\omega^2 \times a = mg \times c + \frac{Mg \pm f}{2}(c+b)$$

8.4.2 Height of the Governor

If length of arms is equal to the length of links and the points *P* and *D* lie on the same vertical line, then

$$\tan\alpha = \tan\beta \text{ or } k = \tan\alpha/\tan\beta = 1$$

If *f* = Frictional force acting on the sleeve

$$h = \frac{m.g + \left(\frac{M.g \pm f}{2}\right)(1+k)}{m.g} \times \frac{895}{N^2}$$

When tanα = tanβ or *k* = 1, then

$$h = \frac{m+M}{m} \times \frac{g}{\omega^2}$$

8.5 Proell Governor

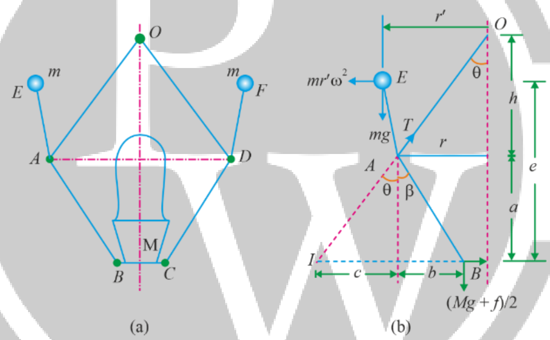


Fig. 8.3 Proell Governor

8.5.1 Equilibrium equation about I centre

$$mr'\omega^2 e = mg(c + r - r') + \frac{Mg \pm f}{2}(c+b)$$

$$N^2 = \frac{895}{h}\frac{a}{e}\left(\frac{2mg + (mg \pm f)(1+k)}{2mg}\right)$$

8.5.2 Height of governor

If $k = 1$, $$h = \frac{895}{N^2}\frac{a}{e}\left(\frac{mg + (Mg \pm f)}{mg}\right)$$

If $f = 0$, $$h = \frac{895}{N^2}\frac{a}{e}\left(\frac{2m + M(1+k)}{2m}\right)$$

If $k = 1, f = 0$ $$h = \frac{895}{N^2}\frac{a}{e}\left(\frac{m+M}{m}\right)$$

8.6 Hartnell Governor

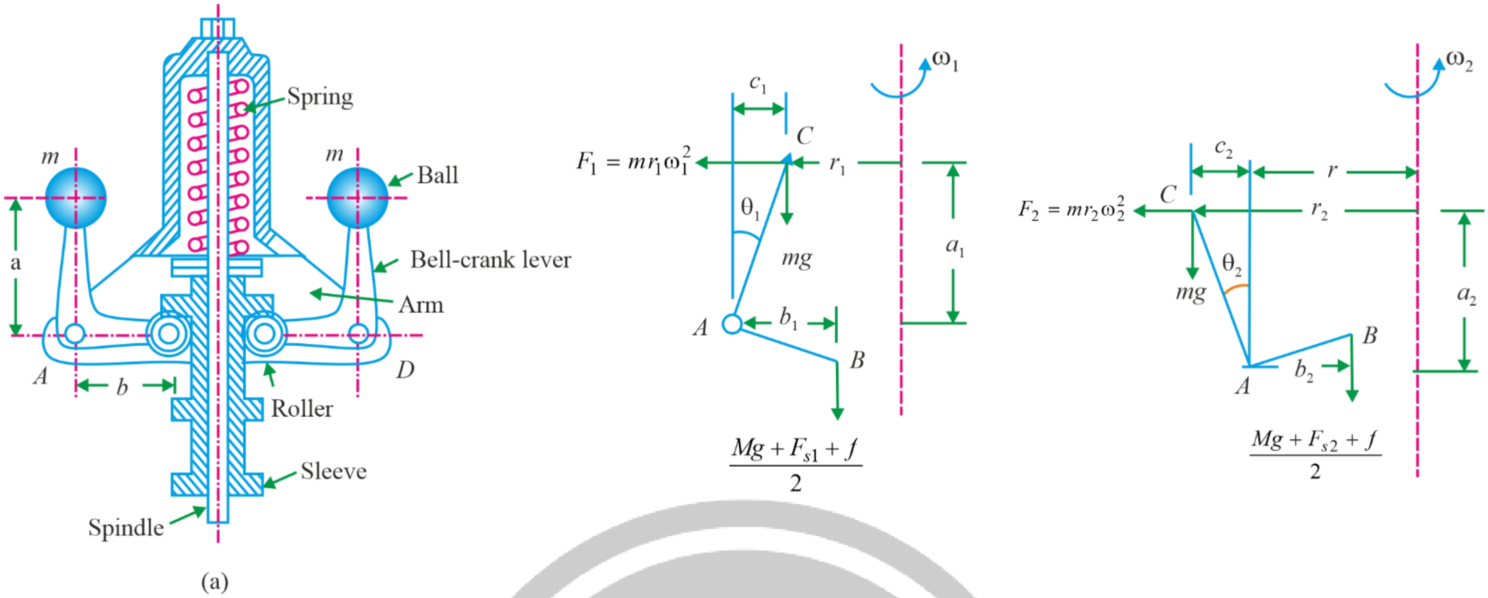


Fig. 8.4 Hartnell governor

8.6.1 Stiffness of Spring

$$s = \frac{2}{r_2 - r_1} \cdot \left(\frac{a}{b}\right)^2 \cdot (F_2 - F_1) \quad = \quad 2\left(\frac{a}{b}\right)^2 \left(\frac{F_2 - F_1}{r_2 - r_1}\right)$$

8.6.2 Lift of Sleeve

$$h_1 \quad = \quad \theta.b = \frac{r_2 - r_1}{a} \cdot b$$

8.7 Parameters related to Governor

8.7.1 Sensitiveness of Governor

It is ratio of the mean equilibrium speed to the difference between the maximum and minimum equilibrium speeds.

N_1 = Minimum equilibrium speed,

N_2 = Maximum equilibrium speed, and

N = Mean equilibrium speed = $\frac{N_1 + N_2}{2}$

Sensitiveness of governor $= \dfrac{N}{N_1 - N_2} \qquad = \dfrac{(N_1 + N_2)}{2(N_2 - N_1)} = \dfrac{\omega_1 + \omega_2}{2(\omega_2 - \omega_1)}$

8.7.2 Stability of Governors

For a stable governor, if the equilibrium speed increases, the radius of governor balls must also increase.

Note:
A governor is said to be unstable if the radius of rotation decreases as the speed increases.

8.7.3 Isochronous Governors

- when the equilibrium speed is constant (*i.e.*, range of speed is zero) for all radii of rotation of the balls within the working range, neglecting friction. The isochronism is the stage of infinite sensitivity.
- **Condition of isochronous in Hartnell Governor:**

$$\frac{M \cdot g + S_1}{M \cdot g + S_2} = \frac{r_1}{r_2}$$

8.7.4 Hunting

- if the speed of the engine fluctuates continuously above and below the mean speed. And results in hitting stopper repeatedly. This is caused by a too sensitive governor which changes the fuel supply by a large amount when a small change in the speed of rotation takes place.

8.7.5 Effort of a Governor

- it is the mean force exerted at the sleeve for a given percentage change of speed (or lift of the sleeve).
- **For porter governor:** $\frac{E}{2} = \frac{cg}{1+k}[2m + M(1+k)]$
- **For watt governor:** $\frac{E}{2} = cmg$
- **For Hartnell governor:** $\frac{E}{2} = c(Mg + F_S)$

Where "c" fraction of change in speed

8.7.6 Power of a Governor

- it is the work done at the sleeve for a given percentage change of speed. It is the product of the mean value of the effort and the distance through which the sleeve moves.

$$\text{Power} = \text{Mean effort} \times \text{lift of sleeve}$$

- **For porter governor power:**

$$= \left[m + \frac{M}{2}(1+k)\right] gh \left(\frac{4c^2}{1+2c}\right)$$

- **For porter governor power (k = 1):**

$$= (m+M)gh\left(\frac{4c^2}{1+2c}\right)$$

- **Coefficient of Insensitiveness** = $\left(\frac{N_1 - N_2}{N}\right)$

8.8 Controlling Force (Fc)

The centrifugal force on each ball of a governor is balanced by an equal and opposite force acting radially inwards known as controlling force.

8.8.1 For Porter Governor Controlling Force

$$= \tan\theta\left[mg + \frac{Mg \pm f}{2}(1+k)\right]$$

8.8.2 For Hartnell Governor Controlling Force

$$= \frac{1}{2}(Mg + F_s \pm f)\frac{b}{a}$$

- The intersection of the speed curves with the controlling force curve provides the speeds of the governor corresponding to the radii. (Assume $f = 0$)

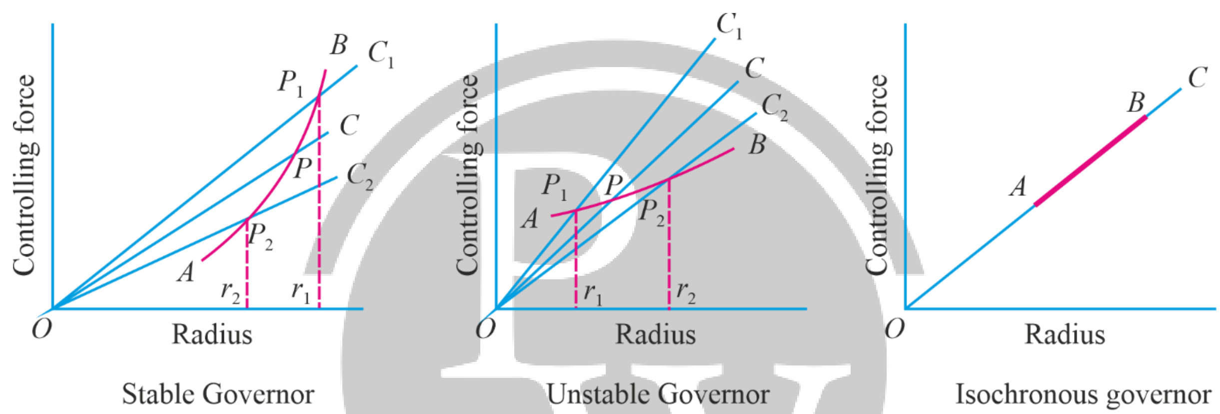


Fig. 8.5 Controlling force (without friction)

Note:

If friction is considered, two more curves of the controlling force are obtained. Thus, in all, three curves of the controlling:

(i) for steady run (neglecting friction)

(ii) While the sleeve moves up (f positive)

(iii) While the sleeve moves down (f negative)

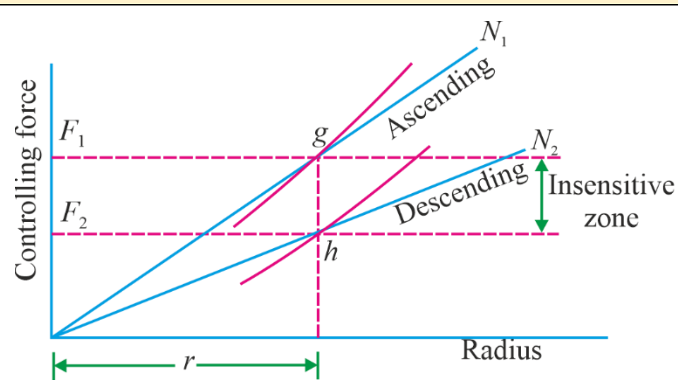


Fig. 8.6 Controlling force with friction

❒❒❒

9 GYROSCOPE

9.1 Introduction

- Gyroscopic effect comes into action when axis of rotation of a rotating body (propeller, wheel etc.) is turned round about an axis perpendicular to axis of spin.
- Axis of spin, axis of precession and axis of gyroscopic couple are perpendicular to each other.

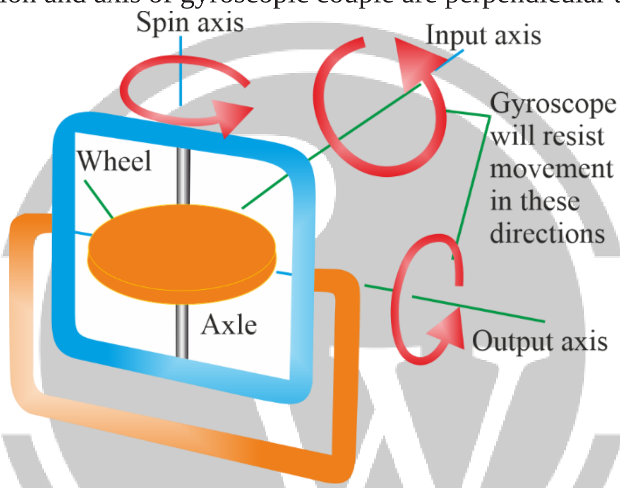


Fig. 9.1 Gyroscope

9.1.1 Analysis

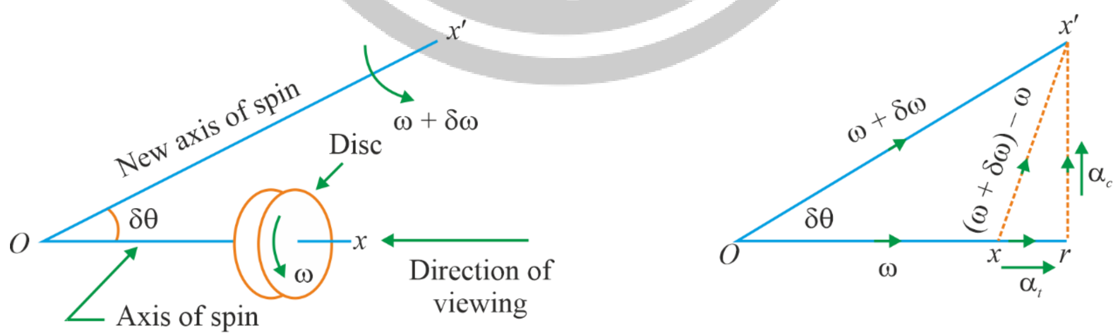


Fig. 9.2 Gyroscope Couple axis

α_c = angular acceleration responsible for changing direction of ω

α_t = Angular acceleration responsible for change in magnitude of ω

$$\alpha_t = \frac{\delta\omega}{\delta t}$$

$$\alpha_c = \omega \times \frac{d\theta}{dt} = \omega \cdot \omega_P$$

- Total angular acceleration of the disc:

$$= \text{vector } xx' = \text{vector sum of } \alpha_t \text{ and } \alpha_c$$

$$= \frac{d\omega}{dt} + \omega \cdot \omega_P$$

- **Angular velocity of precession:** Angular velocity of the axis of spin $\omega_p = \frac{d\theta}{dt}$.
- **Axis of precession:** The axis, about which the axis of spin is to turn
- **Precessional angular motion:** The angular motion of the axis of spin about the axis of precession.

Note:

- The axis of precession is perpendicular to the plane in which the axis of spin is going to rotate.
- If the angular velocity of the disc remains constant at all positions of the axis of spin, then $\frac{d\omega}{dt}$ is zero; and thus, α_t is zero.
- If the angular velocity of the disc changes the direction, but remains constant in magnitude, then angular acceleration of the disc is given by

$$\alpha_c = \omega.d\theta/dt = \omega.\omega_p$$

- The angular acceleration α_c is known as gyroscopic acceleration.

9.2 Gyroscopic Couple

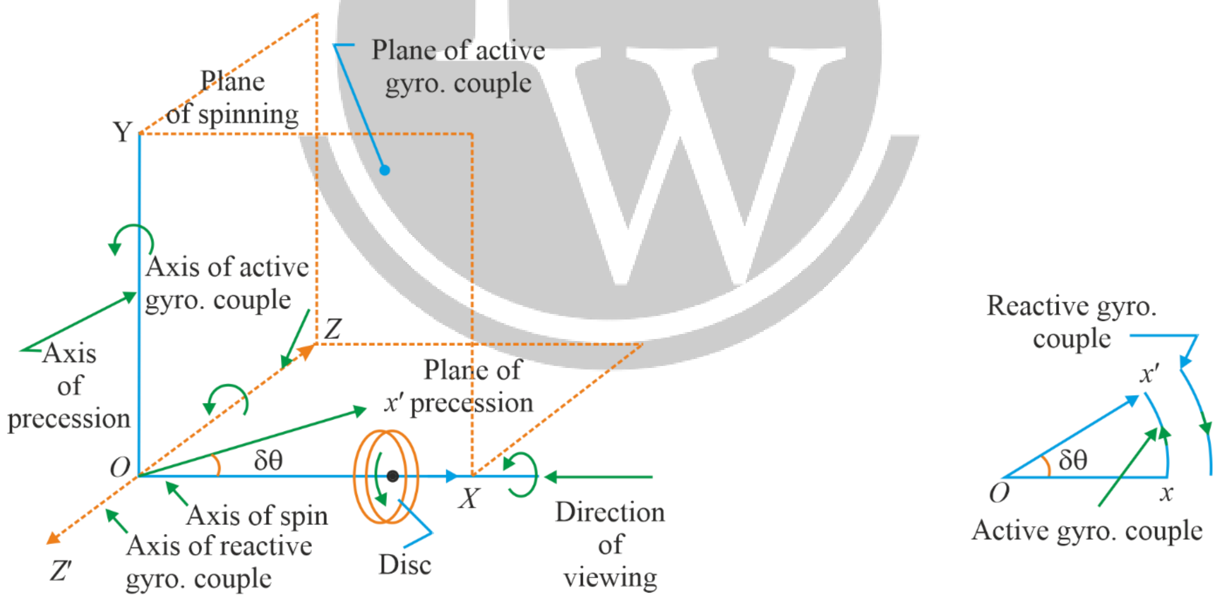


Fig. 9.3 Gyroscope Couple axis

- The rate of change of angular momentum:

$$C = \lim_{\delta t \to 0} I.\omega \times \frac{\delta\theta}{\delta t} = I.\omega \times \frac{d\theta}{dt} = I.\omega.\omega_P \qquad \left(\because \frac{d\theta}{dt} = \omega_P\right)$$

9.3 Effect of the Gyroscopic Couple on an Aeroplane

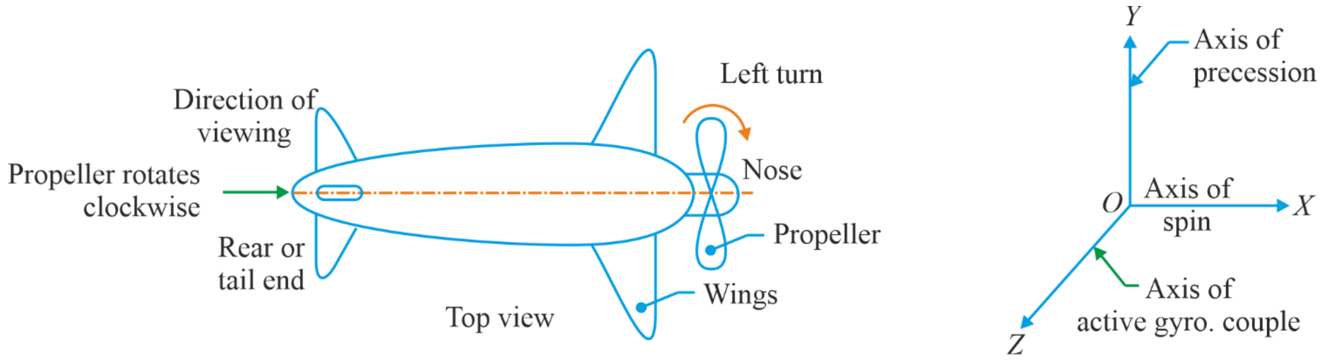


Fig. 9.4 Aeroplan Spin Axis

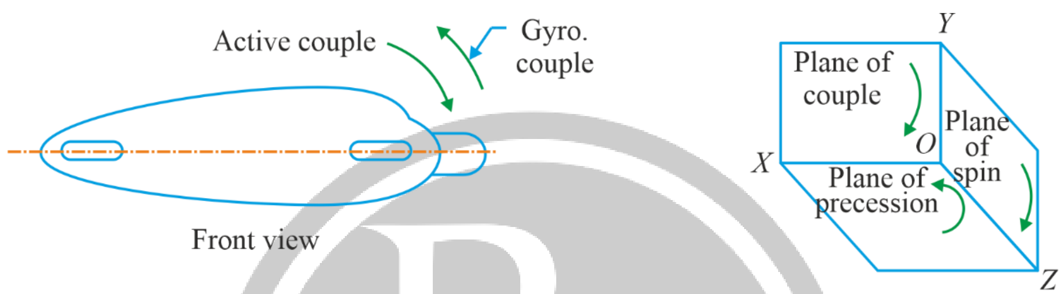


Fig. 9.5 Gyroscope Couple

Let ω = Angular velocity of the engine in rad/s,

m = Mass of the engine and the propeller in kg,

k = Its radius of gyration in metres,

I = Mass moment of inertia of the engine and the propeller in kg-m^2

$= m.k^2$,

V = Linear velocity of the aeroplane in m/s,

R = Radius of curvature in metres, and

ω_p = Angular velocity of precession = $\dfrac{V}{R}$ rad/s

∴ Gyroscopic couple acting on the aeroplane,

(a) Aeroplane taking left turn

(b) Aeroplane taking right turn

9.3.1 Conclusion

- When the aeroplane takes a **right turn** under similar conditions as discussed above (Propeller rotating clockwise when viewed from back side of aeroplane), the effect of the reactive gyroscopic couple will be to **dip the nose** and **raise the tail** of the aeroplane.
- When the engine or propeller rotates in **anticlockwise direction** when viewed from the rear or tail end and the aeroplane takes a **left turn**, then the effect of reactive gyroscopic couple will be to **dip the nose** and **raise the tail** of the aeroplane.
- When the aeroplane takes a **right turn** under similar conditions as mentioned in note 2 above, the effect of reactive gyroscope couple will be to **raise the nose** and **dip the tail** of the aeroplane.
- When the engine or propeller rotates in **clockwise direction** when viewed from the front and the aeroplane takes a left turn, then the effect of reactive gyroscopic couple will be to **raise the tail** and **dip the nose** of the aeroplane.
- When the aeroplane takes a **right turn** under similar conditions as mentioned in note 4 above, the effect of reactive gyroscopic couple will be to **raise the nose** and **dip the tail** of the aeroplane.

9.4 Effect of the Gyroscopic Couple on a Naval ship

- The fore end of the ship is called bow and the rear end is known as **stern** or **aft**. The left hand and right-hand sides of the ship, when viewed from the stern are called **port** and **star-board** respectively.

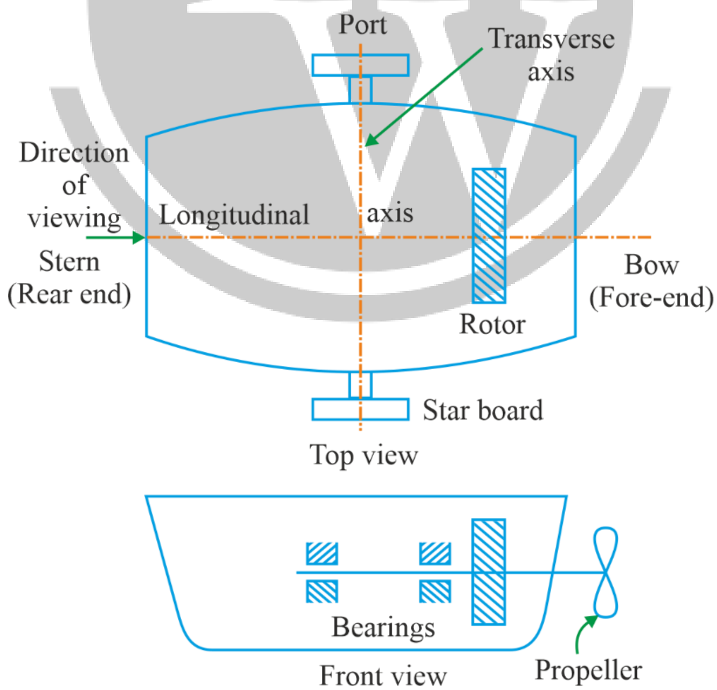


Fig. 9.6 Naval Ship

9.4.1 Naval Ship during Steering (left turn)

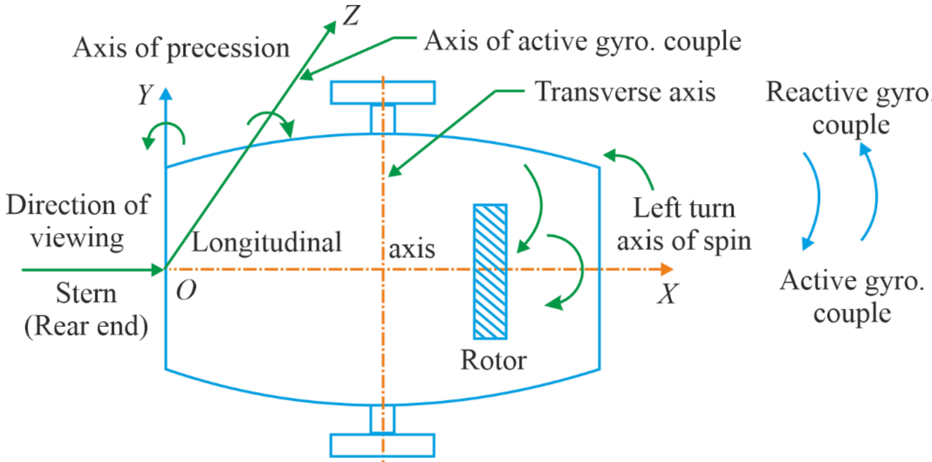
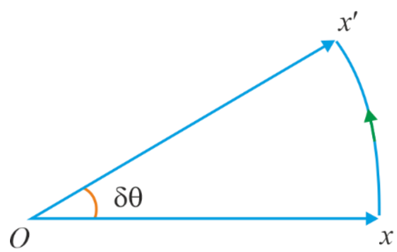


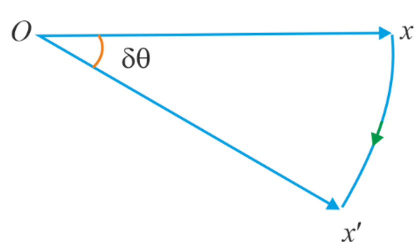
Fig. 9.7 Left turn of Naval Ship

The effect of this reactive gyroscopic couple is to raise the bow and lower the stern.

- When the ship steers to the right under similar conditions as discussed above, the effect of the reactive gyroscopic couple, it will be to **raise the stern** and **lower the bow**.
- When the rotor rates in the anticlockwise direction. When viewed from the stern and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to **lower the bow** and **raise the stern**.
- When the ship is steering to the right under similar conditions, then the effect of reactive gyroscopic couple will be to **raise the bow** and **lower the stern**.
- When the rotor rotates in the clockwise direction when viewed from the bow or fore end and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to **raise the stern** and **lower the bow**.
- When the ship is steering to the right under similar conditions as discussed in note 4 above, then the effect of reactive gyroscopic couple will be to **raise the bow** and **lower the stern**.



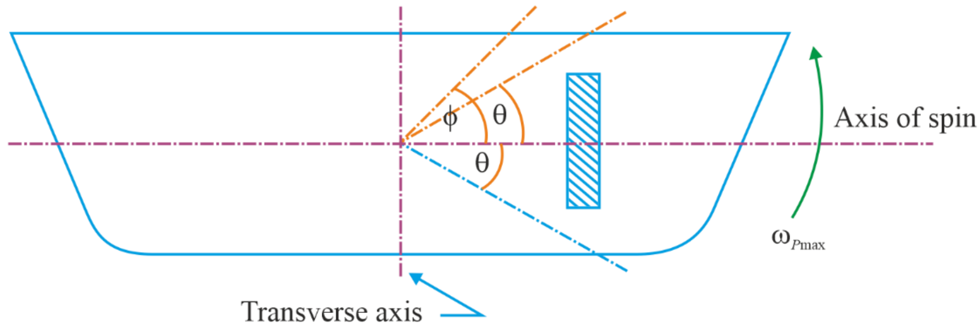
(a) Streeing to the left



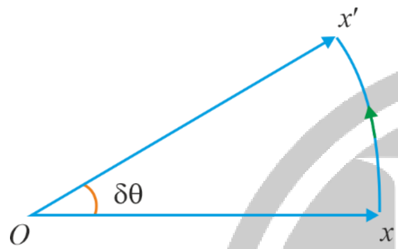
(b) Streeing to the right

9.4.2 Naval Ship during Pitching

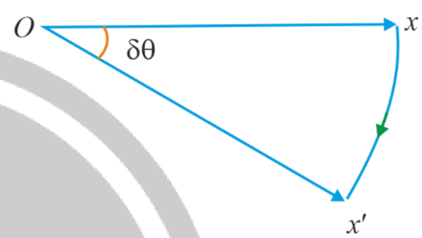
Pitching is the movement of a complete ship up and down in a vertical plane about transverse axis.



(a) Pitching of a naval ship



(b) Piching upward



(c) Pitching downward

Fig. 9.8 Naval Ship with Pitching

- Angular velocity of S.H.M (ω_1)

$$= \frac{2\pi}{\text{Time period of S.H.M. in seconds}} = \frac{2\pi}{t_p}\ \text{rad/s}$$

- Angular velocity of precession (ω_p)

$$\omega_p = \frac{d\theta}{dt} = \frac{d}{dt}(\phi \sin \omega_1 .t) = \phi\omega_1 \cos \omega_1 t \quad [\phi = \text{amplitude of pitching}]$$

- Maximum angular velocity of precession ($\omega_{P\max}$)

$$\omega_{P\max} = \phi \cdot \omega_1 = \phi \times 2\pi / t_p$$

- Assume I = Moment of inertia of the rotor in kg-m^2

  ω = angular velocity of the rotor in rad/s.

- Minimum gyroscopic couple,

$$C_{\max} = I.\omega.\omega_{P\max}$$

- The angular acceleration during pitching,

$$\alpha = \frac{d^2\theta}{dt^2} = -\phi(\omega_1)^2 \sin \omega_1 t$$

- Maximum angular acceleration during pitching,

$$\alpha_{\max} = (\omega_1)^2$$

9.4.3 Naval Ship during Rolling

- In case of rolling of a ship, the axis of precession (*i.e.*, longitudinal axis) is always parallel to the axis of spin for all positions. Hence, there is no effect of the gyroscopic couple acting on the body of a ship.

❒❒❒

10 MECHANICAL VIBRATIONS

10.1 Natural Vibrations for Systems having Single Degree of Freedom

Natural Vibration

Vibration in which there is no kinetic friction at all as well as there is no external force after initial release of system.

10.2 Force Method

D' Alembert's Principle:

- System of forces/system of torques acting on a moving body is in dynamic equilibrium with the inertial force/inertia torque of the body.

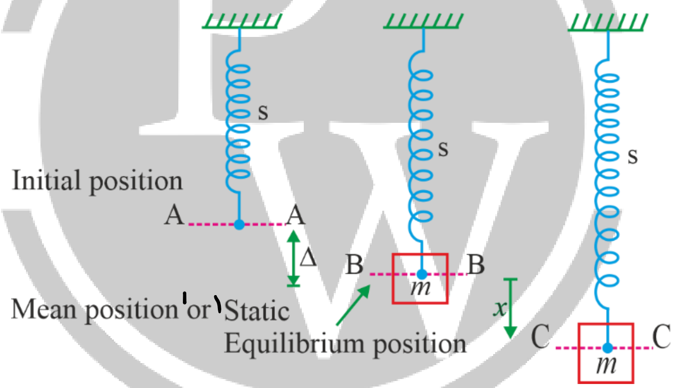
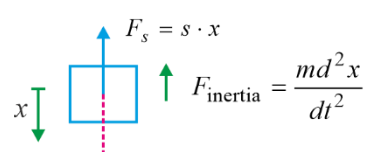


Fig. 10.1 Spring and Mass System

- As mass attached to spring, static deflection generated in spring is Δ. At this position system attains equilibrium position. (B-B).
- At static equilibrium, mg = sΔ
- If system is disturbed by small displacement x from equilibrium position.



- $\frac{md^2x}{dt^2} + sx = 0$ [D'Alembert Principle]

  On comparison with standard SHM equation

$$\boxed{\omega_n = \sqrt{\frac{s}{m}}}\ (\text{rad}/s), \quad \boxed{F = \frac{1}{2\pi}\sqrt{\frac{s}{m}}}\ s^{-1}$$

The solution of equations (*I*) is of the form –

$$\boxed{x = A\sin\left(\sqrt{\frac{s}{m}}t + \phi\right)}$$

A = amplitude (constant)

10.3 Energy Method

- In natural vibrations, the kinetic friction assumed as zero.
- Total energy of system during vibration = Constant

i.e., $\frac{dE}{dt} = 0$

In this case,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}s\,x^2$$

$$\Rightarrow \quad \frac{dE}{dt} = \frac{1}{2}\left[2mv\frac{dv}{dt} + 2s\,x\frac{dx}{dt}\right]$$

$$\frac{dE}{dt} = 0 \quad \Rightarrow \quad \left(\frac{2md^2x}{dt^2} + 2s\,x\right)v = 0$$

$$v \neq 0 \qquad \rightarrow 2m\ddot{x} + 2sx = 0$$

$$\Rightarrow \quad \ddot{x} + \left(\frac{s}{m}\right)x = 0$$

$$\Rightarrow \quad \omega = \sqrt{\frac{s}{m}}$$

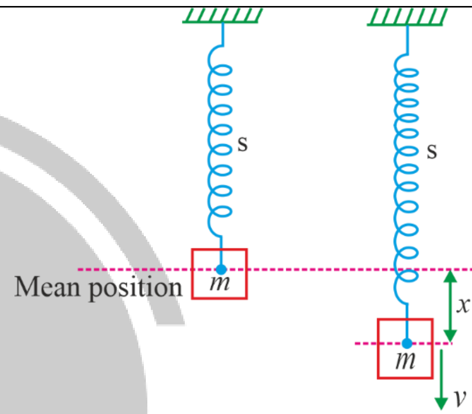


Fig. 10.2 Spring and Mass System

10.4 Torque Method

- This method is very useful for the cases in which body is rotating about a point

Ex. (i) Oscillation of pendulum

(ii) System in pure rolling motion – Pure rotation observed from point of contact.

$$\boxed{I\ddot{\theta} + s\theta = 0}$$

10.5 Combination of Spring

10.5.1 Parallel Combination

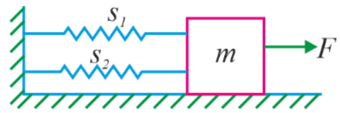


Fig. 10.3 Spring in Parallel

same elongation but different force in each spring.

$$F = F_1 + F_2$$

$$\Rightarrow s_{eq} \cdot x = s_1 x + s_2 x \quad \Rightarrow s_{eq} = s_1 + s_2$$

10.5.2 Series Combination

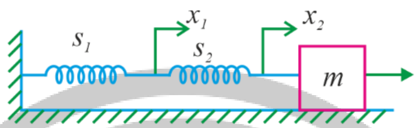


Fig. 10.4 Spring in Series

- same force but different deflection

$$\Rightarrow x = x_1 + x_2$$

$$\frac{F}{s_{eq}} = \frac{F}{s_1} + \frac{F}{s_2} \Rightarrow \left[\frac{1}{s_{eq}} = \frac{1}{s_1} + \frac{1}{s_2}\right]$$

10.5.3 Cutting of Spring in ratio m: n ratio

$$s_A = \left(\frac{m+n}{m}\right)s$$

$$s_B = \left(\frac{m+n}{n}\right)s$$

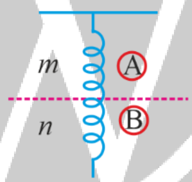


Fig. 10.5 Spring

10.5.4 Inertia Effect of Spring Mass

m_s = mass of spring

- If a simple spring mass system is undergoing natural vibration. Then equivalent mass of system is, $m_{eq} = m + \frac{m_s}{3}$

$$\Rightarrow \quad \omega_n = \sqrt{\frac{s}{m_{eq}}} \qquad \Rightarrow \omega_n = \sqrt{\frac{s}{m + \frac{m_s}{3}}}$$

10.6 Rayleigh Method

Conditions

- Only applied in natural vibration
- Only applied in spring-mass system ‘or’ equivalent
- There should be only one mass in the system that must be point mass.

If Δ = static deflection under suspended Mass = 'm' from static equilibrium

$$mg = s\Delta \quad \Rightarrow \quad \frac{s}{m} = \frac{g}{\Delta}$$

$$\Rightarrow \quad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}}$$

10.7 Free-Damped Vibrations

$(x = 0, t = 0) \rightarrow$ static equilibrium

At $(t = t, x = x)$

$$\boxed{\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{s}{m}x = 0} \qquad \text{...(I)}$$

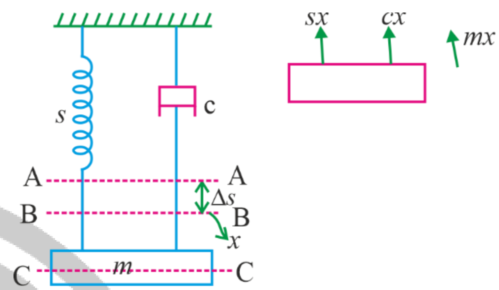


Fig. 10.6 Spring damped Mass System

$$\left(D^2 + \frac{c}{m}D + \frac{s}{m}\right)x = 0$$

$$x \neq 0, \quad \Rightarrow D^2 + \frac{c}{m}D + \frac{s}{m} = 0$$

Auxiliary equation, $\alpha^2 + \frac{c}{m}\alpha + \frac{s}{m} = 0$

$$\alpha_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right)}$$

Solution of equation (I) is –

$$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

10.7.1 Damping Factor or Damping Ratio

$$\zeta = \sqrt{\frac{\left(\frac{c}{2m}\right)^2}{\frac{s}{m}}} = \frac{\frac{c}{2m}}{\sqrt{\frac{s}{m}}} = \frac{c}{2m\omega_n}$$

$$\Rightarrow \quad \zeta = \frac{c}{2\sqrt{ms}}$$

$$\boxed{\zeta = \frac{\text{Actual damping coeff.}}{\text{Critical damping coeff.}} = \frac{c}{c_c}}$$

- If $\zeta < 1 \rightarrow$ Under damped
  $\zeta = 1 \rightarrow$ Critical damped
  $\zeta > 1 \rightarrow$ Over damped
- For $\zeta = 1, \quad c_c = 2\sqrt{ms}$
- In terms of damping factor –

$$\boxed{\frac{d^2x}{dt^2} + (2\zeta\omega_n)\frac{dx}{dt} + \omega_n^2 x = 0}$$

$$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

Where,

$$\alpha_1, \alpha_2 = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$\alpha_1, \alpha_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

- **Case-I: Overdamped System ($\zeta > 1$)**

  Roots of auxiliary equations are real and unequal.

  $$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

  For, $\alpha_1, \alpha_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$

  System can't vibrate due to overdamping. (Aperiodic motion) and $x = 0$ with time.

- **Case-2: Critically Damped ($\zeta = 1$)**

  Roots of auxiliary equations are equal, $\alpha_{1,2} = -\omega_n$

  $\Rightarrow x = (A + Bt)e^{-\omega_n t}$

  As $t \to \infty$, $x \to 0$

  Aperiodic motion. $x \to 0$ in shortest possible time without oscillation.

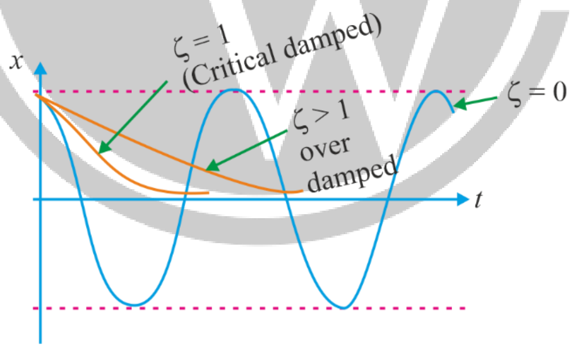


Fig. 10.7 Damping Factor

- **Case-3: Underdamped ($\zeta = 1$)**

  Roots of auxiliary equations are imaginary

  $$\alpha_{1,2} = \left(-\zeta \pm i\sqrt{1 - \zeta^2}\right)\omega_n$$

  Now, $x = Xe^{-\zeta\omega_n t} \cdot \sin(\omega_d t + \phi)$

  X, $\phi \to$ Constant determined by initial conditions.

Note:

In above solution, part $Xe^{-\zeta\omega_n t}$ represents exponentially decreasing of amplitude with time and $\sin(\omega_d t+\phi)$ represents repetition of motion.

10.7.2 Oscillating Frequency of Underdamped Vibration

$$\boxed{\omega_d = \sqrt{1-\zeta^2}\cdot\omega_n} \qquad \{\zeta < 1 \Rightarrow \omega_d < \omega_n\}$$

- Resultant motion is oscillatory with frequency ω_d and decreasing magnitude with time.

$$\boxed{\text{Time period, } T_d = \frac{2\pi}{\omega_d}}$$

X_0 = displacement at start of first cycle

X_1 = displacement at end of cycle

X_n = displacement at end of n[th] cycle

$$x = Xe^{-\zeta\omega_n t}\cdot\sin\left[\frac{2\pi}{T_d}\cdot t+\phi\right]$$

At $t = T_d$

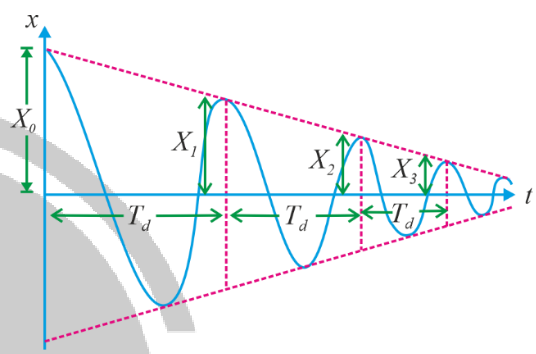


Fig. 10.8 Frequency under damped condition

$$\Rightarrow \quad X_o = X\sin\phi$$

$$X_1 = Xe^{-\zeta\omega_n T_d}\cdot\sin(2\pi+\phi)$$

$$X_1 = X_e{}^{-\zeta\omega_n T_d}\cdot\sin\phi$$

$$X_2 = X_e{}^{-2\zeta\omega_n T_d}\cdot\sin\phi$$

$$\vdots$$

$$X_n = Xe^{-n.\zeta\omega_n T_d}\cdot\sin\phi$$

$$X_{n+1} = Xe^{-(n+1)\zeta\omega_n T_d}\cdot\sin\phi$$

$$\Rightarrow \quad \boxed{\frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3} = ... = \frac{X_n}{X_{n+1}} = e^{\zeta\omega_n T_d}}$$

$\Rightarrow$ Ratio of amplitude of two successive oscillations is constant.

10.7.3 Logarithmic Decrement (δ)

$$\delta = \ln\left(\frac{X_n}{X_{n+1}}\right) = \ln\left(e^{\zeta\omega_n T_d}\right)$$

$$\Rightarrow \quad \delta = \zeta\omega_n T_d = \zeta\omega_n\left(\frac{2\pi}{\omega_d}\right)$$

$$\delta = \frac{2\pi\zeta\omega_n}{\left(\sqrt{1-\zeta^2}\right)\omega_n} \Rightarrow \boxed{\left(\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

10.8 Forced Damped Vibrations

$$m\ddot{x} + c\dot{x} + sx = F_o \sin\omega t$$

$$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{c}{m}\right)\frac{dx}{dt} + \left(\frac{s}{m}\right)x = \left(\frac{F_o}{m}\right)\sin\omega t$$

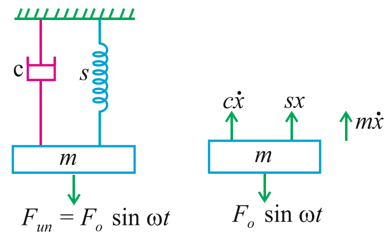


Fig. 10.9 Spring Damped Mass System

10.8.1 Amplitude of Steady State Response

$$\boxed{A = \frac{\frac{F_o}{s}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}}$$

$\frac{F_o}{s} \rightarrow$ static deflection of spring under F_o.

$\phi \rightarrow$ phase l_{ag} of displacement relative to velocity vector.

$$\boxed{\tan\phi = \frac{\left(2\zeta\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}}$$

10.8.2 Magnification Factor

- Ratio of amplitude of steady state response to static deflection under the action of F_o.

$$\boxed{MF = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}}$$

Note:

Irrespective of amount of damping maximum amplitude of vibration occurs before resonance $\left(\frac{\omega}{\omega_n} = 1\right)$

10.8.3 Vibration Isolation and Transmissibility

- Transmissibility (ε) is defined as force transmitted (to foundation) to force applied. It is the measure of effectiveness of vibration is dating material.

$$\varepsilon = \frac{F_t}{F_o} = \sqrt{\frac{1+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

- At resonance, $\omega = \omega_n$

$$\Rightarrow \varepsilon = \frac{\sqrt{1+(2\zeta)^2}}{2\zeta}$$

- If no damper is used – i.e., $\zeta = 0$

$$\Rightarrow \varepsilon = \frac{1}{\pm\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

Note:

For all possible values of g, transmissibility

$\varepsilon = 1$ when $\frac{\omega}{\omega_n} = 0$ and $\frac{\omega}{\omega_n} = \sqrt{2}$

$$\varepsilon = \begin{cases} \text{greater than 1;} & 0 < \frac{\omega}{\omega_n} < \sqrt{2} \\ \text{less than 1;} & \frac{\omega}{\omega_n} > \sqrt{2} \end{cases}$$

Rotary Unbalance force

m_{ro} = mass of rotary part of machine.

e = distance between centre of mass of rotary part and axis of rotation.

ω = forced frequency

$$\Rightarrow F_{un} = \left(m_{ro} \cdot e \cdot \omega^2\right)\sin\omega t$$

Reciprocating Unbalance force

m_{rec} = mass of reciprocating part of machine

r = crank radius = $\frac{\text{stroke}}{2}$

ω = Forced frequency

$$\Rightarrow \quad \boxed{F_{un} = \left(m_{rec} \cdot r \cdot \omega^2\right) \sin \omega t}$$

10.9 Whirling Speed of Shaft

- When a rotor is mounted on a shaft, its centre of mass does not usually coincide with the centre of the shaft. Therefore, upon rotation, shaft bend in the direction of eccentricity of the centre of mass.
- Critical or whirling or whipping speed is the speed at which the shaft tends to vibrate violent in the transverse direction.
- Synchronous whirl –

  Spinning speed = Rotation speed

- Critical/Whirl speed:

  At which shaft tend to vibrate violently in transverse direction.

  s = stiffness of shaft

  e = Initial eccentricity of centre of mass of rotor

  m = mass of rotor.

  y = additional deflection of rotor due to centrifugal force

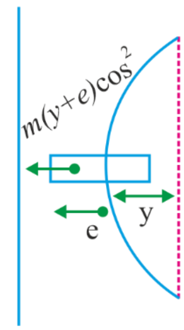


Fig. 10.10 Disk with Shaft

$$\Rightarrow \quad sy = my\omega^2 + me\omega^2$$

$$\Rightarrow \quad y = \frac{me\omega^2}{s - m\omega^2}$$

$$\boxed{y = \frac{e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1}}$$

If $\omega_n = \omega$ (critical speed), resonance occurs, $y \to \infty$

$$\boxed{\omega = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\Delta}}}$$

10.10 Torsional Vibration

I = mass MOI of disc

G = modulus of rigidity of shaft

J = polar MOI (area) of shaft

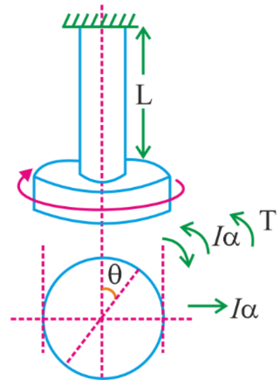


Fig. 10.11 Torsional Vibration System

$$T = \left(\frac{GJ}{L}\right)\theta$$

$$T = q\theta$$

$$\Rightarrow \quad I\alpha + T = 0$$

$$\Rightarrow \quad I\frac{d^2\theta}{dt^2} + \left(\frac{GJ}{L}\right)\theta = 0$$

$$\Rightarrow \quad \boxed{\omega = \sqrt{\frac{q}{I}}}$$

If I′ = mass MOI of shaft, then - $\left[I_{eq} = I + \frac{I'}{3}\right]$

❒❒❒