Continuity and Differentiability, Methods of Differentiation

Properties of Continuous Functions

Here we present two extremely useful properties of continuous functions;

Let $y = f(x)$ be a continuous function $\forall x \in [a,b]$, then following results hold true.

(i) f is bounded between a and b. This simply means that we can find real numbers m_1 and m_2 such $m_1 \le f(x) \le m_2 \forall x \in [a,b]$.

(ii) Every value between $f(a)$ and $f(b)$ will be assumed by the function atleast once. This property is called intermediate value theorem of continuous function.

In particular if $f(a) \cdot f(b) < 0$, then $f(x)$ will become zero atleast once in (a,b). It also means that if $f(a)$ and $f(b)$ have opposite signs then the equation $f(x) = 0$ will have atleast one real root in (a,b).

Types of Discontinuities

Type-1: (Removable type of discontinuities)

(a) **Missing point discontinuity:** Where $\lim_{x \to a} f(x)$ exists finitely but $f(a)$ is not defined.

(b) **Isolated point discontinuity:** Where $\lim_{x \to a} f(x)$ exists & $f(a)$ also exists but; $\lim_{x \to a} f(x) \ne f(a)$.

Type-2: (non-Removable type of discontinuities)

(a) **Finite type discontinuity:** In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

(b) **Infinite type discontinuity:** In such type of discontinuity atleast one of the limit viz. LHL and RHL is tending to infinity.

(c) **Oscillatory type discontinuity:** Limits oscillate between two finite quantities.

Derivability of Function at a Point

If $f'(a^+) = f'(a^-) =$ finite quantity, then $f(x)$ is said to be derivable or differentiable at $\boldsymbol{x = a}$. In such case $f'(a^+) = f'(a^-) = f'(a)$ and it is called derivative or differential coefficient of $f(x)$ at $x = a$.

Note:

(i) All polynomial, trigonometric, inverse trigonometric, logarithmic and exponential function are continuous and differentiable in their domains, except at end points.

(ii) If $f(x)$ and $g(x)$ are derivable at $x=a$ then the functions $f(x)+g(x), f(x)-g(x), f(x)\cdot g(x)$ will also be derivable at $x=a$ and if $g(a)\neq 0$ then the function $f(x)/g(x)$ will also be derivable at $x=a$.

In short, for a function ' f ':

Differentiable $\Rightarrow$ Continuous

Not Differentiable $\Rightarrow$ Not Continuous

But Not Continuous $\Rightarrow$ Not Differentiable

Continuous $\Rightarrow$ May or may not be Differentiable.

Derivability Over an Interval

(a) $f(x)$ is said to be derivable over an open interval (a,b) if it is derivable at each and every point of the open interval (a,b).

(b) $f(x)$ is said to be derivable over the closed interval $[a,b]$ if:

(i) $f(x)$ is derivable in (a,b) and

(ii) for the points a and b, $f'(a^{+})$ & $f'(b^{-})$ exist.

Note:

(i) If $f(x)$ is differentiable at $x=a$ and $g(x)$ is not differentiable at $x=a$, then the product function $F(x)=f(x)\cdot g(x)$ can still be differentiable at $x=a$.

(ii) If $f(x)$ & $g(x)$ both are not differentiable at $x=a$ then the product function; $F(x)=f(x)\cdot g(x)$ can still be differentiable at $x=a$.

(iii) If $f(x)$ & $g(x)$ both are non-derivable at $x=a$ then the sum function $F(x)=f(x)+g(x)$ may be a differentiable function.

(iv) If $f(x)$ is derivable at $x=a \not\Rightarrow f'(x)$ is continuous at $x=a$.

Differentiation of Some Elementary Functions

1. $\frac{d}{dx}\left(x^{n}\right)=nx^{n-1}$

2. $\frac{d}{dx}\left(a^{x}\right)=a^{x}\ln a$

3. $\frac{d}{dx}\left(\ln|x|\right)=\frac{1}{x}$

4. $\frac{d}{dx}\left(\log_{a}x\right)=\frac{1}{x\ln a}$

5. $\frac{d}{dx}\left(\sin x\right)=\cos x$

6. $\frac{d}{dx}\left(\cos x\right)=-\sin x$

7. $\frac{d}{dx}\left(\sec x\right)=\sec x\tan x$

8. $\frac{d}{dx}\left(\operatorname{cosec} x\right)=-\operatorname{cosec} x\cot x$

9. $\frac{d}{dx}\left(\tan x\right)=\sec^{2}x$

10. $\frac{d}{dx}\left(\cot x\right)=-\operatorname{cosec}^{2}x$

Basic Theorems

1. $\frac{d}{dx}\left(f\pm g\right)\left(x\right)=f'\left(x\right)\pm g'\left(x\right)$

2. $\frac{d}{dx}\left(kf\left(x\right)\right)=k\frac{d}{dx}f\left(x\right)$

3. $\frac{d}{dx}\left(f\left(x\right)\cdot g\left(x\right)\right)=f\left(x\right)g'\left(x\right)+g\left(x\right)f'\left(x\right)$

4. $\frac{d}{dx}\left(\frac{f\left(x\right)}{g\left(x\right)}\right)=\frac{g\left(x\right)f'\left(x\right)-f\left(x\right)g'\left(x\right)}{g^{2}\left(x\right)}$

5. $\frac{d}{dx}\left(f\left(g(x)\right)\right)=f'\left(g(x)\right)g'(x)$

Derivative of inverse Trigonometric Functions

$\frac{d\sin^{-1}x}{dx}=\frac{1}{\sqrt{1-x^2}},\frac{d\cos^{-1}x}{dx}=-\frac{1}{\sqrt{1-x^2}}$, for $-1<x<1$.

$\frac{d\tan^{-1}x}{dx}=\frac{1}{1+x^2},\frac{d\cot^{-1}x}{dx}=-\frac{1}{1+x^2}\ (x\in R)$

$\frac{d\sec^{-1}x}{dx}=\frac{1}{|x|\sqrt{x^2-1}},\frac{d\text{cosec}^{-1}x}{dx}$

$=-\frac{1}{|x|\sqrt{x^2-1}}$, for $x\in(-\infty,-1)\cup(1,\infty)$

Differentiation Using Substitution

Following substitutions are normally used to simplify this expression.

1. $\sqrt{x^2+a^2}$ by substituting $x=a\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$
2. $\sqrt{a^2-x^2}$ by substituting $x=a\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$
3. $\sqrt{x^2-a^2}$ by substituting $x=a\sec\theta$, where $\theta\in[0,\pi],\theta\ne\frac{\pi}{2}$
4. $\sqrt{\frac{x+a}{a-x}}$ by substituting $x=a\cos\theta$, where $\theta\in[0,\pi]$.

Parametric Differentiation

If $y=f(\theta)$ and $x=g(\theta)$ where θ is a parameter, then $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}$.

Derivative of one Function with Respect to Another

Let $y=f(x);z=g(x)$ then $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{f'(x)}{g'(x)}$.

If $F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where $f, g, h, l, m, n, u, v, w$ are differentiable functions of x then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+\begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$