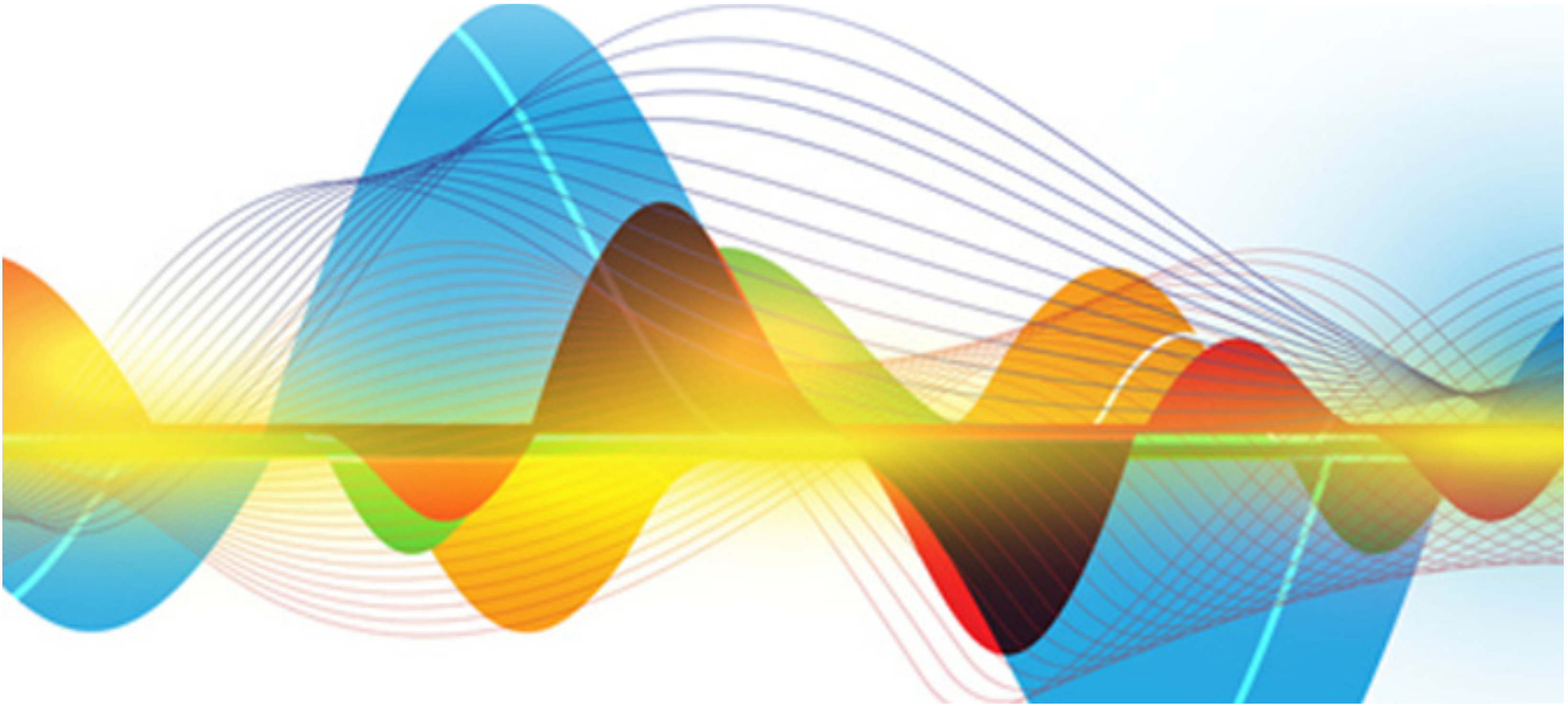
# Signals and Systems



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# SIGNAL AND SYSTEMS

## INDEX

# 1

# BASIC SIGNALS AND SYSTEMS

## 1.1. Introduction

### 1.1.1. Continuous Time Signal

When independent variable is it continuous in time

**Discrete Time Signal:**

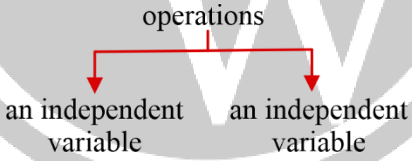
- Obtained from CTS by uniform sampling given a

$$t = nT_S$$

$$x(nT_S) = f(nT_S) \qquad a \le nT_S \le b$$

$$x(n) = f(nT_S) \qquad \frac{a}{T_S} \le n \le \frac{b}{T_S}$$

Continuous Time Signal $x(t)\ v\ \&t$



**On D.V.**

**(1)** **Amplitude:** Given x(t) vs t, plot Ax(t) vs t every vertical axis parameter is multiplied by A

**(2)** **Amplitude Reversal:** Given x(t) vs t, plot $-x(t)\ vs\ t$ Take mirror image w. r. to horizontal axis

**(3)** **Modulus - |x(t)| vs t**

- ➢ Retain graph above horizontal axis.
- ➢ Take the mirror image of graph below horizontal axis.

(1) Addition or subtraction of dc value

Plot $x(t) \pm A\ vs\, t$

$x(t) + A \rightarrow$ Shift up

$x(t) - A \rightarrow$ Shift down

Operation on independent variable: Let x(t) is given

Every operation on t only $(t_0 > 0)$

(1) Time Shifting - Plot $x(t-t_0)$ or $x(t+t_0)$

$x(t-t_0)\ v\ \&t \rightarrow$Shift x(t) vs to unit rightward

(Delay)

$x(t+t_0)\ v\ \&\ t \rightarrow$ Shift x(t) vs t to unit leftward

(Advance)

(2) Time scaling Plot x(at) vs t a > 0

Divide time axis by a

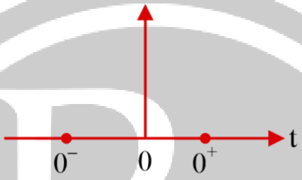
(3) Time Reversal Plot x(-t) v & t

Mirror image w.r. to vertical axis

**Natural :** Time shifting → Time scaling → Time Reversal

## 1.1.1. Standard Signals:

(1) Unit impulse



$$\delta(t)=0 \quad t\neq 0$$
$$\delta(t)=\infty \quad t=0$$
$$\int_{-\infty}^{\infty}\delta(t)dt=1$$

$$\delta(t)=\begin{cases} 0 & t\neq 0 \\ \neq 0 & t=0\rightarrow\infty \\ \int_{0^-}^{0^+}\delta(t)dt=1 & \end{cases}$$

### Properties

(1) $\delta(t)=\delta(-t)$: Even signal

(2) $\delta(t\pm t_o)\Rightarrow$ Not even signal

(3) $\delta(bt)=\dfrac{1}{|b|}\delta(t)$

(4) $\delta(-bt)=\dfrac{1}{|-b|}\delta(t)$

(5) $\delta(-bt+c)=\dfrac{1}{|-b|}\delta\left(t-\dfrac{c}{b}\right)$

(6) $\delta(-bt-c)=\dfrac{1}{|-b|}\delta\left(t+\dfrac{c}{b}\right)$

(7) $\delta(bt-c)=\dfrac{1}{|b|}\delta\left(t-\dfrac{c}{b}\right)$

(8) $\delta(bt+c) = \frac{1}{|b|}\delta\left(t+\frac{c}{b}\right)$

(9) $\delta[g(t)] = \sum_i \frac{\delta(t-t_i)}{|g(t_i)|}$ where $t_i$ is root of $g(t)=0$

(10) $x(t)\delta(t) = x(0)\delta(t)$

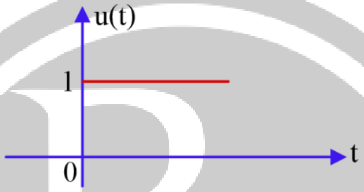
↓

$t=0$

(11) $\int_a^b x(t)\delta(t)dt = x(0)\int_a^b \delta(t)dt$

**Unit step signal:** $u(t) = \begin{cases} 1 : t \geq 0 \\ 0 : t < 0 \end{cases}$

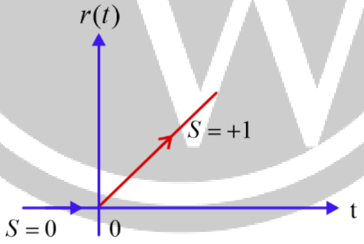
**Property:**



(1) $u(at) = u(t)$

(2) $2u(at) - 1 = Sgn(at)$

**Unit Ramp signal :** $r(t) = tu(t) = t\begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$



$r(at) = ar(t)$

$r(at+b) = ar\left(t+\frac{b}{a}\right)$

$r(-at+b) = ar\left(-t+\frac{b}{a}\right)$

(1) Impulse
divide by a
↓
Horizontal axis

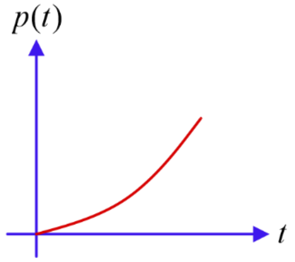
Ramp
Divide by a

(2) Divide by a
(Area)
↓
Vertical axis

multiplied by a
(Slope)

## Unit Parabola Signals:

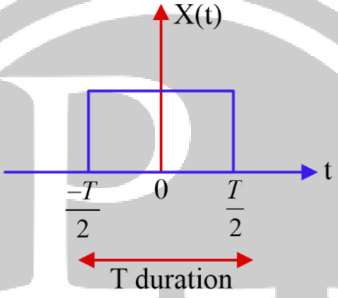
$$p(t) = \frac{t^2}{2} u(t)$$



$$p(t) \xrightarrow{d/dt} r(t) \xrightarrow{d/dt} u(t) \xrightarrow{d/dt} \delta(t)$$

$$\delta(t) \xrightarrow{\int_{-\infty}^{t} dt} u(t) \xrightarrow{\int_{-\infty}^{t} dt} r(t) \xrightarrow{\int_{-\infty}^{t} dt} p(t)$$

## Gate pulse or Rectangular Pulse :

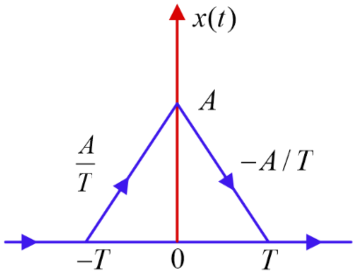


(i) $$x(t) = \begin{cases} A & |t| \le T/2 \\ 0 & \text{else} \end{cases}$$

(ii) $$x(t) = Au\left(t + \frac{T}{2}\right) - Au\left(t - \frac{T}{2}\right)$$

(iii) $$x(t) = \underset{\substack{\downarrow \\ \text{amplitude}}}{A}\text{rect}\left(\underset{\substack{\downarrow \\ \text{duration}}}{t/T}\right)$$

## Triangular Pulse :



(i) $$x(t) = \begin{cases} A\left(1 - \frac{|t|}{T}\right) & : \ |t| \le T \\ 0 & : \ \text{else} \end{cases}$$

(ii) $$x(t) = \underset{\substack{\downarrow \\ \text{peak}}}{A}\ tri\ \underset{\substack{\downarrow \\ \text{duration / 2}}}{\left(\frac{t}{T}\right)}$$

(iii) $$x(t) = \begin{cases} A(1+t/T) & -T \le t < 0 \\ A & t = 0 \\ A(1-t/T) & 0 < t \le T \end{cases}$$

(iv) $$x(t) = \frac{A}{T}r(t+T) - \frac{2A}{T}r(t) + \frac{A}{T}r(t-T)$$

## SINC Function

$$\boxed{\sin ct = \frac{\sin \pi t}{\pi t}} \qquad \boxed{\sin c(Kt) = \frac{\sin(K\pi t)}{K\pi t}}$$

\# $$\frac{\sin at}{bt} = \frac{a}{b}\sin c\left(\frac{at}{\pi}\right)$$

\# $$\frac{\sin t}{t} = \sin c\left(\frac{t}{\pi}\right)$$

**Properties of** $\sin c(t)$ -

(1) $$\lim_{t\to 0}\sin c(t) = \lim \frac{\sin \pi t}{\pi t} = 1 = \sin c(0)$$

(2) $$\lim_{t\to\pm\infty}\sin c(t) = \lim_{t\to+\infty}\frac{\sin \pi t}{\pi t} = 0$$

(3) $\sin c(-t) = \sin c(t)$ Even graph

$$\frac{\sin \pi(-t)}{\pi(-t)} = \frac{\sin \pi t}{\pi t}$$

(4) $t = n \quad n \in I,\ n = \pm 1$

$n \ne 0 \quad n = \pm 2$

(5) $$\int_{-\infty}^{\infty}\sin c(t) = 1 \quad \Rightarrow \quad 2\int_{-\infty}^{\infty}\sin c(t)dt$$

(6) $$\int_{-\infty}^{\infty}\sin c(Kt)dt = 1/K$$

(7) $$\int_{-\infty}^{\infty}\sin c^2(t)dt = 1$$

(8) $$\int_{-\infty}^{\infty}\sin c^2(Kt)dt = \frac{1}{K}$$

## Sampling Function:

$$Sa(t) = \frac{\sin t}{t}, Sa(Kt) = \frac{\sin Kt}{Kt}, \frac{\sin at}{bt} = \frac{a}{b}Sa[at]$$

$$Sa(t) = \frac{\sin t}{t} = \sin c\left(\frac{t}{\pi}\right)$$

## Properties:

(1) $\lim_{t \to 0} Sa(t) = 1$

(2) $\lim_{t \to \pm\infty} Sa(t) = 0$

(3) $Sa(-t) = Sa(t)$

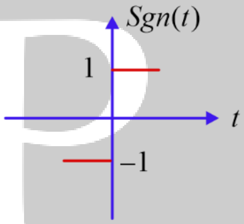
(4) Zero crossover - $t = n\pi, n \in I \quad n \neq 0$

(5) $\int_{-\infty}^{+\infty} Sa(t)dt = \pi$

(6) $\int_{-\infty}^{\infty} Sa^2(t)dt = \pi$

## Signum Function:

$$Sgn(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$



$Sgn\ (Sgn\ (Sgn\ t)) = Sgn(t)$

$$Sgn(t) = 2u(t) - 1 = \frac{t}{|t|}$$

## Discrete Time Signal:

$(DTS) \rightarrow$ ↗ Natural DTS
↘ Converted DTS

## Important Points:

(1) $x(n) = \{\underset{\uparrow_{n=0}}{1}, 2, 3\}$ Finite duration

(2) $x(n) = \{\underset{\uparrow}{1}, 2, 3 - - - - - -\}$ Infinite duration + Right sided

(3) $x(n) = \{- - - - - 3, 2, \underset{\substack{\uparrow \\ n=0}}{1}\}$ Infinite duration + left sided

(4) $x(n) = \{- - - 3, 3, 2, 1, 4, 4 - - -\} \rightarrow$ Duration infinite

\# $x(n - n_0) \rightarrow$ Left
$x(n + n_0) \rightarrow$ Right

\# $x(-n) VS\ \ n \rightarrow$ Mirror image about vertical axis.

**Time Scaling:** plot $x(an)\ VS\ n$

**Case 1.** $a>1$ $x(n)=\left\{1,2,3,\underset{\substack{\uparrow\\ n-0}}{4},5,6,7,9\right\}$ Decimation ,

$x(2n)=\left\{\ 2,\underset{\uparrow}{4},6,8\right\}$

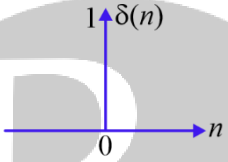
**Case 2.** $a<1$ $x(n)=\left\{1,2,\underset{\uparrow}{3},4\right\}$

$x\left(\frac{n}{2}\right)=\{1\ \ 0\ \ 2\ \ 0\ \ 3\ \ 0\ \ 4\}$

- Interpolation of zero

**Unit Impulse Signal :**

$$\delta=\begin{cases}1 & : \quad n=0\\ 0 & : \quad n\neq 0\end{cases}$$



**Properties:**

(1) $\delta[-n]=\delta[n]:\text{Even}$

(2) $\delta[an]=\delta[n]$

(3) $\delta[-an+b]=\delta[-a(n-b/a)]=\delta\left[n-\frac{b}{a}\right]$

(4) $x(n)\delta(n)=x(0)\delta(n)$

$n=0$

(5) $x(n)\delta(n-n_0)=x(n_0)\delta(n-n_0)$

(6) $x(n)\delta(-an+b)=x\left(\frac{b}{a}\right)\delta\left[n-\frac{b}{a}\right]$

(7) $\delta(n)\times\delta(n)-----=\delta(n)$

(8) $\delta[n]+\delta[-n]=2\delta[n]$

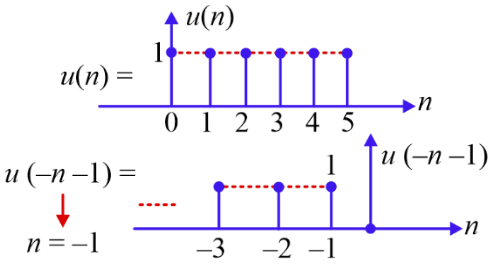
(9) $\delta[n]-\delta[-n]=0$

(10) $\sum\limits_{K=-\infty}^{\infty}\delta(K)=1$

(11) $\sum\limits_{K=n_1}^{n_2}\delta(K)$ $\nearrow$ if $\delta[K]$ lies between $n_1\leq K\leq n_2$
$\searrow 0$ else where

(12) $\sum\limits_{n=n_1}^{n_2}x(n)\delta(-an+b)=x\left(\frac{b}{a}\right)\sum\limits_{n-n_1}^{n_2}\delta\left(n-\frac{b}{a}\right)$ $\nearrow x(b/a)$
$\searrow 0$

## Unit Step Signal:



(1) $u(n)+u(-n-1)=(1)^n$

$u(-t)\xleftrightarrow[\text{Analogy}]{}u(-n-1)$

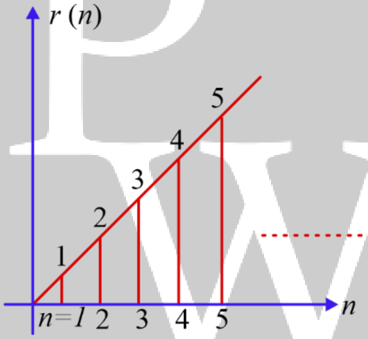
(2) $u(n)u(-n-1)=0$

(3) $u[n]+u[-n]=\begin{cases}2 & : \quad n=0\\ 1 & : \quad n\neq 0\end{cases}$

(4) $u(n)\times u(-n)=\delta(n)$

(5) $u(n)+u(-n-1)=1$

$\boxed{u(n)=\sum_{K=0}^{\infty}\delta[n-K]}$ $\boxed{u(n)=\sum_{K=-\infty}^{n}\delta[K]}$ $\boxed{\delta[n]=u[n]-u[n-1]}$



## Unit Ramp Sequence:

$$r(n)=\sum_{K=0}^{\infty}u[n-K-1]$$

$$r(n)=\sum_{K=-\infty}^{n-1}u[K]$$

## Even /odd | N.E.N.O:

(1) Even - $x(-t)=x(t)$

$x(-n)=x(n)$

graph , must be symmetrical about the vertical axis.

$\int_{-\infty}^{\infty}x(t)dt=2\int_{-\infty}^{0}x(t)dt$ $\nearrow=0$ $\searrow\neq 0$

$Eg-\delta(t),\delta(n),\sin c(t),|t|,$ $\cos t,\ |\sin t|$

(2) Odd Signal, $x(-t)=-x(t)$ Graph Must be Symmetrical about origin.

$x(-n)=-x(n)$

Eg- $\sin t,\ \text{sgn}(t),\ t,\ 1/t,\ n,\ \sin n$

$$\int_{-\infty}^{\infty} x(t)dt = 0, \qquad \sum_{n=-\infty}^{\infty} x(n) = 0$$

(3) Neither Even nor odd –

Eg- $u(t), r(t), u(n), \delta(t-2), \delta(n-2)$

$$x_e(t) = \frac{x(t)+x(-t)}{2}, \qquad x_e(n) = \frac{x(n)+x(-n)}{2}$$

$$x_0(t) = \frac{x(t)-x(-t)}{2}, \qquad x_0(n) = \frac{x(n)-x(-n)}{2}$$

| $x_1(t) \mid x_1(n)$ | $x_2(t) \mid x_2(n)$ | $x_1 \cdot x_2$ | $x_1 \mid x_2$ |
|---|---|---|---|
| E | E | E | E |
| E | 0 | 0 | 0 |
| 0 | E | 0 | 0 |
| 0 | 0 | E | E |

## Conjugate Symmetry :

(1) Even Conjugate

(2) $x(-t) = x^*(t)$ $\quad$ $x(t)$ : Even Conjugate $\Rightarrow \text{Re}\,[x(t)] = \text{Even}$

$x(-n) = x^*(n)$ $\quad$ $x(n)$ : $\quad$ $\text{Im}[x(t)] = \text{odd}$

$x(t) \mid x(n) \rightarrow \text{complex}$

## Conjugate Anti Symmetry :

(1) odd conjugate

(2) $$\left.\begin{matrix} x(-t) = -x^*(t) \\ x(-n) = -x^*(n) \end{matrix}\right] x(t) \mid x(n) \text{ complex}$$

## Periodic & Non periodic Signal :

### For continuous time signal –

(1) Graph must repeat itself from $-\infty$ to $+\infty$ :- $-\infty < t < \infty$

(2) $x(t+T_0) = x(t_0 - T_0) = x(t)$

To = Smallest duration = fundamental Time period

To = +ve and constant , integer or non integer , rational or Irrational

### Complex Exponential

$$x(t) = Ae^{j(\omega_0 t + \phi)}, \text{T}_0 = \frac{2\pi}{\omega_0}$$

$$A\cos(\omega_0 t + \phi) \quad T_0 = \frac{2\pi}{\omega_0}$$

| $x_1(t)$ | $x_2(t)$ | $x(t)=x_1(t)+x_2(t)$ | $x(t)=x_1x_2$ |
|---|---|---|---|
| P | P | ? | ? |
| N | NP | NP | NP |
| NP | P | NP | NP |
| NP | NP | NP | NP |

Continuous time sinusoids or complex exponential are always individually periodic (irrespective of $\omega_0$)
The linear combination of above may or may not be period

## Periodicity of Liner combination of C.T sinusoidal –

$$x(t)=A+B\cos(\underset{\substack{\downarrow\\ T_1=\frac{2\pi}{\omega_1}}}{\omega_1 t+\phi_1})+C\sin(\underset{\substack{\downarrow\\ T_2=\frac{2\pi}{\omega_2}}}{\omega_2 t+\phi_2})-D\cos(\underset{\substack{\downarrow\\ T_3=\frac{2\pi}{\omega_3}}}{\omega_3 t+\phi_3})$$

S-1 $T_1, T_2, T_3$

S-2 $\frac{T_1}{T_2}:R,\ \ \frac{T_1}{T_3}:R\quad x(t)$ is periodic.

S-3 $T_0=LCM(T_1,T_2,T_3)=\dfrac{\text{LCM of Numeratar}}{\text{HCF of Denominator}}$

$$\omega_0=\frac{2\pi}{\omega_0}=HCF\left(\omega_1,\omega_2,\omega_3\right)=\frac{\text{HCF of }N^r}{\text{LCM of }D^r}$$

$\omega_1=K_1\omega_0\ \ K_1th\quad$ Harmonic

$\omega_2=K_2\omega_0\ \ K_{2th}$

$\omega_3=K_3\omega_0$

### Discrete Tie Periodic signal :

Fundamental Time period – Minimum no of samples Which repeats itself

$$\boxed{x(n+n_0)=x(n)}$$

- ➢ $N_0\neq 0,\ N_0\neq\infty,\ N=+ve,\ N_0=$ Integer $N_0$ cannot be negative
- ➢ Discrete time sinusoids and complex exponential are not individually periodic always

Steps – $x(n)=A\cos(\omega_0 n+\phi)$

S-1 $N=\dfrac{2\pi}{\omega_0}\begin{array}{l}\nearrow \text{R:periodic}\\ \searrow \text{IR:Non periodic}\end{array}$

S-2 $FTP=N_0=N\times r$ (r is smallest integer which makes $N_0$ integer)

Periodicity of under combination of discrete time signal –

| $x_1$ | $x_2$ | $\pm x_1 \pm x_2$ |
|---|---|---|
| P | P | P |
| P | NP | NP |
| NP | P | NP |
| NP | NP | NP |

$$x(n) = A(1)^n + B\cos(\underset{\underset{N_{0_1}}{\downarrow}}{\omega_1} n + \phi_1) + C\cos(\underset{\underset{N_{0_2}}{\downarrow}}{\omega_2} n + \phi_2) + D\sin(\underset{\underset{N_{0_3}}{\downarrow}}{\omega_3} n + \phi_3)$$

$$\boxed{N_0 = LCM(N_{0_1}, N_{0_2}, N_{0_3})}$$

**Note:**

| C.T.S | D.T.S |
|---|---|
| $x(t) \to T_0$ | $x(n) = T_0$ |
| $x(-at+b) = \frac{T_0}{\lvert a \rvert}$ | $x(-an+b) \to T_0 = \text{P check}$ |
| $P \times NP = NP$ | $P \times NP = NP$ |
| NP should not be constant | NP should not be constant |

- $x(n) = A\cos[\omega_0 T_S]n$

$$N = \frac{2\pi}{\Omega_0} \to \text{Rational}, \quad \frac{2\pi}{\omega_0 T_s} \to \text{Rational}, \quad \frac{T_0}{T_s} \to \text{Rational}$$

Orthogonal – If inner product of two Signal is zero

$$\int_{-\infty}^{\infty} x_1(t).x_2^*(t)dt = 0, \quad \int_{<T_0>} x_1(t)x_2^*(t)dt, \quad \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = 0$$

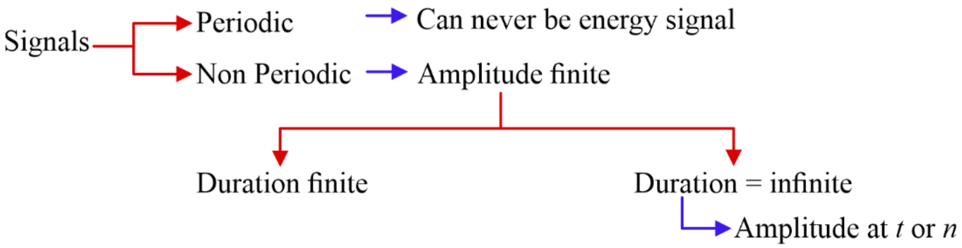
$$\sum_{n=<N_0>} x_1(n)x_2^*(n) = 0$$

**Energy , Power, NENP:**

(1) N.E.N.P → $x(t)/x(n)$ → ±∞ at any signal value of t/n

(2) Energy signal – Must have finite energy for infinite possible duration .

$$\underset{\underset{watt}{\downarrow}}{P} = \frac{E}{T}\,\frac{(\text{Joules})}{\text{sec}} \begin{matrix} \nearrow \text{finite} \\ \searrow \text{Infinite} \end{matrix} = 0$$



- Formula - $E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$, $E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$
- $|x(t)|^2 = x^2(t)$ for real value of $x(t)$ .
- If $x(t) = x_1(t) + x_2(t)$

$$\boxed{E_x = E_{x_1} + E_{x_2} + \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt + \int_{-\infty}^{\infty} x_1^*(t)x_2(t)dt}$$

$$\downarrow$$

If $x_1$ and $x_2$ are orthogonal

$$E_x = E_{x_1} + E_{x_2}$$

**Note :**

| Signal | | Energy |
|---|---|---|
| $x(t)$ | $\longrightarrow$ | $E_x$ |
| $x(t-t_0)$ | $\longrightarrow$ | $E_x$ |
| $x(-t)$ | $\longrightarrow$ | $E_x$ |
| $x(at)$ | $\longrightarrow$ | ${}^{E_x}\!/_{\lvert a\rvert}$ |
| $x(-at+b)$ | $\longrightarrow$ | ${}^{E_x}\!/_{\lvert a\rvert}$ |
| $-Kx(-at+b)$ | $\longrightarrow$ | $\lvert K\rvert^2 \frac{E_x}{\lvert a\rvert}$ |

## Discrete time Energy Signal:

$$\boxed{E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2} \qquad |x(n)|^2 = x^2(n) \text{ for } x(n) \text{ real}$$

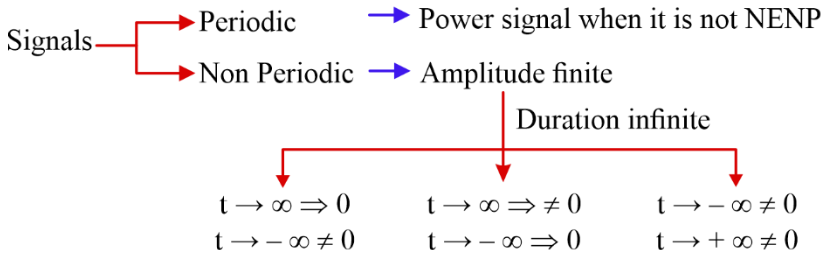
$$E_x = E_{x_1} + E_{x_2} + \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) + \sum_{n=-\infty}^{\infty} x_1^*(n)x_2(n)$$

## Average Value

$${}^{x(t)}\!/_{x(n)} \text{ is periodic } {}^{T_0}\!/_{N_0} \to \overline{x}(t)\frac{1}{T_0}\int_{T_0} x(t)dt, \overline{x}(n) = \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)$$

$${}^{x(t)}\!/_{x(n)} \text{ is non periodic } \Rightarrow \overline{x}(n) = \lim_{N\to\infty}\left[\sum_{n=-N/2}^{n=N/2} x(n)/N\right], \overline{x}(t) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)dt$$

## Power Signal



| **Periodic ($T_0$ / $N_0$)** | **Non Periodic** |
|---|---|
| $P_x = \frac{1}{T_0}\int_{<T_0>} \lvert x(t)\rvert^2\, dt = MSV \;\; [x/\underset{\text{Avrage value of } \lvert x(t)\rvert^2}{\underset{\downarrow}{(t)}}]$ | $P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \lvert x(t)\rvert^2\, dt = \overline{\lvert x(t)\rvert^2}$ |
| $P_x = \frac{E_{xT_0}}{T_0} = \frac{\text{Energy of } 1\,\text{T}_0 \text{ of } x(t)}{T_0}$ | |

$$P_x = P_{x_1} + P_{x_2} + \lim_{T\to\infty}\int_{\frac{-T}{2}}^{T/2} x_1(t)x_2^*(t)dt + \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_1^*(t)x_2(t)dt \quad \text{for non periodic}$$

If $x_1(t)$ and $x_2(t)$ are orthogonal $\rightarrow P_x = P_{x_1} + P_{x_2}$

### Properties for Periodic Signal:

(1) Power signal has finite Energy.

$$\underset{\text{finite}}{\underset{\swarrow}{P}} = \frac{E \to \infty}{T \to \infty}$$

(2) $-Kx(-at+b) = \lvert -K\rvert^2 P$

(3) $Px = \frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2\, dt = \frac{ET_0}{T_0}$

### Discrete Time Power Signal:

$x(n)$ is power signal

$x(n)$ is non periodic signal - $P_x = \lim_{N\to\infty}\frac{1}{(2N+1)}\sum_{n=-N}^{N} \lvert x(n)\rvert^2$

$$P_x = \frac{E_{N_0}}{N_0}$$

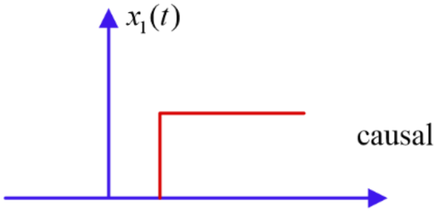
## Causal non causal ant Causal:

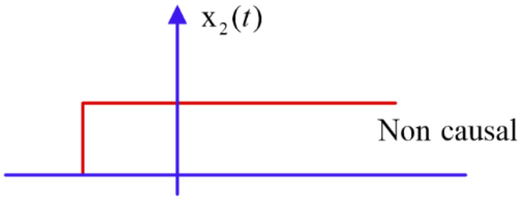
(a) Causal signal $x(t) = 0$ for $t < 0$

$x[n] = 0$ for $n < 0,\; n \le -1$

Part of graph for –ve value of time = 0

(b) Non causal – Which is not causal





(c) Anti causal $\rightarrow$ $\begin{matrix} x(t)=0 & t\geq 0 \\ x(n)=0 & n\geq 0 \end{matrix}$ Graph should be zero for +ve value of time including 0

$u[n]$ – causal Anti causal $\rightarrow$Non causal

$u[-n-1]$ - Anti causal

$u[-n]$ - Non causal

- $x(t)$ $\nearrow \int_{-\infty}^{\infty} x(t)dt \rightarrow$ finite $\rightarrow$Integrable
  $\searrow \int_{-\infty}^{\infty} |x(t)|dt \rightarrow$finite $\rightarrow$Absolutely integrable

- $\sum_{n=-\infty}^{\infty} x(n) =$ finite $\rightarrow$summable

$\sum_{n=-\infty}^{\infty} |x(n)| =$ finite $\rightarrow$ Absolutely summable.

Bounded Signal – x(t) is Bounded

$|x(t)| \leq M < \infty$ $-\infty < t < \infty$
(finite)

$|x(t)| \leq M < \infty$ $-\infty < t < \infty$
(finite)

Ex– $\cos t / \sin t,\ \mathrm{sgn}(t), u(t), dc, e^{-a|t|}\ a > 0,\ \delta[n]$

## Static and Dynamic System :

Static – output should depends only on present value of input

$Ex-$ $y(t) = \sin[x(t)], y(t) = |x^2(t)|$

Dynamic – Not static

$Ex-$ $y(t) = \text{Even }[x(t)], y(t) = \frac{d}{dt}x(t), y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

## Causal and Non causal :

- Causal – output at any instant of time depends on either input at same instant of time or input at past instant of time.

  (OR)
- Output depends on past or present values of input.
- Non causal – which is not causal.
- Anti causal – output depends on future value of input value

## Linear – Non liner:

Linear equation : $y = mx + c$

Non linear : $y^2 = x, \sin x, \cos x$

linear system : Additivity + Homogeneity

S.1 $x(t) \xrightarrow{S} y(t) \searrow$
$x_1(t) \xrightarrow{S} y_1(t) \nearrow \oplus \rightarrow y_1(t) + y_2(t)$ …(i)

S.2 $x_2(t) \xrightarrow{S} y_2(t) \Rightarrow x_1(t) + x_2(t) \longrightarrow y_3(t)$ …(ii)

Equation (i) = equation (ii)

S.3 $A\,x(t) \xrightarrow{S} y_4(t)$ …(iii)

$A\,y(t) = ?$ …(iv)

equation (iii) = equation (iv) → Homogeneity is satisfied

## Time variant and Invariant :

$$x(t-t_0) \longrightarrow \boxed{S/S} \longrightarrow y(t-t_0)$$

Identity definition of system.

$$x(t) \xrightarrow{S} y(t)$$
$$x_1(t) \longrightarrow y_1(t)$$
$$x_1(t) = x(t-t_0)$$
$$y_1(t) = ?\ \_\_\_\_\_\_(\text{i})$$

S-3 Mathematical exp. $y[t-t_0]$ −−−−(iii)

equation (i) = equation (ii) Time Invariant

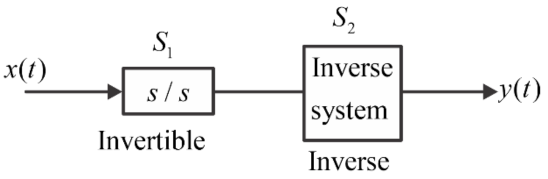
## Invertible and Non Invertible:

Invertible – There must be a one to one mapping between the input and output .

S-1 Replace x and y .

S-2 Obtain y completely in terms of x

S-3



- ➢ Inverse System may or may not be Invertible .

### Stable and Unstable :

**Stable S/S** – Bounded input – Bounded output.

$x(t) / x(n)$ is Bounded –

$$|x(t)| \le M \underset{\to \text{finite}}{<} \infty; -\infty < t < \infty$$

$$|x(n)| \le M < \infty; -\infty < t < \infty$$

Ex-

| | $x(t)$ | $x(n)$ | |
|---|---|---|---|
| $\to$ | $dc$ | $dc$ | |
| $\to$ | $u(t)$ | $u(n)$ | $\delta(n)$ |
| $\to$ | sinusoidal | sinusoidal | |

Then y(t) must be bounded

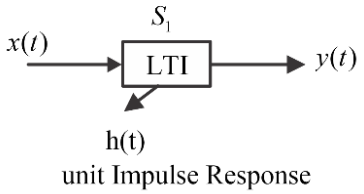
$$y(t) \le N < \infty$$
$$|y(n)| \le N < \infty$$

finite

- ➢ Finite $\to$ time duration

  Bounded $\to$ Amplitude / Magnitude

## 1.2. Continuous Time LTI System



$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y(t) = x(t) * h(t)$$

$\to$ Convolution operator

### Convolution Integral :

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

### Properties Convolution :

(1) $A * B = B * A$

(2) Cumulative: $x(t) * h(t) = h(t) * x(t)$

(3) Distributive: $x(t)*[h_1(t)+h_2(t)]=[x(t)*h_1(t)+x(t)*h_2(t)]$

(4) Associative: $x(t)*[h_1(t)*h_2(t)]=[x(t)*h_1(t)*h_2(t)]$

(5) $y(t)=x(t)*h(t)\Rightarrow A=A_1\times A_2$

(6) $x(t-a)*h(t-b)=y[t-a-b]$

(7) $x(-t)*h(-t)=y(-t)$

(8) $x(at)*h(at)=\frac{1}{|a|}y(at)$

(9) $Ax(t)*Bh(t)=ABy(t)$

(10) $\left(\frac{d^n x(t)}{dt^n}\right)\times\left(\frac{d^m h(t)}{dt^m}\right)\Rightarrow\frac{d^{m+n}y(t)}{dt^{m+n}}$

## Standard Result :

(1) $x(t)*\delta(t)=x(t)$

(2) $x(t-a)*\delta(t-b)=x(t-a-b)$

(3) $\delta(t)*\delta(t)=\delta(t)$

(4) $\delta(t)*\delta(t)-------=\delta(t)$

(5) $\delta(t-a)*\delta(t-b)=\delta(t-a-b)$

(6) $x(t)*u(t)=\int_{-\infty}^{t}x(\tau)d\tau$

(7) $\delta(t)*u(t)=u(t)$

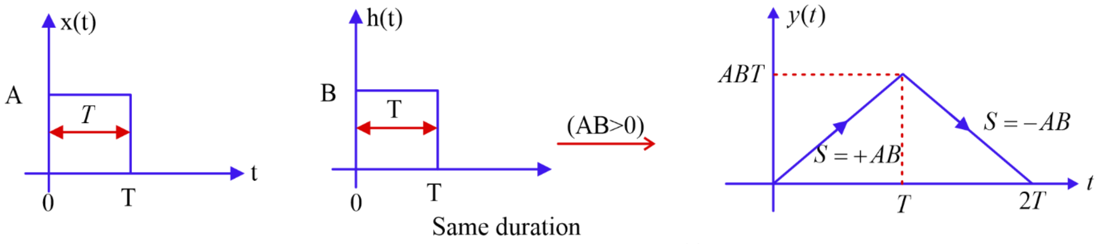
(8) $u(t)*u(t)=r(t)$

(9) $u(t-a)*u(t-b)=r(t-a-b)$

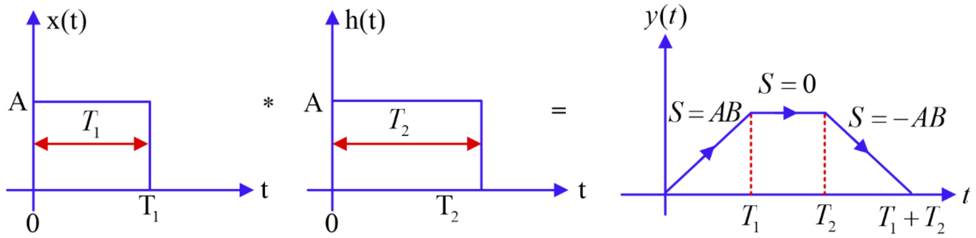
(10) $u(t)\times r(t)=p(t)$

(11) $u(t-a)*r(t-b)=p(t-a-b)=\frac{(t-a-b)^2}{2}u(t-a-b)$

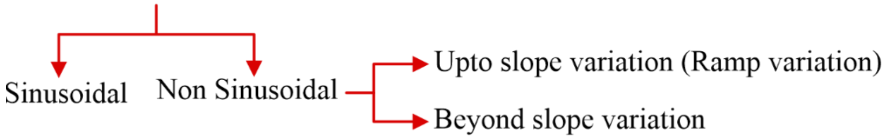
(12)

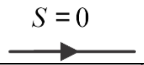
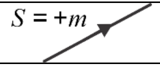
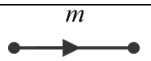
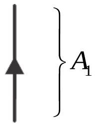
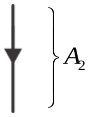


(13)



## Differential of a Signal :



| $x(t)$ | Slope | $Dx(t)dt \to$ Slope |
|---|---|---|
|  | $S=0 \longrightarrow$ | Part of time axis |
|  | $S=+m \longrightarrow$ |  |
|  | $S=+\infty \longrightarrow$ | Upward Impulse $=A_1$ |
|  | $S=-\infty \longrightarrow$ | Downward Impulse $=-A_2$ |

**Integration:** $x(t), y(t)$

$y(t)=\int_{-\infty}^{t}=x(t)dt$ ↗ RunningIntegratio

↘ Area of $x(t)$ from $-\infty$ upto t

## Convolution Method :

**Method (1)** $x(t)*u(t)=\int_{-\infty}^{t} x(\lambda)d\lambda$

**Method (2)** Rectangular pulse → ↗ Same duaration (Triangle)

↘ Different duration (Trapezoidal)

**Method (3)** $y(t)=\int_{-\infty}^{t}[x(t+1)+x(t-1)]dt$

**Method (4)** Timeline Method

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

S -1 Given : $x(t)$ and $h(t)$

S – 2 $x(\tau)$ and $h(t-\tau)$

S – 3 Make time line of $x(\tau)$ $vs$ $\tau$ and $h(t-\tau)$ $vs$ $\tau$

S – 4 Vary $t$ and determine the integration

**Method (5)** **Graphical Method**

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

S -1 Given : $x(t)$ and $vs\ t$ and $h(t)\ vs\ t$

S – 2 $x(\tau)\ vs\ t$ and $h(\tau)\ vs\ \tau$

S – 3 $h(t-\tau)\ vs\ \tau$

$$h(\tau) vs\ \tau \xrightarrow{\text{fold}} h(\tau)\ vs\ \tau \xrightarrow[\text{by t}]{\text{Right Shift}} h(t-\tau) vs\ \tau$$

S – 4 Vary $t$ and calculate integration

**Note:** Before solving the problem of convolution decide the range of convolution

## 1.2.1. Discrete Time L.T.I. System

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$\boxed{y(n) = x(n) * h(n) = \sum_{K=-\infty}^{\infty} x(K)h(n-K)}$$

$h(n)$: unit impulse response of D. T LTI system

Or

Mathematical representation of D. T LTI system

Or

D.T LTI system parameter

$$x(n) * \delta(n) = \sum_{K=-\infty}^{\infty} x(K)\delta(n-K) = x(n)$$

$$x(n) * u(n) = \sum_{K=-\infty}^{\infty} x(K)u(n-K) = \sum_{K=-\infty}^{n} x(K)$$

**Standard Result :**

(1) $\delta(n-n_1) * \delta(n-n_2) = \delta(n-n_1-n_2)$

(2) $x(n-n_1) * \delta(n-n_2) = x(n-n_1-n_2)$

(3) $u(n) * u(n) = (n+1)u(n)$

(4) $u(n+\alpha) * u(n+\beta) = r(n+\alpha+\beta+1) = (n+\alpha+\beta+1)u(n+\alpha+\beta+1)$

**Method Of Discrete Time Convolution:**

Either $x(n)$ or h $(n)$ or both are having infinite duration

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)h(n-K)$$

Both $x(n)$ and h $(n)$ are of finite duration Tabular Method

**Basic Methods :**

(1) By using standard Method

(2) Time line Method : $y(n)\sum_{K=-\infty}^{\infty} x(K)h(n-K)$

S.1 $x(K), h(n-K)$

S. 2 vary n and calculate summation .

(3) Graphical Method: $y(n)=\sum_{K=-\infty}^{\infty} x(K)h(n-K)$

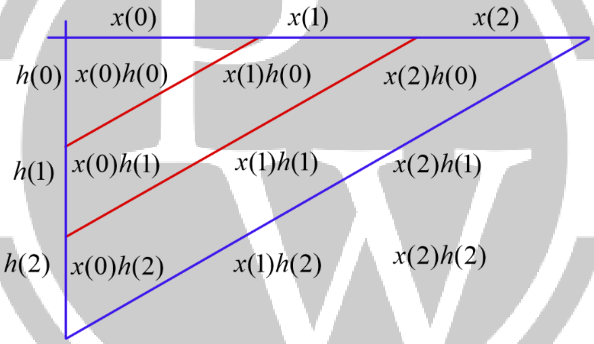
S.1 $x$ (1) VS K

S.2 $h(K)\ VS\ K \xrightarrow{\text{fold}} h(-K)\ VS\ K \xrightarrow[\text{shift by n}]{\text{Right}} h(n-K)\ VS\ K$

S.3 very n and calculation summation .

$x(n)=l,\ \ h(n)=m,\ y(n)=l+m-1$

**Tabulation**



$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \begin{cases} \nearrow t<\tau(\text{above system behaves as non causal system}) \\ \searrow t\geq\tau(\text{above system behaves as casual system}). \end{cases}$$

**For an LTI system to be causal system:**

$h(t-\tau)=0 \qquad t<\tau$

$h(t-\tau)=0 \qquad t-\tau<0\, t-\tau=p \qquad\qquad h(t)=0, \text{for } t<0$

$h(p)=0 \qquad p<0$

$h(t)=0 \qquad t<0$

$h(n-K)=0\ \ ;\ \ n<K\ \ ;\ \ n\leq K-1$

$h(n-K)=0\ \ ;\ \ n-K<0\ \ ;\ \ n-K\leq -1 \qquad\qquad h(n)=0 \text{ for } n<0$

$h(p)=0\ \ ;\ \ p<0\ \ ;\ \ p\leq -1 \qquad\qquad \text{n} \leq -1$

$h(n)=0\ \ ;\ \ n<0\ \ ;\ \ n\leq -1$

### Stability of LTI System :

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Let $|x(t)| \le M < \infty$

$$|x(t-\tau)| \le M < \infty$$

$$\boxed{|y(t)| \le \int_{-\infty}^{\infty} M\,|h(\tau)|\,d\tau} \quad N$$

$$N = \int_{-\infty}^{\infty} \underset{\text{finite}}{\underset{\downarrow}{M}}\,|h(\tau)|\,d\tau \begin{array}{l} \nearrow N : \text{finite} \\ \searrow \int_{-\infty}^{\infty} |h(\tau)|\,d\tau \rightarrow \text{finite} \end{array}$$

### For discrete :

$$\boxed{|y(n)| \le \sum M\,|h(K)|} \rightarrow N \qquad |x(n-K)| \le M$$

$$N = M\sum_{-\infty}^{\infty} |h(K)| \rightarrow \text{finite}, \qquad \sum_{-\infty}^{\infty} |h(K)| \rightarrow \text{finite}$$

**Note :** $h(t) : e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$ : stable system when $a > 0$

$h(n) : a^{|n|} = a^{n}u(n) + a^{-n}u[-n-1]$ stable system when |a| < 1

## 1.3. Static and Dynamic System

For an L.T.I system to be static the unit impulse response $h(t)\,/\,h(n)$ must be an impulse signal.

### Invertible and Non Invertible system–

$$x(t) \longrightarrow \boxed{h(t)} \xrightarrow[y_1(t)]{} \boxed{h_I(t)} \longrightarrow y(t) = x(t)$$

$$y_1(t) = [x(t) * h(t)]$$

$$y(t) = y_1(t) * h_I(t) = x(t) * [\underbrace{h(t) * h_I}_{S(t)}(t)]$$

$$y(t) = x(t)$$

$$h(t) * h_I(t) = S(t) \Rightarrow \boxed{H_I(S) = \frac{1}{H(S)}}$$

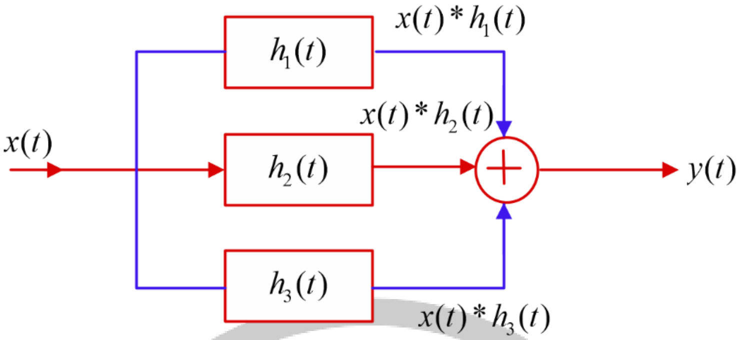
- For discrete $H_I(z) = \dfrac{1}{H(z)}$
- Unit step Response : $s(t) \Rightarrow \dfrac{ds(t)}{dt} = h(t)$ unit impulse Response
- Unit impulse Response : $h(t) \Rightarrow \int_{-\infty}^{t} h(\tau)\,d\tau = s(\tau)$ unit step response

### For discrete :

Unit step - $s(n),\ s(n)-s(n-1)=h(n):$ unit impulse response

Unit Impulse - $h(n),\ \sum_{K=-\infty}^{n} h(K)=s[n]$ unit step response

### LTI System in Cascaded :



$$x(t)\longrightarrow\boxed{ht}\longrightarrow yt$$

$$ht=h_1(t)*h_2(t)*h_3(t)$$

### LTI System in Cascaded:

$$x(t)\longrightarrow\boxed{h_1(t)}\longrightarrow\boxed{h_2(t)}\longrightarrow y(t)$$

$$x(t)\longrightarrow\boxed{h_1(t)*h_2(t)}\longrightarrow y(t)$$

* * *

# 2

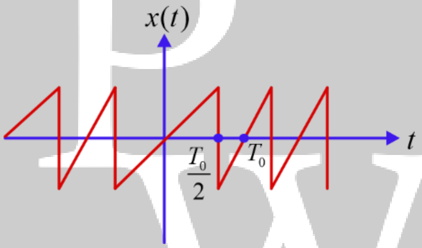
# CONTINUOUS TIME FOURIER SERIES

## 2.1. Introduction1

$$x(t) = A\sin\omega_0 t \begin{array}{l} \nearrow \text{Sinusoidal} \\ \searrow \text{Periodic } C \to \omega_0 \end{array}$$

Fourier series is the representation of time domain non sinusoidal periodic signal as the weighted sum of harmonically related , mutually orthogonal sinusoids .

### 2.1.1. Trigonometric Fourier Series:



$$x_{FS}(t) = a_0 + \sum_{n=1}^{\infty}[(a_n \cos n\,\omega_0 t) + (b_n \sin n\omega_0 t)]$$

$$T_0 = \frac{2\pi}{\omega_0}$$

$a_0, a_n, b_n \to$ Trigonometric Fourier series coefficient

$$\boxed{a_0 = \frac{\int_{T_0} x(t)dt}{T_0}} \; \frac{\text{area } of \;\; x(t) \text{ in } T_0}{T_0} \Rightarrow a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

$a_0$ D.C value or avg value or mean value of x(t)

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt = f(n\omega_0) : n \geq 1$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt = g(n\omega_0) : n \geq 1$$

| **x(t)** | $a_0$ | $a_n$ | $b_0$ |
|---|---|---|---|
| Real | Real | Real | Real |
| Purely Imaginary | P.I | P.I | P.I |
| Complex | Complex | Complex | Complex |

$$\boxed{a_n = a_{-n}} \qquad n \geq 1$$

$$\boxed{b_{-n} = -b_n}$$

$$\boxed{x(t) = r_0 + \sum_{n=1}^{\infty} r_n \cos[n\omega_0 t - \phi_n]}$$

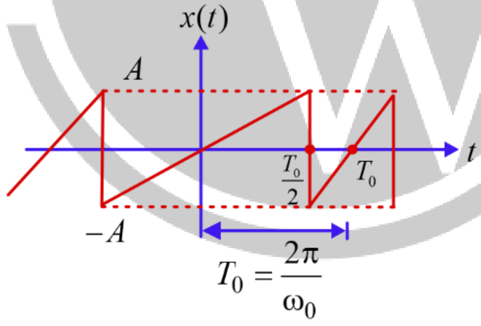
## Polar from of T.F.S.

- $r_0 \rightarrow$dc component of time domain nonsinusoidal periodic signal x(t).
- frequency of $dc$ component = 0Hz
- Amplitude = $r_0$
- Power = $r_0^2$
- rms value = $r_0$

$r_K \cos(K\omega_0 t - \phi_K) \rightarrow K^{th}$ Harmonic of time domain nonsinusoidal periodic signal .

- Frequency of K^th^ harmonic = $K\omega_0$ rad/sec , $Kf_0$ Hz
- Amplitude of K^th^ harmonic $= r_K = \sqrt{a_K^2 + b_K^2}$
- rms value of K^th^ harmonic $= r_k / \sqrt{2}$
- MSV value of or power of K^th^ harmonic = $\dfrac{r_K^2}{2}$

$$X_{FS}(t) = r_0 + r_1\cos(\omega_0 t - \phi_1) + r_2\cos(2\omega_0 t - \phi_2) + r_3\cos(3\omega_0 t - \phi_3) + \_\_\_\_\_$$



$$\boxed{x_{FS}^2(t) = r_0^2 + \frac{r_1^2}{2} + \frac{r_2^2}{2} + \frac{r_3^2}{2} + \_\_\_} = \frac{A^2}{3} \quad \text{Parseval Theorem}$$

## How to calculate absent harmonic in Time domain nonsinusoidal periodic signal:

$S-1 \quad \omega_0, T_0$

$S-2 \quad a_0, a_n, b_n$

$S-3 \quad r_0 = a_0,\ r_n = \sqrt{a_n^2 + b_n^2}\ n \geq 1$

S – 4 find value of n for which $r_n = 0$

$r_K = 0 \quad K^{th}$ harmonic is absent .

Complex or Exponential Fourier series – $x$(t) is real .

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\,\omega_0 t}$$

$$C_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = a_0 = r_0 a$$

$$C_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j_n\omega_0 t} \quad -\infty < n < \infty$$

(1) $\boxed{C_n = \frac{a_n}{2} - j\frac{b_n}{2}} : n \geq 1$

(2) $\boxed{C_{-n} = \frac{a_n}{2} + j\frac{b_n}{2}} : n \geq 1$

(3) $\boxed{C_0 = a_0}$

(4) $\boxed{C_n = C_{-n}^{*}}$

(5) $|C_n| = \frac{r_n}{2} : n \geq 1$

$\angle C_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right) : n \geq 1$

(6) $|C_{-n}| = \frac{r_n}{2} \quad : \quad n \geq 1$

$\angle C_{-n} = \tan^{-1}\left(\frac{b_n}{a_n}\right) : n \geq 1$

(7) $|C_n| = |C_{-n}| \rightarrow$ Magnitude spectrum : Even

(8) $\angle C_n = -\angle C_{-n} \rightarrow$ Phase spectrum : Odd

$$\boxed{C_1 = -|C_1|}$$

**Note:** **As Long as x(t) is real .**

$\rightarrow |C_n|$ *vs* $n\omega_0 \rightarrow$ Even

$\angle C_n$ *vs* $n\omega_0 \rightarrow$ Odd $\rightarrow$ It may looklike even when $\angle C_n$ is multiple of π.

(6) absent frequency – If $|C_n| = 0,\ C_n = 0$

→ n<sup>th</sup> harmonic will be absent.

(7) Amplitude of K<sup>th</sup> harmonic : $r_K = \sqrt{a_K^2 + b_K^2} = 2|C_K|$

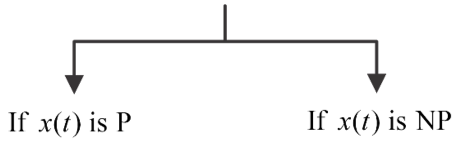
rms value of K<sup>th</sup> harmonic : $\frac{r_K}{\sqrt{2}} = \sqrt{2}|C_K|$

Power of K<sup>th</sup> harmonic : $\frac{r_K^2}{2} = 2|C_K|^2$

## Numerical :

### Type 1 – validity of Trigonometric Fourier series and calculation of harmonies –

- **Procedure** Check the periodicity of given signal



- Given exp is valid F.S
- Given exp is not valid F.S
- Calculate harmonics

### Type 2 – Calculation of complex F.S.C of sinusoid or combination of sinusoidal :

S.1 Calculate $\omega_0 \begin{array}{l} \nearrow 2\pi / T_0 \\ \searrow \omega_0 = HCF(\omega_1, \omega_2, \_\_\_\_\_\_) \end{array}$

S.2 Write x(t) in exponential from .

S.3 $x(t) = \sum C_n e^{jn\omega_0 t}$ replace $\omega_0$

S.4 Compare S.2 and S.3

➢ Calculation of T.F.S coefficient when sinusoids are mentioned-

S-1 Calculate $\omega_o$

S-2 Calculate the harmonics $\begin{array}{l} \omega_1 = K_1\omega_o \\ \omega_2 = K_2\omega_o \end{array}$

S-3 Final values of $a_n, b_n$

### Type 3 – Questions based on properties of Fourier series w.r.t complex F.S.C

1. Linearity- $g(t) = A\underset{C_n}{\underset{\downarrow}{x_1}}(t) + B\underset{d_n}{\underset{\downarrow}{x_2}}(t) \longleftrightarrow g_n = AC_n + Bd_n$

2. Time shifting property - $g(t) = x(t+t_0) \overset{FSC}{\longleftrightarrow} d_n = e^{jn\omega_0 t_0} C_n$

$$C_n \longrightarrow x(t)$$

$$|g_n| = |C_n|,\ \angle g_n = \underset{x(t)}{\underset{\downarrow}{\angle C_n}} - n\omega_0 t_0$$

3. Time Reversal - $x(t) \overset{FSC}{\longleftrightarrow} C_n \Rightarrow C_n \text{ vs } n\omega_0$

$$g(t) = x(-t) \longrightarrow \underset{g(n\omega_0) = f(-n\omega_0)}{\underset{\downarrow}{g_n}} = C_{-n} \Rightarrow g_n \text{ vs } n\omega_0$$

| $x(t)$ | $C_n$ |
|---|---|
| E | E |
| 0 | 0 |
| NENO | NENO |

4. Time Scaling – $T_0,\omega_o \;\; x(t)\xleftrightarrow{FSC}C_n \Rightarrow C_n$ vs $n\omega_o$

$$\left(\frac{T_0}{a}\right),(a\omega_0)\; g(t)=x(at)\xleftrightarrow{FSC}C_n \Rightarrow C_n \text{ vs } n(a\omega_0)$$

| **Time domain** | | **Frequency Domain** |
|---|---|---|
| Compression | $\longleftrightarrow$ | Expansion |
| Expansion | $\longleftrightarrow$ | Compression |

5. Complex conjugate –

$$\omega_o,T_o \quad x(t)\xleftrightarrow{FSC}C_n \text{ vs } \quad n\omega_o$$

$$\omega_o,T_o\;:g(t)=x^*(t)\xleftrightarrow{FSC}g_n=C^*_{-n}\Rightarrow g_n \text{ vs } n\omega_o$$

| $x(t)$ | $C_n$ | |
|---|---|---|
| Real $\longrightarrow$ | Conjugate symmetry | $\Rightarrow C_n=C^*_{-n}\Rightarrow \lvert C_n\rvert=\lvert C_{-n}\rvert,\angle C_n=-\angle C_{-n}$ |
| Imaginary $\longrightarrow$ | Conjugate Summitry | $\Rightarrow C_n=-C^*_{-n}\Rightarrow \lvert C_n\rvert=\lvert C_{-n}\rvert,\angle C_n=-\angle C_{-n}\pm 180^0$ |
| Conjugate Symmetry $\longrightarrow$ | Real | |
| Conjugate anti Symmetry $\longrightarrow$ | Imaginary | |

| **$x(t)$** | **$C_n$** |
|---|---|
| R E | R E |
| R O | I O |
| I E | I E |
| I O | R O |

(6) Multiplication by complex exponential function.

$$T_o,\omega_o \quad x(t)\xleftrightarrow{FSC}C_n\longrightarrow C_n \text{ vs } n\omega_o$$

$$g(t)=e^{jm\omega_o t}\;x(t)\longleftrightarrow g_n=C_{n-m}\Rightarrow g_n \text{ vs } n\omega_o$$

$$g(t)=e^{-jm\omega_o t}\;x(t)\longleftrightarrow g_n=C_{n+m}\Rightarrow g_n \text{ vs } n\omega_o$$

(7) Differentiation : $T_o,\omega_0 : x(t) \xleftrightarrow{FSC} C_n \Rightarrow C_n \text{ vs } n\omega_o$

$$T_o,\omega_o : \frac{d^3x(t)}{dt^3} \longleftrightarrow (jn\omega_o)^3 C_n$$

(8) Integration Property : $T_o,\omega_o : x(t) \xleftrightarrow{FSC} C_n \Rightarrow C_n \text{ vs } n\omega_o$

$$T_0,\omega_0 : g(t) = \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FSC} \frac{C_n}{jn\omega_0} = g_n = g_n \text{ vs } n\omega_0$$

(9) Periodic convolution - $x_1(t)$ and $x_2(t)$ are both periodic with some time period To.

$$x_1(t) * x_2(t) = \int_{T_o} x_1(\tau)x_2(t-\tau)d\tau$$

**Multiplication in time domain :**

$$T_0,\omega_0 \quad x_1(t) \rightarrow C_n$$

$$T_0,\omega_0 \quad x_2(t) \rightarrow d_n$$

$$g(t) = x_1(t)\cdot x_2(t) \longleftrightarrow g_n = C_n * d_n$$

$\longrightarrow$ Tabular Method

## Type 4 – Symmetry :

(a) Even :- Even in $\left(-\frac{T_0}{2},\frac{T_0}{2}\right) or \left(-\frac{T_0^+}{2},\frac{T_0^+}{2}\right) or \left(-\frac{T_0^-}{2},\frac{T_0^-}{2}\right)$

(b) Odd :- odd in $\left(\frac{-T_0}{2},\frac{T_0}{2}\right)$

(c) Half wave symmetry .

(a) Odd HWS - $x\left(t \pm \frac{T_0}{2}\right) = -x(t)$

(b) Even HWS - $x\left(t \pm \frac{T_0}{2}\right) = x(t)$

Effect of symmetry on T.F.S Coefficients .

**Case 1:** $x$(t) is even

$a_0 \begin{matrix} \nearrow = 0 \\ \searrow \neq 0 \end{matrix}$ but $b_n = 0$ always , $a_n$ : will not be zero for all value of n.

- dc value may or may not be present.
- Harmonic of cosine decided by $a_n$

- All sine harmonics are absent.
- Frequency – 0HZ →decide by $a_n$
- Other frequency →decide by $a_n$

**Case 2:** $x(t)$ is odd

$a_n = 0$ , $a_0 = 0$, $b_n \to$ will not be zero always.

- dc is absent, all cosine harmonics absent, sine harmonic decided by $a_n$
- $0\text{HZ} \to \text{absent}$

Other frequency →decided by $a_n$

**Case 3:** $x(t)$ is HWS-

$$a_0 = 0$$
$$a_n = 0 \quad \text{for } n \text{ even}$$
$$= \frac{4}{T_0} \int_{-T_o/2}^{T_o/2} x(t) \cos n\omega_o t \; dt \;\; n : \text{odd}$$

$$b_n = 0 \; n : \text{even}$$
$$b_n = \frac{4}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \sin + \omega_0 t \, dt$$

- dc is absent
- all even harmonic of sine / cosine are absent.
- all odd harmonic of sine /cosine are present .
- 0HZ: absent and $f_0, 3f_0, 5f_0$________ will be present .

**Case 4:** $x(t)$ is Even + HWS (odd)

$$a_0 = 0, \quad a_n = 0 \;\; n : \text{even} \qquad a_n \neq 0 \;\; n : \text{odd} \quad \Big| \quad b_n = 0 \quad \forall_n$$

- dc absent
- all harmonic of sine and even harmonic of cosine are absent.
- all odd harmonic of cosine are present.
- OHZ → absent , $f_0, 3f_0, 5f_0 ---$ prestent

**Case 5:** $x(t)$ is odd +HWS

$a_0 = 0$ $\qquad$ $b_n = 0 \; n$:even

$a_n = 0 \;\; \forall_n$ $\qquad$ $b_n \neq 0 \quad n : \text{odd}$

- dc absent
- all harmonic of cosine and even harmonic of sine $\rightarrow$absent .
- odd harmonic of sine will be present.
- 0HZ $\rightarrow$ absent , $f_0, 3f_0, 5f_0 \rightarrow$ presens

## Fourier Transform:

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt,\ X_{To}(\omega) = \int_{-To/2}^{To/2} x(t)e^{-j\omega t}dt$$

$$X_{To}(n\omega_0) = \int_{-T_0/2}^{T_o/2} x(t)e^{-jn\omega_o t}dt$$

$$x(t) \xleftrightarrow{BLT} X(S) \qquad \underset{L.T}{X(S)} \xrightarrow{S=j\omega} x(\omega)F.T$$

$$X(S) = \int_{-\infty}^{\infty} x(t)e^{st}dt \longrightarrow \text{ROC} \qquad s = j\omega \text{ is part of ROC.}$$

Property $= x(t) \longrightarrow x(\omega)$ then

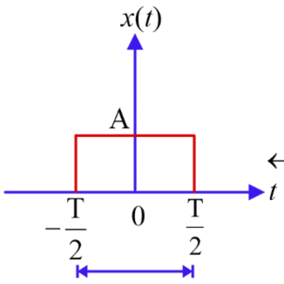
$$x(t) \longleftrightarrow X(s)$$

(1) $x(t-t_o) = e^{-j\omega to}X(\omega)$ | $x(t-t_o) \longleftrightarrow e^{-St_o}X(s)$

(2) $x(t+to) = e^{j\omega to}X(\omega)$ | $x(t+to) \longleftrightarrow e^{St_o}X(s)$

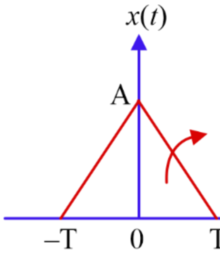
(1) $\delta(t) \xleftrightarrow{F.T} 1$

(2)



$$\longleftrightarrow X(\omega) = \underset{\text{Area}}{\underset{\downarrow}{AT}}\ \text{Sa}\underset{\substack{\downarrow \\ \text{Half of duration T}}}{\left[\frac{\omega T}{2}\right]}$$

(3)



$$AT \longleftrightarrow \underbrace{AT}_{\text{Area}}\ \text{Sa}^2\left[\frac{\omega T}{2}\right] = X(\omega)$$

$$\frac{T}{2} = \frac{\text{duration}}{4}$$

(4) $u(t) \xleftrightarrow{L.I} \frac{1}{s}$

(5) $tu(t) \longleftrightarrow \frac{1}{s^2}$

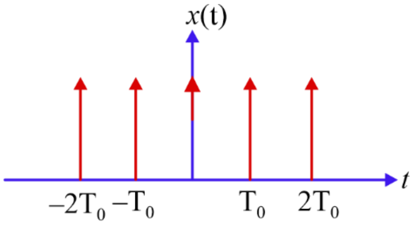
(6) $t^n u(t) \longleftrightarrow \frac{n!}{s^{n+1}}$

(7) $\sin\omega_0 t \; u(t) \longleftrightarrow \frac{\omega_0}{s^2+\omega_0^2}$

(8) $\cos\omega_0 t \; u(t) \longleftrightarrow \frac{s}{s^2+\omega_0^2}$

## Important Observation :

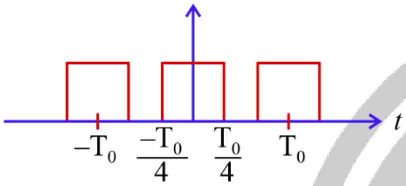
(1)



$\xleftrightarrow{\text{FSC}} C_n = \frac{1}{T_0}$

(2)

$x(t)$ : HWS + Step



$\longleftrightarrow C_n = C_0 \sin C\left(\frac{n}{2}\right)$

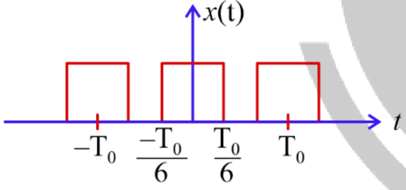
$\longrightarrow |C_n| \alpha \frac{1}{n}$

n : even $\rightarrow C_n = 0$

n : odd $\rightarrow |C_n| = \left(\frac{2C_0}{\pi}\right) \times \frac{1}{n}$

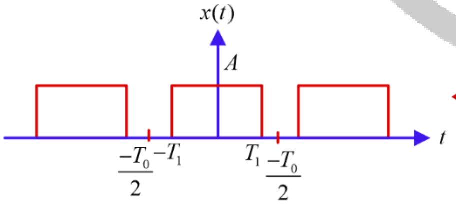
(3)

NO HWS + Step



$\longleftrightarrow C_n = C_0 \sin C\left(\frac{n}{3}\right)$

$|C_n| \alpha \frac{1}{n}$



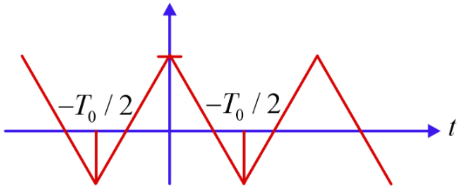
duration

$\longleftrightarrow C_n = C_0 \sin C\left[\frac{n(2T_0)}{T_0}\right]$

F.T.P

$|C_n| \alpha \frac{1}{n}$

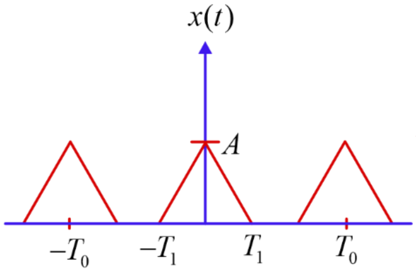
$x(t)$ : Similar + HWS



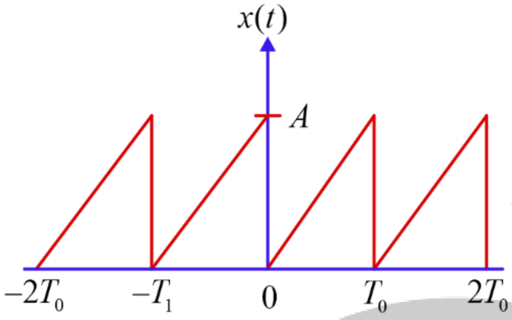
$\longleftrightarrow C_n = \frac{2A}{n^2\pi^2}\left[1-(-1)^n\right]$

$C_n = 0 \quad n : \text{even}$

$|C_n| \alpha \frac{1}{n^2} \quad C_n = \frac{4A}{n^2\pi^2} \quad n : \text{odd}$



$\longleftrightarrow$ $C_n = C_0\left[\text{Sin}\,C\left(\frac{nT_1}{T_0}\right)\right]^2$



$\longleftrightarrow C_n = \frac{jA}{n\pi}$

**Type 6:** Parseval Theorem

$x(t)$: Power signal, which is periodic with F.T.P $T_0$ absolute or Exact power $x(t)$:

$$P_x = \frac{1}{T_0}\int_{T_0} x^2(t)dt \qquad \text{(If } x(t) \text{ is real)}$$

$$= \frac{1}{T_0}\int_{T_0} |x(t)|^2\ dt$$

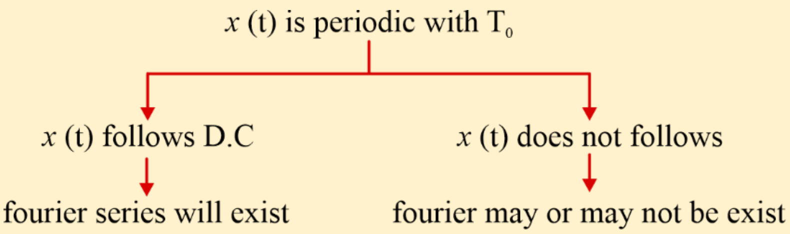
$$P_x = a_0^2 + \sum_{n=1}^{\infty}\left[\frac{a_n^2}{2} + \frac{b_n^2}{2}\right]$$

**Note:** $x(t) \xleftrightarrow{FSC} C_n$

(1) $P_x = \sum_{n=-\infty}^{\infty} |C_n|^2$

(2) $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \Rightarrow x(0) \sum_{n=-\infty}^{\infty} |C_n| e^{j\angle C_n}$

Drichlet's condition - Only sufficient condition not necessary



x (t) is periodic with $T_0$

x (t) follows D.C

fourier series will exist

x (t) does not follows

fourier may or may not be exist

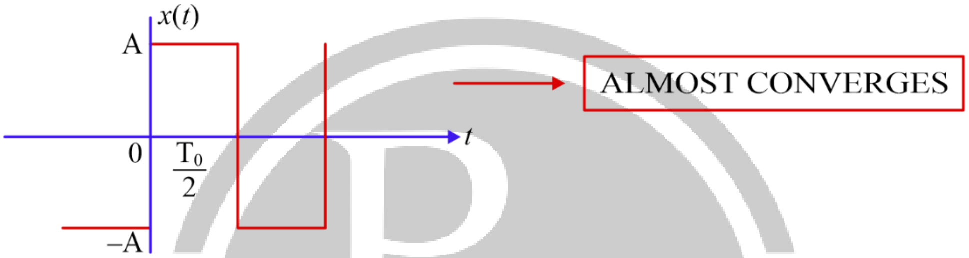
**Statement :**

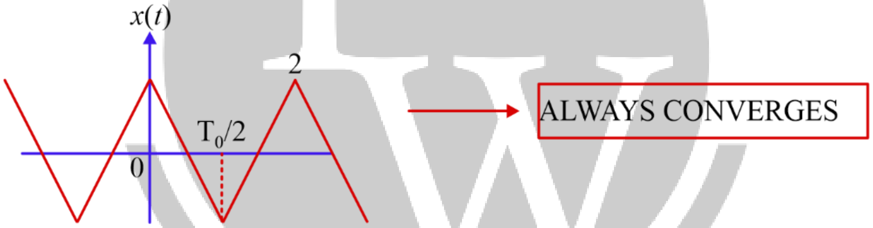
(1) Any nonsinusoidal time domain periodic signal can always be Exactly written as weighted sum of Harmonically related naturally orthogonal sinusoids is not completely true.

(2) Fourier series a nonsinusoidal time domain periodic signal converges at all points on the nonsinusoidal time domain periodic signal is not Exactly True.

(3) The Fourier series representation of T.D. non sinusoidal periodic signal converge at ALMOST all the points on time domain non sinusoids periodic signal, except at the point of discontinuity

| **$x$(t) : N.S. + P** | **Fourier Series** |
|---|---|
| Continuous in Amplitude ⟶ | Fourier Series converges at all points |
| Discontinuous in Amplitude ⟶ | Fourier Series converges at almost all the point except the point of discontinue |





# 3

# FOURIER TRANSFORM

## 3.1. Continuous Time Fourier Transform

- $x(t)$ is non periodic signal
- $x(t)\xleftrightarrow{F.T}X(\omega)$
- $X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$ or $X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$
- $X(\omega)=\delta(\omega)$
- $X(f)=\delta(2\pi f)=\frac{1}{2\pi}\delta(f)$

**Note :** For applying F.T formula $x(t)$ should be N.P and absolutely integrable.

| x(t) | Formula of F.T | F.T Exist |
|---|---|---|
| Energy | Applicable | Yes (always) |
| Power | Not Applicable | Always Exist |
| NENP except $\delta(t)$ | Not applicable | No |
| $\delta(t)$ | Applicable | Always Exist |

$\longrightarrow$Limited sense

$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt, X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$

$x(t)\xleftrightarrow{F.T}X(\omega)$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{X(\omega)}_{\text{volt/(rad/sec)}}\ e^{j\omega t}d\omega\rightarrow \text{rad/sec}$$

$$\underset{\substack{\downarrow\\ volt}}{x(t)}=\int_{-\infty}^{\infty}\underbrace{X(f)}_{\text{volt/Hz}}e^{j2\pi ft}df$$

$$X(\omega)=|X(\omega)|e^{j}\angle x(\omega)$$

$$X(f)=|X(f)|e^{j}\angle x(f)$$

## Properties

(1) Linearity

$$x_1(t)\longleftrightarrow X_1(\omega)$$
$$x_2(f)\longleftrightarrow X_2(\omega)$$

$$g(t)=Ax_1(t)+Bx_2(f)\longleftrightarrow G(\omega)=AX_1(\omega)+BX_2(\omega)$$

Time shift - $x(t)\longleftrightarrow X(\omega)$

$$x(t-t_0)\longleftrightarrow e^{-j\omega t_o}X(\omega)=e^{-j2\pi f\,t_o}X(f)$$

$$x(t+t_0)\longleftrightarrow e^{j\omega t_o}X(\omega)=e^{j2\pi ft_o}X(f)$$

- Does not affect the magnitude .
- $\dfrac{x(t-a)+x(t+a)}{2}\longleftrightarrow X(\omega)\cos a\omega,\dfrac{x(t+a)-x(t-a)}{2j}\longleftrightarrow X(\omega)\sin a\omega$

$$\frac{x(t-a)+x(t-a)}{2}\longleftrightarrow X(f)\cos(2\pi a)f,\frac{x(t+a)-x(t-a)}{2j}\longleftrightarrow X(f)\sin(2\pi a)f$$

**Frequency Shifting** $x(t)\longleftrightarrow X(\omega)$

$$e^{j\omega_0 t}x(t)\longleftrightarrow X\left(\omega-\omega_0\right)$$

$$e^{-j\omega_0 t}x(t)\longleftrightarrow X(\omega+\omega_0)$$

$$\cos\omega_0 t\,x(t)\longleftrightarrow\frac{X(\omega-\omega_0)+X(\omega+\omega_0)}{2}$$

$$\sin\omega_0 t\,x(t)\longleftrightarrow\frac{X(\omega-\omega_0)-X(\omega+\omega_0)}{2j}$$

$$\cos 2\pi f_0 tx(t)\longleftrightarrow\frac{X(f-f_0)+x(f+f_0)}{2}$$

$$\sin 2\pi f_0 tx(t)\longleftrightarrow\frac{X(f-f_0)-X(f+f_0)}{2j}$$

**Modulation Property** $x(t)\longrightarrow X(f)$

$$\cos 2\pi f_0 t\,x(t)\longleftrightarrow\frac{X(f-f_0)+X(f+f_0)}{2}$$

$$\sin 2\pi f_0 t\,x(t)\longleftrightarrow\frac{X(f-f_0)-x(f+f_0)}{2j}$$

**Time Reversal**

| | |
|---|---|
| $x(t) \longleftrightarrow X(\omega)$ | $x(t) \longleftrightarrow X(f)$ |
| $x(-t) \longleftrightarrow X(-\omega)$ | $x(-t) \longleftrightarrow X(-f)$ |

**Time Scaling**

| | |
|---|---|
| $x(t) \longleftrightarrow X(\omega)$ | $x(t) \longleftrightarrow X(f)$ |
| $x(at) \longleftrightarrow \frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$ | $x(at) \longleftrightarrow \frac{1}{\lvert a \rvert} X\left(\frac{f}{a}\right)$ |

**Differentiation Property:**

$$x(t) \longleftrightarrow X(\omega)$$

$$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega) \longrightarrow \text{valid only when } \overline{x(t)} = 0$$

$$\frac{dx(t)}{dt} \longleftrightarrow (2j\pi f) X(f)$$

If $\bar{x}(t) \neq 0, \bar{x}(t) = K$ then $X(\omega) = \frac{G(\omega)}{j\omega} + \text{F.T of } [K]$

(1) $\delta(t) \xleftrightarrow{F.T} 1$

(2) $\frac{\delta(t-a) + \delta(t+a)}{2} = \cos(a\omega)$

(3) $\frac{\delta(t+a) - \delta(t-a)}{2j} = \sin(a\omega)$

(3) One sided exponential , $x(t) = e^{-at}u(t), a > 0$

$$X(\omega) = \frac{1}{(a + j\omega)}$$

$$x(t) = e^{at}u(-t) \longleftrightarrow \frac{1}{(a - j\omega)}$$

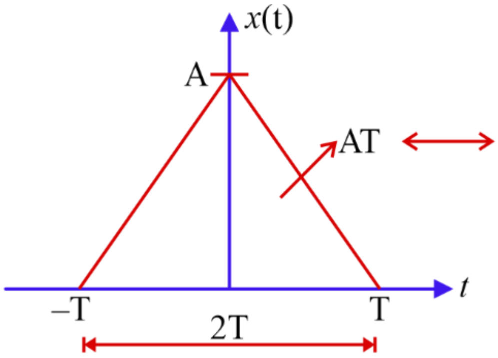
(4) Two sided exponential = $x(t) = e^{-a|t|} \longleftrightarrow X(\omega) = \frac{2a}{a^2 + \omega^2}$

(5) $x(t) = e^{-a|t|}\, sgn(t) \qquad a > 0, \longleftrightarrow X(\omega) = \frac{-2j\omega}{a^2 + \omega^2}$

(6) Multiplication - $tx(t) = +j\frac{dx(\omega)}{d\omega}$

$$t^n[e^{-at}u(t)] = \frac{n!}{(a + j\omega)^{n+1}}$$

(7) Even Triangular pulse:–



$$\longleftrightarrow X(\omega) = \underbrace{AT}_{\text{Area}}\ Sa^2\underbrace{\left[\frac{\omega T}{2}\right]}_{\frac{\text{Width}}{4}}$$

Fourier Transform of power signal (Type II)

or

Periodic + Non periodic

- ➢ Formula not applicable , properties applicable .
- ➢ Limitedly defined F.T so can . not be calculated by L.T.
- ➢ Obtained by limiting Type 1 signal.

(1) $1 \xleftrightarrow{F.T} 2\pi\delta(\omega)$

$1 \xleftrightarrow{F.T} \delta(f)$

(2) $\dfrac{dx(t)}{dt} \longleftrightarrow j\omega[X(\omega) - F.T(\overline{x}(t))]$

(3) $\cos\omega_0 t \longleftrightarrow \pi\delta(\omega-\omega_0)+\pi\delta(\omega-\omega_0)$

or

$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$

(4) $\sin\omega_0 t \longleftrightarrow \frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

or

$\frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$

**Duality** $x(t) \xleftrightarrow{F.T} X(\omega)$ $x(t) \xleftrightarrow{F.T} X(f)$

$X(t) \xleftrightarrow{F.T} 2\pi x(-\omega)$ $X(t) \xleftrightarrow{F.T} x(-f)$

**Steps :**

(1) Identify the $x(t)$ and try to obtain $X(\omega)$ from $x(t)$

(2) If step 1 fails then

$$x(t)\xrightarrow{t=\omega}G(\omega)$$

or

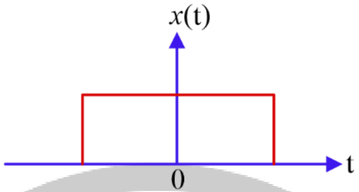
$$x(t)\Big|_{t=\omega}=G(\omega)$$

(3) $g(t)\xleftrightarrow{F.T}G(\omega)$

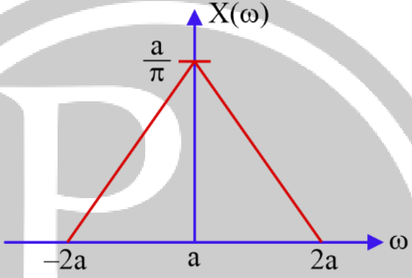
$$G(t)\xleftrightarrow{F.T}2\pi g(-\omega)$$

**Important Result:**

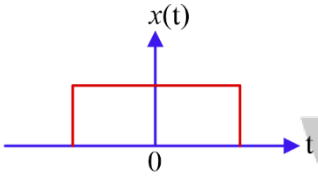
$$\left(\frac{\sin at}{\pi t}\right)\xleftrightarrow{F.T}$$



$$\left(\frac{\sin at}{\pi t}\right)^2\xrightarrow{F.T}$$

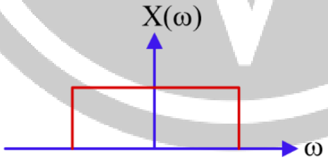


(1)

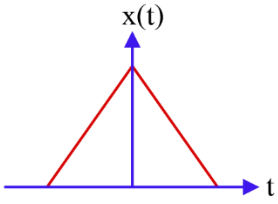


$\longleftrightarrow Sa(\omega)=\frac{\sin\omega}{\omega}$ or $\sin c(\omega)$ or $sa(\omega)$

(2) $\frac{\sin at}{\pi t}$ or $sa(t)$ or $\sin c(t)\longleftrightarrow$

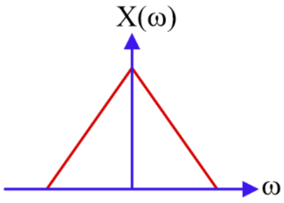


(3)



$\longleftrightarrow\left(\frac{\sin\omega}{\omega}\right)^2$ or $\sin c^2(\omega)$ or $sa^2(\omega)$

(4) $\sin\,c^2(t)$ or $sa^2(t)$ or $\left(\frac{\sin at}{\pi t}\right)^2\longleftrightarrow$



## Area Property:

(1) $$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(\omega)\big|_{\omega=0} = \int_{-\infty}^{\infty} x(t)dt$$

Area of $x(t) \Rightarrow \int_{-\infty}^{\infty} x(t) \xrightarrow{\quad} F.T \quad X(\omega)\big|_{\omega=0}$

(2) (i) Area of $X(\omega)=\int_{-\infty}^{\infty} X(\omega)d\omega = 2\pi x(0)$

(ii) $x(t)\big|_{t=0} = \int_{-\infty}^{\infty} X(f)df$

Convolution $x_1(t) \longleftrightarrow X_1(\omega)$

$x_2(t) \longleftrightarrow X_2(\omega)$

$x_1(t) * x_2(t) \xleftrightarrow{F.T} X_1(\omega)X_2(\omega)$

**Note:** A sin c $(\alpha t) * B\sin c(\beta t) = AB\left[\frac{1}{m}\sin c(kt)\right]$ $m = \max(\alpha, \beta)$

$K = \min(\alpha, \beta)$

Multiplication in time domain

$$x_1(t).x_2(t) \longleftrightarrow \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)X_2(\omega-\lambda)d\lambda$$

$$x_1(t).x_2(t) \longleftrightarrow X_1(f) * X_2(f) = \int_{-\infty}^{\infty} X_1(\lambda)X_2(f-\alpha)d\lambda$$

Integration Property –

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t) \longleftrightarrow X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Complex conjugate - $x^*(t) \longleftrightarrow X^*(\omega)$ or $X^*(-f)$

## Important table:

| ***x*(*t*)** | ***X*(ω)** |
|---|---|
| Even | Even |
| Odd | Odd |
| NENO | NENO |

| **x(t)** | **X(ω)** |
|---|---|
| Real | Conjugate symmetry |
| Imaginary | Conjugate anti symmetry |
| Conjugate Symmetry | Real |
| Conjugate anti symmetry | Imaginary |

| **x(t)** | **X(ω)** |
|---|---|
| RE | RE |
| RO | IO |
| IE | IE |
| IO | RO |

## Parseval's Energy Theorem –

(1) $$\int_{-\infty}^{\infty} x(t)h(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(-\omega)d\omega = \int_{-\infty}^{\infty} X(f)H(-f)df$$

(2) $$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X(-\omega)d\omega = \int_{-\infty}^{\infty} X(f)X(-f)df$$

(3) $$\int_{-\infty}^{\infty} x(t)h^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H^*(\omega)d\omega = \int_{-\infty}^{\infty} X(f)H^*(f)df$$

(4) $$\int_{-\infty}^{\infty} x(t)x^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega = \int_{-\infty}^{\infty} X(f)X^*(f)df$$

F.T of Gaussian Pulse

$$\int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi} \qquad e^{-at^2} \longleftrightarrow \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$

$$e^{-\pi t^2} \longleftrightarrow e^{-\pi f^2}$$

LTI System

$$x(t)\longrightarrow\boxed{h(t)}\longrightarrow y(t) = x(t)*h(t)$$

$$X(\omega) \qquad H(\omega) \qquad Y(\omega)$$

$$|Y(\omega)| = |H(\omega)|\,|X(\omega)|$$

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

➢ $$E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} |y(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2\, |X(\omega)|^2\, d\omega$$

$$= \int_{-\infty}^{\infty} |Y(f)|^2\, df = \int_{-\infty}^{\infty} |H(f)|^2 |X(f)|^2\, df$$

Eigen values and eigen function –

$\nearrow$ eigen value of LTI System

Eigen function of LTI S/S $\xrightarrow{x(t)} \underset{LTI}{\boxed{ht}} \longrightarrow y(t) = \quad Kx(t)$

$\searrow$ Real or complex or 1

$$x(t) = e^{S_0 t} \longrightarrow \boxed{H(S)} \longrightarrow y(t) = e^{S_0 t} H(S_0)$$

$$x(t) = e^{j\omega_0 t} \longrightarrow \boxed{H(\omega)} \longrightarrow y(t) = e^{j\omega_0 t} H(\omega_0)$$

$$A\cos\omega_0 t \longrightarrow \underset{even}{\boxed{h(t) \to H(\omega)}} \longrightarrow y(t) = A\cos\omega_0 t \underset{\text{eigen value}}{\boxed{H(\omega_0)}}$$

$$A\sin\omega_0 t \longrightarrow \boxed{h(t) \leftrightarrow H(\omega)} \longrightarrow y(t) = A\sin\omega_0 t \boxed{H(\omega_0)}$$

| $h(t)$ | $H(\omega)$ |
|---|---|
| R E | R E |
| R O | I O |
| I E | I E |
| I O | R O |

$$A\cos(\omega_0 t + \theta) \longrightarrow \boxed{h(t) \leftrightarrow H(\omega)} \longrightarrow A\cos(\omega_0 t + \theta) H(\omega_0)$$

$$A\sin(\omega_0 t + \theta) \longrightarrow \boxed{h(t) \leftrightarrow H(\omega)} \longrightarrow A\sin(\omega_0 t + \theta) H(\omega_0)$$

$$\begin{matrix} A\cos(\omega_0 t + \theta) \\ A\sin(\omega_0 t + \theta) \\ \downarrow \\ \text{not an eigen function} \end{matrix} \longrightarrow \overset{\text{Real}}{\boxed{h(t) \leftrightarrow H(\omega)}} \longrightarrow \begin{matrix} A|H(\omega_0)|\cos(\omega_0 t + \theta + \angle H(\omega_0)) \\ A|H(\omega_0)|\sin(\omega_0 t + \theta + \angle H(\omega_0)) \end{matrix}$$

**Case 1** $h(t)$ is even / $H(\omega)$ is even

- ➢ Both A sin $(\omega_o t + \theta)$, A cos $(\omega_o t + \theta)$ will be eigen function with same eigen value $H(\omega_o)$ not necessarily real .

**Case 2** $h(t)$ is real and even

- ➢ A $\sin(\omega_0 t + \theta)$ and A $\cos(\omega_0 t + \theta)$ are eigen function with same real eigen value $H(\omega_0)$

**Case 3** $h(t)$ is real .

- ➢ A $\cos(\omega_0 t + \theta)$ and A $\sin(\omega_0 t + \theta)$ is not an eigen function.

* * *

# 4

# LAPLACE TRANSFORM

## 4.1. Introduction

Bilateral T.F $X(S)=\int_{-\infty}^{\infty} x(t)e^{-st}dt = F.T[x(t)e^{-\sigma t}]$

Unilateral T.F $X(S)=\int_{0^-}^{\infty} x(t)e^{-st}dt$

Note: for $X(S)$ to be finite or for $X(S)$ to converge

$S-1$ $x(t)e^{-\sigma t}$ must be absolutely integrable .

$$\int_{-\infty}^{\infty} \left| x(t)e^{-\sigma t} \right| dt \rightarrow \text{finite}$$

S – 2 $X(S)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$

$$x(t)=e^{-s_0 t}u(t) \xleftrightarrow{B.L.T} \begin{cases} X(s)=\dfrac{1}{S+S_0} & \text{When } \operatorname{Re}\{S\} > -\sigma_0 \\ X(s)=\infty & \text{When } \operatorname{Re}\{S\} \le -\sigma_0 \end{cases}$$

➢ $e^{-St_0}u(t) \longleftrightarrow X(S)=\dfrac{1}{S+S_0}$ ROC: $\operatorname{Re}\{S\} = -\operatorname{Re}\{S_0\}$

Pole : $S=-S_0$ $\operatorname{Re}\{S\} > -\operatorname{Re}\{S_0\}$ RHP

$e^{St_0}u(t) \longleftrightarrow X(S)=\dfrac{1}{S-S_0}$

Pole : $S=S_0$ $\operatorname{Re}\{S\} = \operatorname{Re}\{S_0\}$

RHP $\operatorname{Re}\{S\} > \operatorname{Re}\{S_0\}$

$-e^{Sto}u(-t) \longleftrightarrow \dfrac{1}{S-S_0} \Rightarrow ROC:$ $\operatorname{Re}\{S\} < \operatorname{Re}\{S_0\}$

## Properties

(1) Linearity - $x_1(t) \longleftrightarrow X_1(S)$ $ROC: R_1$

$x_2(t) \longleftrightarrow X_2(S)$ $ROC: R_2$

Case 1 $g(t) = Ax_1(t) + Bx_2(t) \longleftrightarrow G(S) = AX_1(S) + BX_2(S) : ROC: R_1 \cap R_2$

$\rightarrow R.S.R$

$\rightarrow L.S.S$

$\rightarrow$ Double sided

**Case 2:** $g(t) = Ax_1(t) + Bx_2(t) \longleftrightarrow G(S) = AX_1(S) + BX_2(S)$

Finite duration + absolutely ROC – entire S plane

## Inferable too

(2) Time Shifting - $x(t) \overset{BLT}{\longleftrightarrow} X(S)$ $ROC: R_1$

$x(t-t_o) \longleftrightarrow e^{-st_o} X(S)$ $ROC: R_1$

$x(t-t_o) \longleftrightarrow e^{st_o} X(S)$ $ROC: R_1$

(3) Multiplication with complex exponential

$x(t) \longleftrightarrow X(S)$ $ROC: \text{Re}[S]$

$e^{S_o t} x(t) \longleftrightarrow X(S - S_0)$ $ROC: \text{Re}\{S - S_0\}$

$e^{-S_o t} x(t) \longleftrightarrow X(S + S_0)$ $ROC: \text{Re}\{S + S_0\}$

➢ B.L.T always have associated ROC with them .

## Properties of R.O.C

(1) R.O.C may or may not include zeros of x(s).

(2) R.O.C can not includes poles of x(s)

be cause $X(S = S_p) \longrightarrow \infty$ ROC is either (1) Right ward of pole

(2) Left ward of pole

(3) Bounded between poles

(3) If $x(t)$ is absolutely integrable then ROC of X(s) must include $j\omega$ axis.

(4) $x(t) \rightarrow$ finite duration + absolutely integrable . ROC of X(s) will be entire s plane

$(-\infty < \sigma < +\infty)$

(i) Impulse signal

(ii) finite duration + finite amplitude

↗ $X(S)$ does not exist even for single value of $\sigma$

(5) $x(t)$ *is R.S.S*

↘ *If* $X(S)$ exist then ROC will be right of right most pole

↗ $X(S)$ does not exist even for single value of $\sigma$

(6) $x(t)$ is L.S.S

↘ *If* $X(S)$ exist then ROC is left of the left most pole.

↗ $X(S)$ does not exist even for single value of $\sigma$

(7) $x(t)$ is B.S.S

↘ If $X(S)$ exist then ROC will be in strip form bounded between poles.

**Some Important Results:**

(1) $\delta(t) \longrightarrow 1$ ROC:entire S plane

(2) $u(t) \longrightarrow \frac{1}{S}$ $\text{Re}\{S\} > 0$

(3) $-u(-t) \longrightarrow \frac{1}{S}$ $\text{Re}\{S\} < 0$

(4) $e^{-at}u(t) \longrightarrow \frac{1}{S+a}$ $\text{Re}\{S\} > -a$

(5) $e^{at}u(t) \longrightarrow \frac{1}{S-a}$ $\text{Re}\{S\} > a$

(6) $-e^{-at}u(-t) \longrightarrow \frac{1}{S+a}$ $\text{Re}\{S\} < -a$

(7) $e^{-a|t|} \longrightarrow \frac{2a}{a^2 - S^2}$ $-a < \text{Re}\{S\} < a$

(8) $e^{-j\omega_0 t}u(t) \longrightarrow \frac{1}{S + j\omega_0}$ $:\text{Re}\{S\} > 0$

(9) $\cos\omega_o t\, u(t) \longrightarrow \frac{S}{S^2 + \omega_0^2}$ $\text{ROC}: \text{Re}\{S\} > 0$

(10) $\sin\omega_0 t\; u(t) \longrightarrow \frac{\omega_0}{S^2 + \omega_0^2}$ $\text{ROC}: \text{Re}\{S\} > 0$

$e^{-at}\cos\omega_0 t u(t) \xleftrightarrow{B.L.T} \frac{(S+a)}{(S+a)^2 + \omega_0^2}$ $\text{Re}\{S+a\} > 0$

$e^{-at}\sin\omega_0 t u(t) \xleftrightarrow{B.L.T} \frac{\omega_0}{(S+a)^2 + \omega_0^2}$ $\text{Re}\{S+a\} > 0$

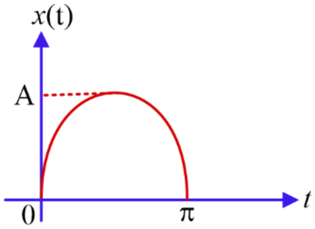
Time Reversal - $x(t) \longleftrightarrow X(S)$ $ROC: \text{Re}\{S\}$

$x(-t) \longleftrightarrow X(-S)$ $ROC: \text{Re}\{-S\}$

Multiplication by t $x(t) \longleftrightarrow X(S)$

$$t^n u(t) \longleftrightarrow \frac{n!}{S^{n+1}} \qquad ROC: \text{Re}\{S\} > 0$$

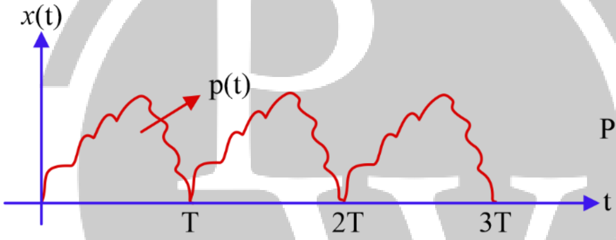
$$tx(t) \longrightarrow -\frac{d}{ds} X(S)$$



$$x(t) = A \sin t[u(t) - u(t-\pi)]$$

$$X(S) = \frac{A(1+e^{-\pi S})}{1+S^2} \qquad \text{ROC : entire S plane .}$$

### Laplace Transform of Causal Periodic Signal :



P↔P(S)

$$x(t) = p(t) + p(t-T) + p(t-2T) - - -$$

$$\boxed{X(S) = \frac{P(S)}{1-e^{-ST}}} \quad \text{only when } \sigma > 0$$

**Time Scaling** $x(t) \xleftrightarrow{BLT} X(S)$ $\qquad ROC: \text{Re}[S]$

$$x(at) \xleftrightarrow{BLT} \frac{1}{|a|} X\left(\frac{S}{a}\right) \qquad ROC: \text{Re}\left\{\frac{S}{a}\right\}$$

**Divide by T property** $x(t) \longleftrightarrow X(S)$ $\qquad \text{ROC}: \text{Re}(S)$

$$\frac{x(t)}{t} \longleftrightarrow \int_{-\infty}^{\infty} X(S)ds \qquad \text{ROC}: \text{Re}[S]$$

### Inverse Laplace Transform:

(1) $\frac{1}{(S+a)}$ $\nearrow$ $e^{-at}u(.t)$ When $\text{Re}\{S\} > -a$

$\searrow$ $-e^{-at}u(t)$ When $\text{Re}\{S\} < -a$

(2) $\frac{1}{(S+a)^2}$ $\nearrow$ $te^{-at}u(t)$ $\text{Re}\{S\} > -a$

$\searrow$ $-te^{-at}u(-t)$ $\text{Re}\{S\} < -a$

(3) $$\frac{1}{S}\begin{matrix}\nearrow & u(t) & \qquad \text{Re}\{S\}>0\\ \searrow & -u(-t) & \qquad \text{Re}\{S\}<0\end{matrix}$$

(4) $$\frac{\omega_0}{S^2+\omega_0^2}\begin{matrix}\nearrow & \sin\omega_0 t\,u(t) & \qquad \text{Re}\{S\}>0\\ \searrow & -\sin\omega_0 t\,u(-t) & \qquad \text{Re}\{S\}<0\end{matrix}$$

(5) $$\frac{S}{S^2+\omega_0^2}\begin{matrix}\nearrow & \cos\omega_0 t\,u(t) & \qquad \text{Re}\{S\}>0\\ \searrow & -\cos\omega_0 t\,u(-t) & \qquad \text{Re}\{S\}<0\end{matrix}$$

## Important Tables:

**(1) Table 1 : X(S) : Rational/ Irrational**

ROC is known and x(t) to be calculated

| ROC | x(t) |
|---|---|
| R. H. P $\longrightarrow$ | R. S. S |
| L. H. P $\longrightarrow$ | L. S.S |
| STRIP $\longrightarrow$ | B.S.S |

**(2) Table 2 : X(S): Rational/ Irrational**

Nature of x(t) is known and ROC to be decided.

| **x(t)** | **ROC** |
|---|---|
| R. S. S | R. H. P |
| L. S.S | L. H. P |
| B.S.S | STRIP |

**(3) Table 3 : X(S): Rational**

ROC is known and x(t) to be calculated

| **ROC** | **x(t)** |
|---|---|
| R. H. P $\longrightarrow$ | Causal |
| L. H. P $\longrightarrow$ | Anti causal |
| STRIP $\longrightarrow$ | Non causal<br>(causal + Anti causal) |

**(4) Table 4 X(S): Rational**

Nature of x(t)is known and ROC is to be decided

| **x(t)** | **ROC** |
|---|---|
| Causal | R. H. P |
| Anti causal | L. H. P |

| | |
|---|---|
| Non causal | STRIP |

**Note :** (1) If ROC is entire s plane then x(t) will be finite duration finite amplitude

(2) If X(S) is irritation then always calculate x(t) to check causal, anti-causal non causal nature.

$$\text{No. of R.O.C} = \text{No of I.L.T} = \frac{\begin{pmatrix}\text{no.of non repeated}\\ \text{complex conjugate}\\ \text{poles}\end{pmatrix}}{2} + (\text{no of non Repeated Real poles}) + 1$$

## LTI System

$$X(S):R_1 \longrightarrow \boxed{H(S):R_2} \longrightarrow Y(S)$$

$\nearrow D.N.E\ \ \text{ROC} \rightarrow R_1 \cap R_2 = \{\phi\}$

$\searrow$ Exist

$Y(S) = X(S)H(S) \qquad \text{ROC}: R_1 \cap R_2$

Differentiation in time domain .

$$x(t) \xrightarrow{B.L.T} X(S) \qquad ROC: R_1$$

$$\frac{dx(t)}{dt} \xrightarrow{B.L.T} SX(S) \qquad ROC: \text{at least } R_1$$

Integration in time domain .

$$\int_{-\infty}^{t} x(\tau)d\tau \longrightarrow$$

$$x(t) * u(t) \longrightarrow$$

$R_1 \qquad \text{Re}\{S\} > 0$

$\nearrow D.N.E$

$\searrow \dfrac{X(S)}{S}\ ROC: R_1 \cap [\text{Re}\{S\} > 0]$

**Stability of an LTI system** – for an LTI system to be stable

(1) $h(t)$ must be absolutely integrable

(2) For $h(t)$ must be absolutely integrable, $H(S)$ must include $j\omega$ axis.

**Causality of an LTI system-**

(1) $h(t)$ must be causal signal .

(2) For an LTI system having rational H(S) : ROC of H(S) must be right of right most pole.

Anti causal of an LTI system - $h(t) \longrightarrow$ anti causal

ROC of rational $H(S) \longrightarrow$ Left of left most pole

Non causality of an LIT system - $h(t) \longrightarrow$ Non casual

For rational $H(S)$:ROC must be in strip form.

Causal and stable - $H(S)$ rational $\rightarrow$All the poles of $H(S)$ must be in left hand side S plane

$H(S)$ Irrational $\rightarrow$ (ROC include $j\omega$ axis) $\bigcap h(t)$ is causal .

Anti causal and Stable $H(S)$ rational : All poles of $H(S)$ must be strictly on right half side of S – plane.

$H(S)$Irrational $\Rightarrow$(ROC include jω axis)$\bigcap$($h(t)$is anti causal)

Non casual and stable - $H(S)$ rational : Poles of $H(S)$ must be located on either side of $j\omega$ axis

$H(S)$ Irrational : (ROC includes $j\omega$ axis) $\bigcap$ ( $h(t)$ is non causal)

## Important Table

**(1) H(S) : Rational**

| ROC | LTI System |
|---|---|
| R.H.P | Causal |
| L.H.P | Anti causal |
| STRIP | Non causal |

(2)

| LTI System | ROC |
|---|---|
| Causal | RHP |
| Anti causal | LHP |
| Non causal | STRIP |

Unilateral L.T $X(S) = \int_{0^-}^{\infty} x(t)e^{-St}dt$ No ROC exist

$$\boxed{ULT\{x(t)\} = BLT\{x(t)u(t)\}}$$

## Properties of ULT

(1) Differentiation property

$$\frac{dx(t)}{dt} \xleftrightarrow{ULT} SX(S) - x(0^-)$$

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{ULT} S^2X(S) - Sx(0^-) - \frac{dx(0^-)}{dt}$$

(2) Integration Property –

$$\int_{-\infty}^{t} x(\tau)d\tau \longleftrightarrow \frac{X(S)}{S} + \frac{\int_{0^-}^{\infty} x(\tau)d\tau}{S}$$

(3) Time Shift –

$$\underset{\substack{\downarrow\\ Causal}}{x(t-t_0)} \xleftrightarrow{ULT} e^{-st_0}X(s)$$

(4) Convolution: $x(t) = u(t) * u(t+1) = r(t+1)$

$$X(S) = \frac{1}{S} \times \frac{1}{S} = \frac{1}{S^2}$$

## Linear constant coefficient differential equation –

A D.E will represent a liner system if and only if

(i) No higher power of x(t) and its derivative and y(t) and its derivative are allowed.

(ii) No product term of x(t) and y(t) and their derivatives are allowed.

(iii) No addition of constant term

## Transfer function by ULT

$$X(S) \longrightarrow \boxed{H(S)} \longrightarrow Y(S)$$

$$H(S) = \frac{Y(S)}{X(S)}$$

If initial conditions are zero:

(1) T.F can be calculated

(2) $y(t)$ can be calculated from T.F

(3) If initial condition not zero – T.F can be calculated but $y(t)$ can not be calculated from T.F.

## Types of Responses :

**Case 1.**

$$\begin{tikzcd} & & \text{Transient Response} \\ x(t)=0 \arrow[r] & \boxed{h(t)\rightarrow H(s)} \arrow[r] \arrow[d] & y(t)=y_{ZIR}(s) \arrow[u] \\ & \text{I.C} \neq 0 & \text{Zero input Response} \end{tikzcd}$$

$$\begin{tikzcd} & & \text{Steady state Response} \\ x(t)\neq 0 \arrow[r] & \boxed{h(t)\leftrightarrow H(S)} \arrow[r] \arrow[d] & y(t)=Y_{ZSR}(S) \arrow[u] \\ & \text{I.C} = 0 & \text{Zero State Response} \end{tikzcd}$$

$$x(t)\longrightarrow\boxed{h(t)\leftrightarrow H(S)}\longrightarrow y_1(t):$$ Poles of input forced Response,

$$x(t)\longrightarrow\boxed{h(t)\leftrightarrow H(S)}\longrightarrow y(t):$$ Poles of system Natural Response,

**Initial value Theorem on ULT –**

(1) Applicable only when x(t) is causal.

(2) Helps in calculation of initial value $x(0^+)$ not initial condition $x(0^-)$

$$X(s)=\frac{N(s)}{D(s)}$$

**Note:** while applying I.V.T common factors in N(S) and D(S) must be cancelled out .

$$\boxed{\lim_{t\to o^+} x(t)=\lim_{S\to\infty} SX(S)}\quad \begin{array}{l} x(t) \quad \text{is casual} \\ x(s)\to D^r > N^r \end{array}$$

## 4.2. Final value Theorem

(1) Applicable only when x(t) is casual .

(2) While applying F.V.T common factor must cancelled out.

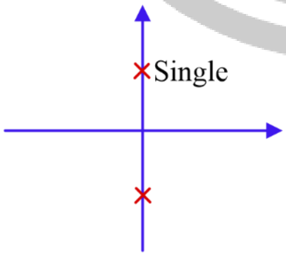
$$\boxed{\lim_{t\to\infty} x(t)=\lim_{S\to 0} SX(S)}$$

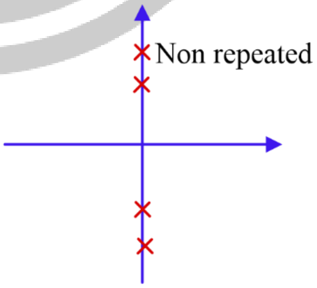
**Case : 1.** If all poles of $SX(S)$ lies strictly in LHP .

(i) Final value is finite

(ii) FVT applicable

**Case : 2.** If poles location of is $SX(S)$ as shown below .





(i) Final value is indeterminate.

(ii) FVT is not applicable.

**Case : 3.** In all other cases

(i) Final value is $\infty$

(ii) F.V.T is not applicable

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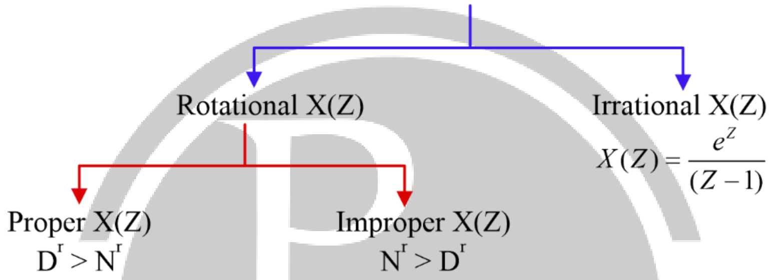
# 5

# Z TRANSFORM

## 5.1. Introduction

- Z domain signal
- $X(Z)=\dfrac{N(Z)}{X(Z)}$



| Laplace Tx | Z.T |
|---|---|
| $S=\sigma+j\omega$ | $Z=re^{j\omega}$ |
| $S=a+jb$:Point | $Z=r_oe^{j\omega_o}$ : Point |
| $\text{Re}[s]=a$ : Line parallel to $j\omega$ axis | $\lvert Z\rvert=r_o$ :Circle concentric to unity circle $\lvert Z\rvert=1$ |
| $\text{Re}\{S\}>a$ :Region parallel to $j\omega$ axis | $\lvert Z\rvert>r_o$ Region concentric to unity circle. |

**Relation between Z.T and L.T**

$$Y(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n} \quad \boxed{Z=e^{st_s}}$$

$$\lvert Z\rvert=e^{\sigma T_s}$$

$$\angle Z=\omega T_s$$

**Mapping**

| | | | |
|---|---|---|---|
| $\sigma>0$ | Left Half of s plane | $0\le\lvert Z\rvert<1$ | Family of circles having radius less then 1. |
| $\sigma>0$ | Right half of s plane | $1<\lvert Z\rvert\le\infty$ | Family of circles having radius greater then 1. |
| $\sigma=0$ | $j\omega$ axis | $\lvert Z\rvert=1$ | Unity circle |

(1) Vertical line in s plane → A circle in A.C.W in Z-plane

(2) Left half side of s plane → Inside unity circle in Z-plane

(3) Left side nature → ln ward nature in Z – plane

(4) Right hand side of s plane – outside of unity circle in z – plane

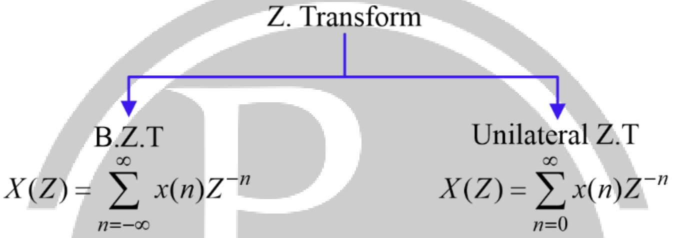
(5) Right side nature → outside nature in z- plane.

(6) $j\omega$ axis mapped onto unity circle.

(7) origin in s plane is mapped $z = e^{ST} = 1$

## Important Analogy

| C.T signal | D.T Signal | | |
|---|---|---|---|
| u(t) | u(n) | | |
| u(-t) | u(-n-1) | $e^{-at}u(t)$ | $a^n u(n)$ |
| $-e^{-at}u(t)$ | $-a^n u(-n-1)$ | | |



$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} x(n)Z^{-n}$$

## B.Z.T

$(z_0)^n u(n) \longrightarrow \frac{Z}{Z - Z_0}$ $\quad ROC: |Z| > |Z_0|$

$-(z_0)^n u(-n-1) \longleftrightarrow \frac{Z}{Z - Z_0}$ $\quad ROC: |Z| < |Z_0|$

(1) $a^n u(n) \longleftrightarrow \frac{Z}{Z - a}$ $\quad ROC: |Z| > |a|$

(2) $a^{-n} u(n) \longleftrightarrow \frac{Z}{Z - \left(\frac{1}{a}\right)}$ $\quad ROC: |Z| > \frac{1}{|a|}$

(3) $(-a)^n u(n) \longleftrightarrow \frac{Z}{Z - (-a)}$ $\quad ROC: |Z| > |-a|$

(4) $(-a)^{-n} u(n) \longleftrightarrow \frac{Z}{Z - \left(\frac{-1}{a}\right)}$ $\quad ROC: |Z| > \frac{1}{|-a|}$

(5) $-a^n u(-n-1) \longleftrightarrow \frac{Z}{(Z - a)}$ $\quad ROC: |Z| < |a|$

(6) $-(a)^{-n}u(-n-1)\longleftrightarrow\dfrac{Z}{Z-\left(\dfrac{1}{a}\right)}$ $\quad ROC:|Z|<\dfrac{1}{|a|}$

(7) $-(-a)^{n}u(-n-1)\longleftrightarrow\dfrac{Z}{Z-(-a)}$ $\quad ROC:|Z|<|-a|$

(8) $-(-a)^{n}u(-n-1)\longleftrightarrow\dfrac{Z}{Z-\left(\dfrac{-1}{a}\right)}$ $\quad ROC:|Z|<\left|\dfrac{1}{-a}\right|$

(9) $u(n)\longleftrightarrow\dfrac{Z}{(Z-1)}$ $\quad ROC:|Z|>1$

(10) $-u(-n-1)\longleftrightarrow\dfrac{Z}{(Z-1)}$ $\quad ROC:|Z|<1$

## Properties

**(1) Linearity:** $x_1(n)\longleftrightarrow X_1(z)\quad ROC:R_1$

$x_2(n)\longleftrightarrow X_2(z)\quad ROC:R_2$

**Case :1** $g(n)=Ax_1(n)+Bx_2(n)\quad (R_1\cap R_2)=\{\theta\}\,Z.T\qquad D.N.E$

L.S.S $\qquad \neq\{\theta\}$

R.S.S $\qquad X(Z)\,\text{exist}\Rightarrow AX_1(z)+BX_2(z)$

B.S.S

**Case:2** $\underbrace{g(n)=Ax_1(n)+Bx_2(n)}_{F.D+Abs\sum}\longrightarrow G(z)=AX_1(z)+BX_2(z)$

ROC:entire z plane except

**(2) Time Shifting:** $x(n)\longleftrightarrow X(Z)\qquad ROC:R_1$

$x(n+1)\longleftrightarrow ZX(Z)\qquad ROC:R_1$, except possibly

$|Z|=0$ or $|Z|=\infty$

inclusion/declusion.

**(3) Multiplication by complex exponential:**

$x(n)\longleftrightarrow X(Z)\qquad ROC:|Z|$

$Z_0^n x(n)\longleftrightarrow X\left(\dfrac{Z}{Z_0}\right)\qquad ROC:\left|\dfrac{Z}{Z_0}\right|$

$u(n)=\dfrac{Z}{Z-1},\qquad |Z|>1$

$$\left(e^{j\omega_o}\right)^n u(n) \longleftrightarrow \frac{Z}{Z-e^{j\omega_0}} \qquad |Z|>1$$

$$\cos\omega_o n\, u(n) \longleftrightarrow \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad ROC:|Z|>1$$

$$\sin\omega_0 n\, u(n) \longleftrightarrow \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad |Z|>1$$

$$a^n \cos\omega_0 n\, u(n) \longleftrightarrow \frac{Z^2 - az\cos\omega_0}{Z^2 - 2az\cos\omega_0 + a^2} |Z|>|a|$$

$$a^n \sin\omega_0 n\, u(n) \longleftrightarrow \frac{az\sin\omega_o}{Z^2 - 2az\, cas\,\omega_0 + a^2} |Z|>|a|$$

## Properties of ROC

(1) ROC may or may not include zeros of x(z).

(2) Will not include poles of x(z).

(3) If x(n) absolutely summable $\rightarrow$ROC of x(z) includes unity circle.

(4) $x(n)$ $\longrightarrow$ ROC of $X(Z)$, will be entire Z plane
F.D + Abs $\Sigma$ except possibly $|Z| = 0$ AND / OR $|Z| = \infty$

(5) $x(n)$ is $L.S.S$
$\nearrow$ X(z) may not exist, even for signal Value of $|Z|$
$\searrow$ If X(z):exist, ROC will inside the innermost circle having radius formed by magnitude of innermost non-zero pole

(6) $x(n)$ is L.S.S
$\nearrow$ $X(Z)$ may not exist, even for signal value of $|Z|$
$\searrow$ If $X(Z)$: exist, ROC will inside the innermost circle having radius formed by magnitude of innermost non - zero pole

(7) $x(n)$ is B.S.S
$\nearrow$ $X(Z)$ may not exist, even for signal value of $|Z|/r$
$\searrow$ If $X(Z)$: exist, ROC will be in form of ring bounded by magnitude of finite/non zero poles

## Time Scaling

$$x(n) \longleftrightarrow X(Z) \qquad ROC:|Z|$$

$$x\left(\frac{n}{K}\right) \longleftrightarrow X(Z^K) \qquad ROC:|Z^K|$$

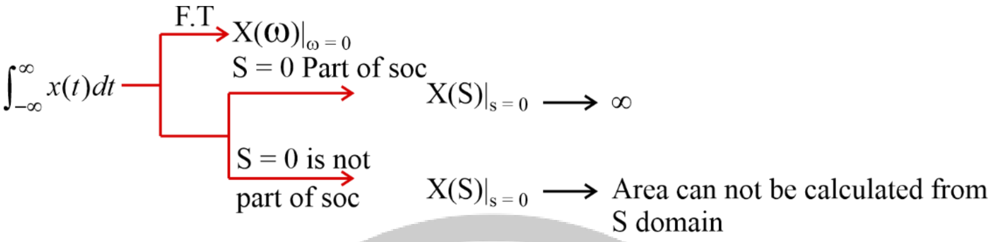
## Area or Summation property-

$$X(S)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\nearrow \int_{-\infty}^{\infty} x(t)dt \text{ when } S=0 \text{is part of ROC}$$

$$X(S=0)$$

$$\searrow \infty, \text{ when S = 0 is not part of ROC}$$



- $a^n u(n) \to \dfrac{Z}{Z-a} \qquad |Z|>|a|$

$$na^{n-1}u(n) \to \frac{Z}{(Z-a)^2} \qquad |Z|>|a|$$

## Multiplication by *n*

$$nx(n) \longleftrightarrow -z\frac{dx(z)}{dz} : \text{ROC-Remains Same}$$

- $na^n u(n) = \dfrac{az}{(z-a)^2} \qquad |z|>|a|$

Or

$$na^n u(n-1)$$

- $(n+1)a^{n+1}u(n+1) \longleftrightarrow \dfrac{az^2}{(z-a)^2} \quad |z|>|a|$

Or

$$(n+1)a^{n+1}u(n)$$

- $a^n u(n) \longleftrightarrow \dfrac{z}{(z-a)} \qquad |z|>|a|$

$$\frac{na^{n-1}u(n)}{1!} \longleftrightarrow \frac{z}{(z-a)^2} : |z|>|a|$$

$$\frac{n(n-1)(n-2)a^{n-3}u(n)}{3!} \longleftrightarrow \frac{z}{(z-a)^4} : |z|>|a|$$

## Analogy between L.T and Z. T

$S \leftrightarrow (1-z^{-1})$ analogy

$z = e^{ST}$ equivalent

## Inverse Z.T

**Table 1 X(Z) : Rational , ROC Known and x(n) to be Calculated**

| ROC | $x(n)$ |
|---|---|
| Outside outmost finite pole | R.S.S |
| Inside Innermost nonzero pole | L.S.S |
| Ring from, bounded by non zero and finite poles | B.S.S |

**Table 2 X(Z) : Rational x(n) is given and ROC is to be decided .**

| $x(n)$ | R.O.C |
|---|---|
| R.S.S | Outside outermost finite pole |
| L.S.S | Inside Innermost nonzero pole |
| B.S.S | Ring from bounded by finite non zero pole |

**Table 3 : X(Z) : Rational nature of ROC known and x(n) to be calculated .**

| ROC | $x(n)$ |
|---|---|
| Outside outermost finite pole, including $\|Z\| = \infty$ | Causal |
| Inside Innermost non zero pole, including $\|Z\| = 0$ | Anti casual |
| Ring form bounded by non zero and finite pole | Non causality |

**Table 4 : X(Z) : Rational**

| $x(n)$ | R.O.C |
|---|---|
| Casual | Outside outermost finite pole including $\|Z\| = \infty$ |
| Anti causal | Inside innermost non – zero pole including $\|Z\| = 0$ |
| Non causal | Ring from bounded by finite and non zero pole . |

## Methods to calculate I.Z.T

$X(Z) = (D)/D(Z)$

(1) By Long division

(i) $D(Z) \geq N(Z)$

$\nearrow$ casual : $N(Z),\ D(Z) \rightarrow$ decreasing power of $Z$.

(ii) $x(n)$

$\searrow$ Anticausal: $N(Z),\ D(Z) \rightarrow$ Increasing power of $Z$.

(2) Partial fraction

(i) $X(Z)$:pole – zero cancellation .

(ii) Plot Pole diagram and obtain all possible ROC.

(iii) Perform partial fraction of $\left\{\frac{X(Z)}{Z}\right\}$ if needed and calculate I.Z.T for each ROC.

## Convolution Property:

$$x(n) \leftrightarrow X(Z)\ R_1$$

$$h(n) \leftrightarrow H(Z)\ R_2$$

$$\text{y(n)=x(n)*h(n)} \longrightarrow R_1 \cap R_2 = \{\phi\} Y(Z) D.N.E$$

$$R_1 \cap R_2 \neq \{\phi\}\ y(z) = X(z)H(Z)$$

ROC : $R_1 \cap R_2$

## Accumulation

$$x(n) \longleftrightarrow X(Z) : ROC - R$$

**Case 1.** $x(n) * u(n)$

$$\sum_{K=-\infty}^{n} x[K] \longleftrightarrow \frac{x(z)}{(1-Z^{-1})} \quad ROC : R \cap (|z|>1)$$

**Case 2.** $x(n) = 0,\ n<0$ or $n \leq -1$

$$x(n) * u(n)$$

$$\sum_{K=-\infty}^{n} x[K] = \sum_{K=0}^{n} x[K] \longleftrightarrow \frac{X(z)}{(1-Z^{-1})}$$

$$n \leq -1$$

## Generalized eigen function for D.T LTI s/s-

D.T LTI system : exponential $\left(Z_0^n\right)$

$$x(n) = \underset{\text{eigen function}}{\underset{\downarrow}{Z_0^n}} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = z_0^n \ \underset{\text{scalar eigen value}}{\underset{\downarrow}{H(z_o)}} \begin{matrix} \nearrow \text{Real} \\ \searrow \text{Complex} \end{matrix}$$

$$\boxed{y(n) = z_0^n \sum_{K=-\infty}^{\infty} \underset{H(z_o)}{\underset{\downarrow}{h[K] Z_0^{-K}}}}$$

## Important Table:

| $x(n)$ | ROC |
|---|---|
| R.S.S + causal | Outside outermost finite pole including $\|Z\| = \infty$ |
| Finite duration + causal | Entire Z plane including $\|Z\| = \infty$ and possibly including $\|Z\| = 0$ |
| L.S.S + Anti causal | Inside Innermost +Non zero pole including $\|Z\| = 0$ |
| Finite duration + Anti causal | Entire Z plane including $\|Z\| = 0$ |
| R.S.S + Non causal | Outside outmost finite pole , including $\|Z\| = \infty$ |
| L.S.S + Non Causal | Inside innermost non zero pole not including $\|Z\| = 0$ |
| B.S.S + Non causal | Ring from bounded by finite & Non zero pole. |
| Finite duration + Non causal | Entire Z plane not including $\|Z\| = 0 \& \|Z\| = \infty$ |

### Stability of an LTI S/S.

$h(n) \to$ must be absolutely summable

ROC $\to$ will include unity circle.

### Causality:

$h(n) \to$ Must be causal signal

ROC $\to$ Either outside of outmost pole including $|Z| = \infty$ or entire Z plane including $|Z| = \infty$

### Anti Causality :

$h(n) \to$ Anti causal

ROC $\to$ Either inside the innermost pole or entire z plane including $|Z| = 0$

### Non Causality:

$h(n) \to$ non causal

$H(Z) \to$ Has finite and non zero poles $\nearrow RSS + NC$, $\to LSS + NC$, $\searrow BSS + NC$

$H(Z) \to$ Does not have any finite – non zero pole. ROC entire Z plane not including $|Z| = 0$ & $|Z| = \infty$

Causal + Stable – All poles must be strictly inside unity circle H(Z) has finite and non zero pole, if not then decide based on common portion of ROC [causal ⋂ stable]

### Anti causal + Stable

H(Z) finite and non zero pole $\longrightarrow$ All the poles must be strictly outside unity circle.

H(Z) does not have finite and non zero pole $\longrightarrow$ (ROC of Stable) ⋂ (ROC of anti causal)

## Unilateral Z. T

$$x(n)\longleftrightarrow X(Z)$$

$$X[Z]=\sum_{n=0}^{\infty}x(n)Z^{-n}$$

$$UZT\{x(n)\}=BZT\{x(n)u(n)\}$$

(1) $1\longrightarrow\dfrac{Z}{Z-1}$

(2) $2^{n}\longrightarrow\dfrac{Z}{Z-2}$

(3) $\cos\omega_0 n\xrightarrow{UZT}\dfrac{Z^2-Z\cos\omega_0 n}{Z^2+2Z\cos\omega_0 n+1}$

## Properties of UZT

**(1) Time Shifting**

$$x(n-1)\longleftrightarrow Z^{-1}X(Z)+x(-1)$$

$$x(n-2)\longleftrightarrow Z^{-2}X(Z)+Z^{-1}x(-1)+x(-2)$$

Types of Response

$$x(n)\longrightarrow\boxed{h(n)}\longrightarrow y(n)\quad \text{ZIR}$$

$$I.C\neq 0$$

$$\underset{\neq 0}{x(n)}\longrightarrow\boxed{h(n)}\longrightarrow y(n)\quad \text{ZSR}$$

I.C = 0

If y(n) is only due to input ⇒ Forced Response y(n) is only due to system pole ⇒ Natural response

## Transfer function

If I.C = 0

$$H(z)=\frac{Y(Z)}{X(Z)}$$

**Note :**

(1) I.C = 0

- (a) H(z) can be calculated.
- (b) Y(n) can be calculated from T.F

(2) $I.C\neq 0$

- (a) H(z) can be calculated
- (b) Y(n) can not be calculated from T.F

| Initial Value Theorem | Final Value Theorem |
|---|---|
| $\lim\limits_{n\to 0} x(n) = \lim\limits_{Z\to\infty} X(z)$<br>Valid only when<br>(1) $x(n)$ is casual $D^r \ge N^r$<br>(2) $X(z) = N(Z)/D(Z)$ | $\lim\limits_{n\to\infty} x(n) = \lim\limits_{Z\to 1}(1-Z^{-1})X(Z)$<br>$\boxed{\lim\limits_{n\to\infty} x(n) = \lim\limits_{Z\to 1}(Z-1)X(Z)}$<br>Valid it (a) x(n) is causal<br>(b) all the poles of<br>$(1-z^{-1})X(z)$ or $(z-1)X(z)$<br>Should strictly be inside unity circle |

**Note:** Before using this theorem , common factors must be cancelled out in $X(Z)$ .

### Multiplication by *n*

$$nx(n) \longleftrightarrow -z\frac{dX(z)}{dz}$$

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# 6

# DTFT

## 6.1. Introduction

**Important Table:**

| Time domain | Frequency domain |
|---|---|
| Continuous | Non Periodic |
| Discrete | Periodic |
| Periodic | Discrete |
| Non Periodic | Continuous |

| Transform | Time domain | Frequency domain |
|---|---|---|
| C.T.F.S | C + P | Discrete + Np |
| C.T.F.T | C + Np | C + Np |
| DTFS | D + p | D + p |
| DTFT | D + Np | C + p |

$X(e^{j\omega}) = \sum_{n-\infty}^{\infty} x(n) e^{-j\omega n}$ well defined DTFT, calculates from B.Z.T at unity circle

- For well defined DTFT to converge $x(n)$ must be absolutely summable.

**For well defined DTFT**

(a) Includes all energy signal .

(b) Formula of DTFT applicable

(c) Properties of DTFT applicable .

(d) $X(e^{j\omega})$ will be defined for each and every value of $\omega$.

**Limitedly defined DTFT**

(a) Includes all power signal

(b) Formula not applicable .

(c) properties applicable.

(d) $X(e^{j\omega})$ will be $\longrightarrow \infty$ for any one value of ω.

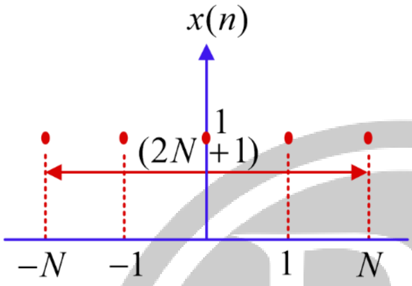
**Note:** $X(e^{j\omega})$ is periodic with $-\pi \le \omega \le \pi$,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$a^n u(n) \longleftrightarrow \frac{1}{1-ae^{-j\omega}} \quad \text{When } |a|<1$$

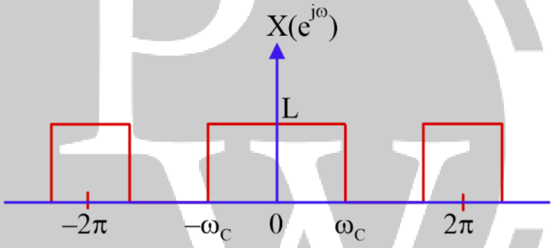
$$\boxed{X(e^{j\omega}) \longleftrightarrow \frac{1}{1-ae^{-j\omega}}} \quad \text{periodic with } 2\pi$$

## DTFT of signals



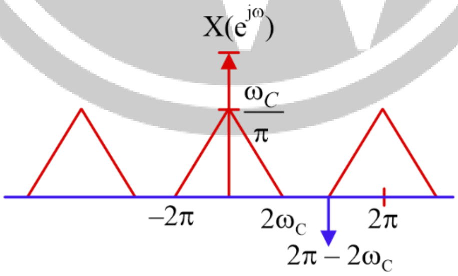
$$\longleftrightarrow X(e^{j\omega}) = \frac{\sin\left\{\frac{\omega}{2}(2N+1)\right\}}{\sin\left(\frac{\omega}{2}\right)}$$

- ➢ $x(n) = \dfrac{\sin \omega_c n}{\pi n} \xleftrightarrow{DTFT}$



Only valid when $\omega_c < \pi$

- ➢ $x(n) = \left(\dfrac{\sin \omega_c n}{\pi n}\right)^2$



$\omega_c < \dfrac{\pi}{2}$

## Properties of DTFT :

(1) Linearity - $Ax_1(n) + Bx_2(n) \longleftrightarrow Ax_1(e^{j\omega}) + BX_2(e^{j\omega})$

(2) Time shifting $x(n-n_0) \longleftrightarrow e^{-j\omega n_o} X(e^{j\omega})$

$x(n+n_0) \longleftrightarrow e^{j\omega n_o} X(e^{j\omega})$

(3) Frequency shifting

$$e^{j\omega_o n}x(n)\longleftrightarrow X(e^{j(\omega-\omega_o)})$$

$$e^{-j\omega_o n}x(n)\longleftrightarrow X(e^{j(\omega+\omega_o)})$$

$$\cos\omega_o n\longleftrightarrow \pi[\delta(\omega-\omega_o)+\pi\delta(\omega+\omega_o)]-\pi\le\omega\le\pi$$

$$\sin\omega_o n\longleftrightarrow \frac{\pi}{j}\left[\delta(\omega-\omega_o)-\delta(\omega+\omega_o)\right]-\pi\le\omega\le\pi$$

$$(-1)^n x(n)=e^{j\pi n}x(n)\longleftrightarrow x(e^{j(\omega-\pi)})\longleftrightarrow X(-e^{j\omega})$$

(4) Time Reversal - $x(-n)\longleftrightarrow x(e^{-j\omega})=X\left((e^{j\omega})*\right)$

(5) Complex conjugate - $x*(n)\longleftrightarrow X^*\left((e^{j\omega})*\right)=X^*(e^{-j\omega})$

| ***x*(*n*)** | **X(*e*^jω^)** |
|---|---|
| E<br>O<br>NENO | E<br>O<br>NENO |

| ***x*(*n*)** | **X(*e*^jω^)** |
|---|---|
| R+E | R+E |
| R+O | I+O |
| I+E | I+E |
| I+O | R+O |

| ***x*(*n*)** | **X(*e*^jω^)** |
|---|---|
| Real | C.S |
| I | C.A.S |
| C.S | Real |
| C.A.S | I |

(1) Time Expansion - $x\left[\frac{n}{K}\right]\longleftrightarrow X(e^{j\omega K})$

1st difference or successive difference –

$$x(n)-x(n-1)\longleftrightarrow(1-e^{-j\omega})X(e^{j\omega})$$

$$u(n)\xleftrightarrow{DTFT}\pi\delta(\omega)+\frac{1}{(1-e^{-j\omega})}-\pi\le\omega\le\pi$$

or

$$\sum_{K=-\infty}^{\infty}\pi\delta(\omega-2\pi K)+\frac{1}{(1-e^{-j\omega})}$$

Multiplication with n - $nx(n)\longleftrightarrow +j\dfrac{d}{d\omega}X(e^{j\omega})$

Convolution - $y(n)=x(n)\times h(n)\longleftrightarrow y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})$

### 6.1.1. Parseval Energy Theorem

(1) $$\sum_{n=-\infty}^{\infty}x(n)h(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})H(e^{-j\omega})d\omega$$

(2) $$\sum_{n=-\infty}^{\infty}x(n)h^{*}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})H^{*}(e^{j\omega})d\omega$$

(3) $$\sum_{n=-\infty}^{\infty}x(n)x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})X(e^{-j\omega})d\omega$$

(4) $$\sum_{n=-\infty}^{\infty}x(n)x^{*}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^{2}d\omega$$

# 7

# SAMPLING

## 7.1. Introduction

### Instantaneous sampling in time domain:

$$m_s(t) = m(t)c(t)$$

$$m_s(t) = m(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_s) \qquad T_S = \frac{1}{f_s}$$

$T_s$ : sampling interval

$f_s$ : Sampling frequency

### Instantaneous sampling in frequency domain

$$m(t) \longleftrightarrow M(\omega)$$

$$m(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_S) \longleftrightarrow f_s\sum_{n=-\infty}^{\infty}M(\omega-n\omega_s)$$

$$m(t) \longleftrightarrow M(f)$$

$$m(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_S) \longleftrightarrow f_S\sum_{n=-\infty}^{\infty}M(f-nf_S)$$

### Spectral analysis of Instantaneous Frequency

$$\sum\delta(t-nT_S) \longleftrightarrow \frac{2\pi}{T_S}\sum\delta(\omega-n\omega_S)$$

$$\sum_{n=-\infty}^{\infty}\delta(t-nT_S) \longleftrightarrow f_S\sum_{n=-\infty}^{\infty}\delta(f-nf_S)$$

If $f_S > 2f_m$ :– oversampling

Tx : No aliasing PBG = $T_S$

Rx: practical LPF , Ideal LPF with $f_m \le f_c \le f_s - f_m$

Recovery - $(f_S > 2f_m)\bigcap(f_m \le f_C \le f_S - f_m)$

If $f_S = 2f_m$: critical sampling

Tx : Aliasing on verge (No aliasing)

Rx : Ideal LPF with $(f_s = f_m)$ & PBG $= T_S$

Recovery - $(f_S = 2f_m) \cap (f_C = f_m)$

Case 3: $f_S < 2f_m$ under sampling

Tx : Aliasing

Rx : Recovery not possible.

### Low Pass Sampling Theorem–

A lowpass signal bandlimited to $f_m$ $Hz$ can be sampled and reconstructed form its samples if and only

If $\boxed{f_S \geq 2f_m} \cap \boxed{f_m \leq f_c \leq (f_S - f_m)}$

Sampling rate. $\boxed{f_s \geq 2f_m}$

Nyquist rate = minimum sampling rate

$$(f_S)_{\min} = 2f_m$$

Nyquist interval $T_s = \dfrac{1}{(f_s)_{\min}} = \dfrac{1}{2f_m}$

| $m(t)$ | $f_{NY}$ |
|---|---|
| $\sin c\ (t)$ | $1Hz$ |
| $\sin c(at)$ | $a\,Hz$ |
| $\sin c^K(at)$ | $Ka\ Hz$ |
| $\sin c(at) + \sin c(bt)$ | $Max(aHz, bHz)$ |
| $\sin c(at) \times \sin c(bt)$ | $(a+b)Hz$ |
| $\sin c(at) * \sin c(bt)$ | $\min(aHz, bHz)$ |
| $\frac{d}{dt}\sin c(t)$ | 1Hz |
| $\int_{-\infty}^{t} \sin c(\tau)d\tau$ | 1Hz |

Sampling using general carrier pulse train–

$$m(t) \longleftrightarrow M(f)$$

$$c(t) \longleftrightarrow C(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s)$$

$$M_s(f) = \sum_{n-\infty}^{\infty} C_n M(f - nf_s)$$

If $(f_s > 2f_m) \cap (f_m \le f_c \le f_s - f_m)$

| L.P.F(P.B.G) | y(t) |
|---|---|
| 1 | $c_0 m(t)$ |
| $1/C_o$ | $m(t)$ |
| L | $L\,C_0 m(t)$ |

When c(t) is rectangular pulse train –

$$C_n = \frac{2A}{a} \sin c\left[n\left(\frac{2}{a}\right)\right]$$

$$M_S(f) = \sum_{n=-\infty}^{\infty} \left(\frac{2A}{a}\right) \sin c\left(\frac{2n}{a}\right) \delta(f - nf_s)$$

## Sampling of Sinusoidal Signal:

**Note:** $f_s < 2f_m$ Recovery is possible through BPF

$f_s < 2f_m$ Recovery not possible through BPF

## Calculation of Frequency:

(i) $m(t) = A_m \cos 2\pi f_m t$

$C(t)$: Impulse train with period $T_S \rightarrow 0, f_s, 2f_s, 3f_s, ---$

$$m_s(t) = m(t)c(t) \longrightarrow \quad 0 \pm fm \begin{matrix} \nearrow 0 + f_m \searrow \\ \searrow |0 - f_m| \nearrow \end{matrix} \text{same}$$

$$f_s \pm f_m \begin{matrix} \nearrow f_s + f_m \\ \searrow |f_s - f_m| \end{matrix}$$

$$2f_s \pm f_m \begin{matrix} \nearrow 2f_s + f_m \\ \searrow |2f_s - f_m| \end{matrix}$$

(ii) $m(t) = A_1 \cos 2\pi f_1 t + A_2 \cos 2\pi f_2 t \longrightarrow f_1, f_2$

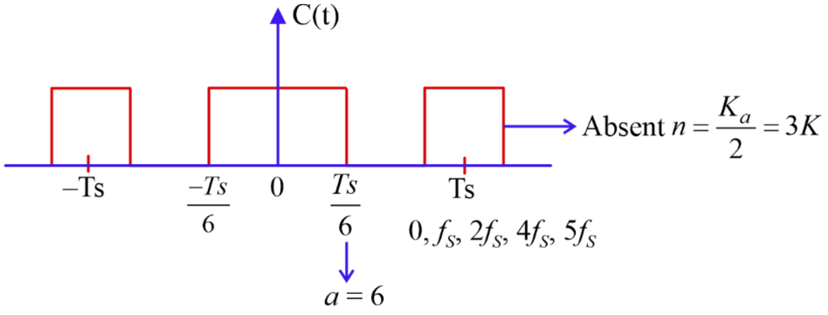
$C(t) =$ Impulse train , $0, f_s 2f_s, 3f_s$

$0 \pm f_1 \quad 0 \pm f_2$

$f_s \pm f_1 \quad f_s \pm f_2$

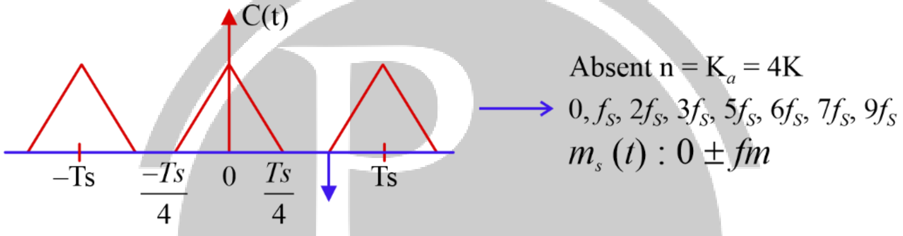
$2fs \pm f_1 \quad 2fs \pm f_2$

(iii) $m(t) = A_m \cos 2\pi f_m t \longrightarrow f_m$



$$m_S(t) = 0 \pm f_m, f_s \pm f_m, 2f_s \pm f_m, 3f_s \pm f_m$$

(iv) $m(t) = Am \cos 2\pi f_m t$



$f_s \pm f_m$

$2f_s \pm f_m$

$3f_s \pm f_m$

## Band pass sampling

m(t) is lowpass signal
S(t) = m (t) cos $2\pi f_c$t

Lowpass signal
(Low pass S.T)

Bandpass signal
(Bandpass S.T)

$\boxed{f_S \ge \frac{2f_H}{K}}$ $\boxed{K = \left[\frac{f_H}{f_H - f_L}\right]}$ $[\cdot] \rightarrow GIF$

$\boxed{\text{Nyauist rate} = 2f_H}$

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# 8

# MISCELLANEOUS

## 8.1. DFT (Disrete Fourier Transform)

$$\underset{\text{discrete in time}}{\underset{\downarrow}{x(n)}} \xleftrightarrow{DTFT} \underset{\text{Continuous in frequency}}{\underset{\downarrow}{X(e^{j\omega})}}$$

**DFT:**

Discrete in time + discrete in frequency .

$$x(n) \xleftrightarrow{DFT} X(K)$$

(i) x(n) periodic with length n .

(ii) x(K) periodic with length K

(iii) Information of one period of either $x(n)$ or $X(K)$ will be given .

N point $x(n)$ is given calculate $n$ point X(K)

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\left(\frac{2\pi}{N}\right)Kn} \qquad K = 0,1,2,---N-1$$

$$x(n) \xleftrightarrow{DFT} X(K)$$

$$\boxed{x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\left(\frac{2\pi}{N}\right)Kn}} \quad n = 0,1,2,---N-1$$

$$x(K) \xleftrightarrow{IDFT} x(n)$$

**Twiddle factor:** $\boxed{W_N = e^{-j\frac{2\pi}{N}}}$

| | | | |
|---|---|---|---|
| $W_N^0 = 1$ | $W_n^{N+1} = W_N$ | $W_N^{(n+lN)} = W_N^n$ | $W_N = e^{-j\frac{2A}{N}}$ |
| $W_N^N = 1$ | $W_N^{n+\frac{N}{2}} = -W_N^n$ | $W_N^{lN} = W_N^N = 1$ | $W_N^{-1} = W_N^*$ |
| $W_N^{N/2} = -1$ | $W_N^{n+N} = W_N^n$ | $W_N^{(2l+1)\frac{N}{2}} = -1$ | |

## Matrix Method :

- DFT: $$[X(K)]=[W_N^n][x(n)]$$
- IDFT: $$[x(n)]=\frac{1}{N}[W_N^n]^{-1}[X(K)]=\frac{1}{N}[W_N^n]^{*}[X(K)]$$

**2 point DFT / IDFT** (N=2)

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix}=\begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \end{bmatrix}=\frac{1}{2}\begin{bmatrix} W_N^{-0} & W_N^{-0} \\ W_N^{-0} & W_N^{-1} \end{bmatrix}\begin{bmatrix} X(0) \\ X(1) \end{bmatrix}$$

**3 point DFT / IDFT** N=3

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^1 & W_3^2 \\ 1 & W_3^2 & W_3^1 \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-4} \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix}$$

**4 point DFT / IDFT** N=4

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}=\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & +j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

➢ If $x(n)=x(-n)\rightarrow$ circle

$$DFT\left[DFT\{x(n)\}\right]\xrightarrow{\quad}\left(\sqrt{N}\right)\left(\sqrt{N}\right)\{x(n)\}$$

$$DFT\left[DFT\left[DFT\left[DFT\{x(n)\}\right]\right]\right]=\left(\sqrt{N}\right)^4 x(n)$$

➢ If $X(-K)=X(K)$

$$IDFT[IDFT[IDFT[IDFT[x(K)]]]]=\left(\frac{1}{\sqrt{N}}\right)^4[X(K)]$$

- $$X(K) = \frac{1}{N^2}\sum_{K=0}^{N-1} x(n) W_N^{-Kn}$$

If $x(n) = x(-n)$

$$DFT[DFT(x(n))] = \left(\sqrt{N}\right)^2 \left(\frac{x(n)}{N^4}\right) = \frac{x(n)}{N^3}$$

## Properties of DFT:

(1) Linearity: $Ax_1(n) + Bx_2(n) \longleftrightarrow AX_1(K) + BX_2(K)$

(2) Periodicity: $x(n+N) = x(n)$

$X(K+N) = X(K)$

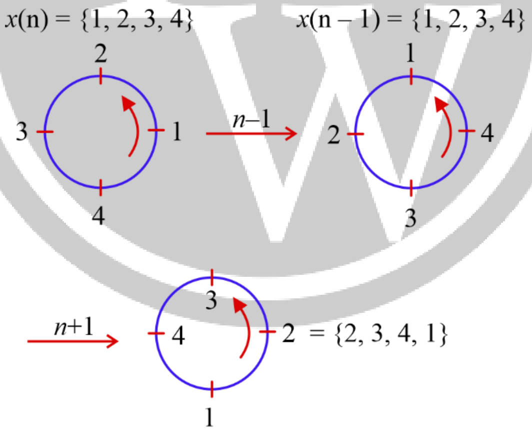
(3) Time Reversal: $[x(n)]_N \longleftrightarrow [X(K)]_N$

$(x(-n))_N \longleftrightarrow (X(-K))_N$

$x(N-n) \longleftrightarrow X(N-K)$

(4) Circular frequency shift $x(n) \longleftrightarrow X(K)$

$$e^{j\left(\frac{2\pi}{N}\right)(l)n} x(n) \longleftrightarrow (x(K-l))_{MODN}$$



Complex conjugate property $x(n) \longleftrightarrow X(K)$

$x^*(n) \longleftrightarrow X^*(-K)$

$(x^*(n))_{MODN} \longleftrightarrow (X^2(-K))_{MODN} = X^*(N-K)$

| **x(n)** | **X(K)** |
|---|---|
| R+E | R+E |
| R+O | I+O |
| I+E | I+E |
| I+O | R+O |

| ***x*(*n*)** | ***X*(*K*)** |
|---|---|
| Real | C.S |
| Image | CAS |
| C.S | Real |
| C.A.S | Img. |

## Circular convolution

**Case 1:** Column Method

$$x_1(n) = \{a, b, c, d\}$$

$$x_2(n) = \{p, q, r, s\}$$

$$x(n) = x_1(n) * x_2(n) = \{\alpha, \beta, \gamma, \delta\}$$

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$$

**Case 2:** Row Method

$$[\alpha, \beta, \gamma, \delta] = [p \ q \ r \ \&] \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix}$$

- $x_1(n) \otimes x_2(n) \xleftrightarrow{DFT} X_1(K) X_2(K)$

  $x(n) \otimes x(n) \xleftrightarrow{DFT} X^2(K)$

## Multiplication in time domain:

$$x_1(n).x_2(n) \xleftrightarrow{DFT} \frac{1}{N}[X_1(K) \otimes X_2(K)]$$

$$x^2(n) \xleftrightarrow{DFT} \frac{1}{N}[X(K) \otimes X(K)]$$

## Parseval's Theorem

(1) $\displaystyle\sum_{n=0}^{N-1} x_1(n) x_2(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_1(K) X_2(K)$

(2) $\displaystyle\sum_{n=0}^{N-1} x_1(n) x_2^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_1(K) X_2^*(K)$

(3) $\displaystyle\sum_{n=0}^{N-1} x^2(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) X(-K)$

$$\boxed{\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|}$$

(4) $\displaystyle\sum_{n=0}^{N-1} x(n) x^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) X^*(K)$

## Time Expansion

- N point $\quad x(n) \xleftrightarrow{D.F.T} \{X(K)\}^{N\text{ point}}$

  2N point $\quad x\left(\frac{n}{2}\right) \xleftrightarrow{D.F.T} \{X(K), X(K)\}^{2N\text{ point}}$

- N point: $\quad X(K) \xleftrightarrow{IDFT} \{x(n)\}$

  2N point: $\quad X\left(\frac{K}{2}\right) \xleftrightarrow{IDFT} \frac{1}{2}[x(n), x(n)]$

## Discrete Time Fourier Series

$$\underset{\text{Periodic N}}{\underset{\downarrow}{x(n)}} = \sum_{K=0}^{N-1} C_K e^{jn}\left(\frac{2\pi}{N}\right)K$$

$$C_K = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-jn}\left(\frac{2\pi}{N}\right)K$$

$$\boxed{C_K = \frac{X(K)}{N}} \qquad \boxed{C_{K+N} = C_K}$$

$$N \quad x(n) \xleftrightarrow{DFT} X(K) = N(C_K)$$

$$2N \quad [x(n), x(n)] \longleftrightarrow 2X\left(\frac{K}{2}\right) = 2\left[2N \ \frac{C_K}{2}\right]$$

**FAST-FOURIER TRANSFORM : (F.F.T)**

↙ Decimation in Time (D.I.T)

↘ Decimation in frequence (D.I.F)

## Drawback of DFT Calculation :

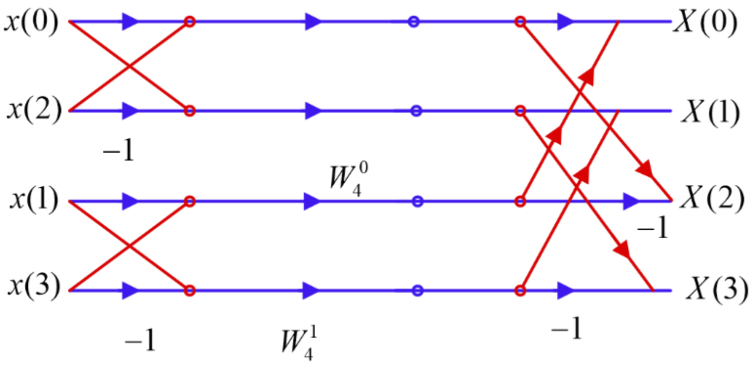
$$\boxed{X(K) = \sum_{n=0}^{N-1} x(n)W_n^{Kn}}$$

N Point DFT
↗ $N^2$ Complex multiplication $\longrightarrow 4N^2$ Real Multiplication
↘ $N(N-1)$ Complex $\rightarrow N(4N-2)$ Real

addition additions

## DIT algorithm in FFT :

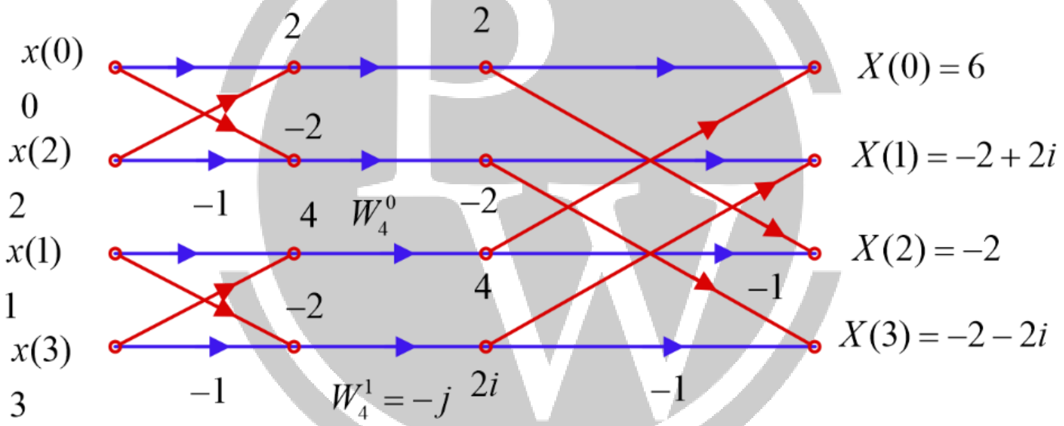
4 point DFT : $x(n) = \{x(o), x(1), x(2), x(3)\}$

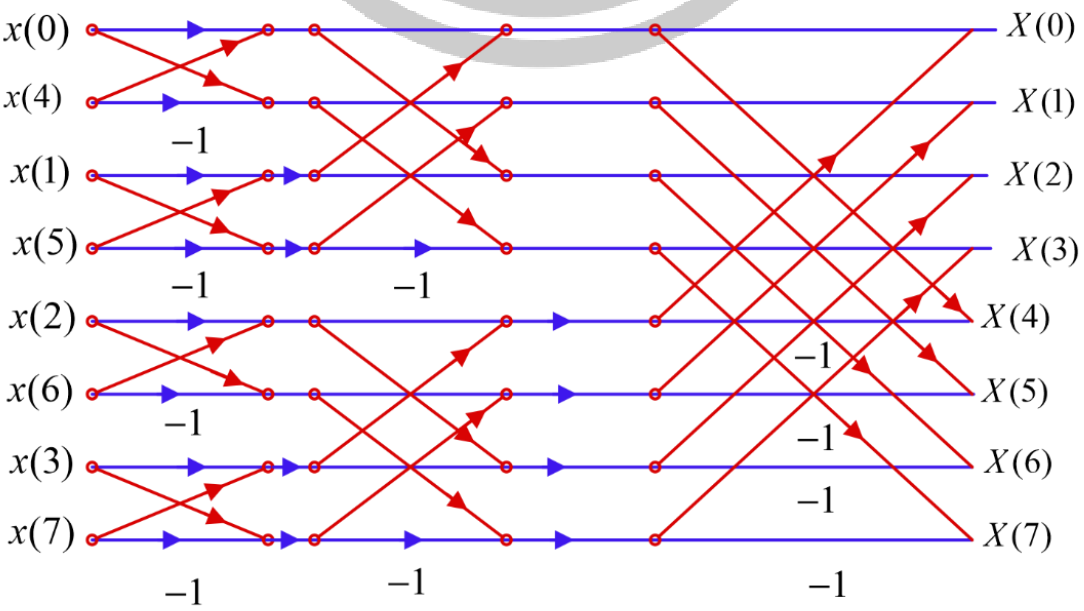


- $$X(K) = \sum_{n=0}^{3} x(n) W_N^{Kn}$$

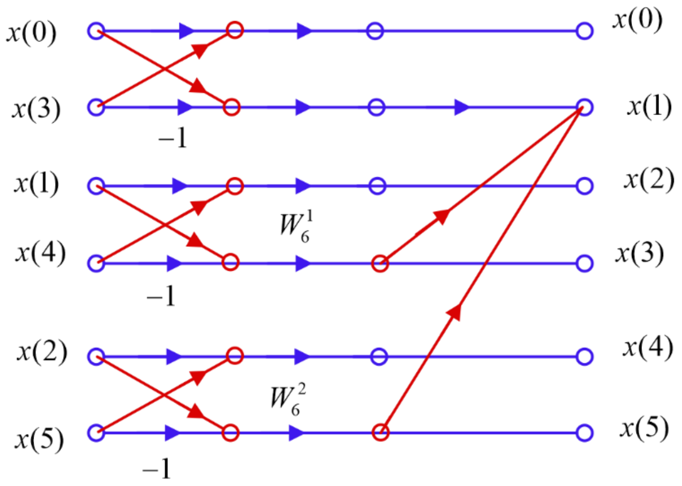
$$X(1) = \sum_{n=0}^{3} x(n) W_4^{n} = [x(0) - x(2)] + W_4^1 [x(1) - x(3)]$$

$$x(n) = \{0, 1, 2, 3\}$$





6 point DFT : $x(n)=\{x(0),x(1),x(2),x(3),x(4),x(5)\}$



$\boxed{DFT\ w.r.t\ X(1)}$

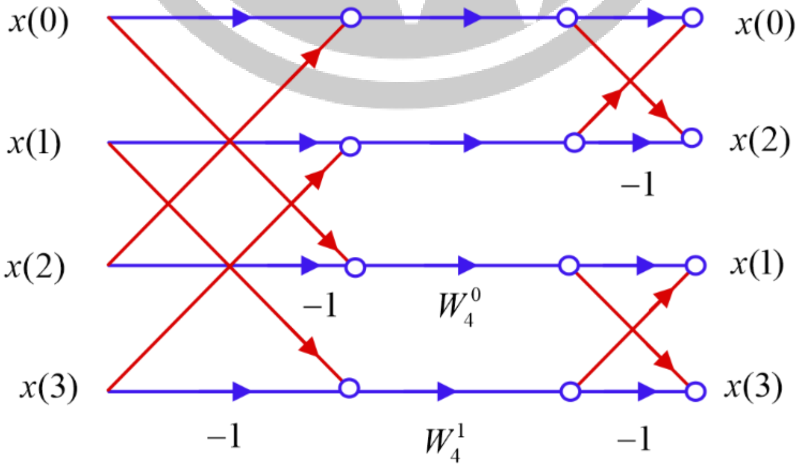
$$X(K)=\sum_{n=0}^{5}x(n)W_6^{Kn}$$

$$X(1)=\sum_{n=0}^{5}x(n)W_6^{n}=[x(0)-x(3)]+(x(1)-x(4))W_6^1+(x(2)-x(5))W_6^2$$

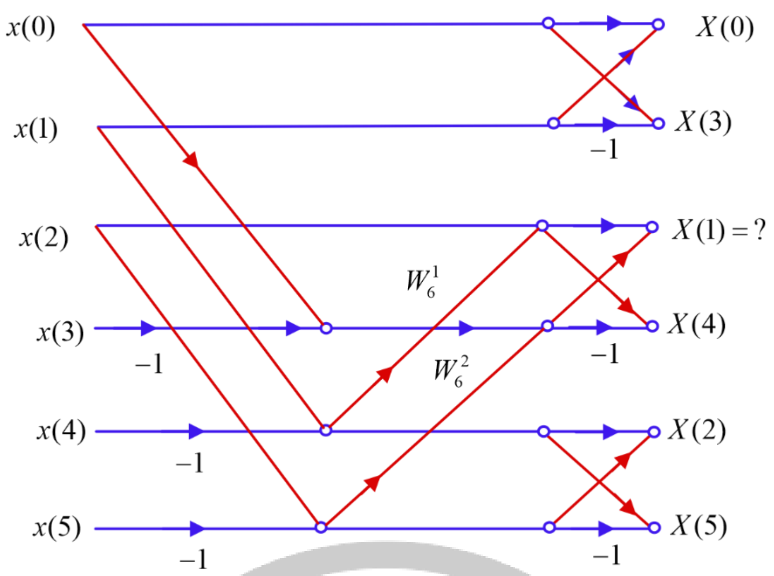
**Summary:**

| | |
|---|---|
| Radix 2 | Radix Non – 2 |
| ↓ | ↓ |
| Symm Butlerfly | use formula to generate Butterfly |

**DIF algorithm**



### 6 point DIF :



### For Radix N Butterfly for calculation of N point DFT

- No of stages = $\log_2^N$
- No of Butterfly in each stage $= N/2$
- Total no. of Butterflies $= \frac{N}{2}\log_2^N$
- Total no of complex multiplication $= \frac{N}{2}\log_2^N$
- Total number of complex addition $= N\log_2 N$

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