Definite Integration

The Fundamental Theorem of Calculus Part 1:

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \quad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

The Fundamental Theorem of Calculus, Part 2:

If f is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Note: If $\int_a^b f(x)dx = 0 \Rightarrow$ then the equation $f(x) = 0$ has at least one root lying in (a, b) provided f is a continuous function in (a, b).

❖ $\int_a^b f(x)dx$ algebraic area under the curve $f(x)$ from a to b.

Properties of Definite Integral:

1. $\int_a^b f(x)dx = \int_a^b f(t)dt$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. $\int_{-a}^a f(x)dx = \int_0^a (f(x) + f(-x))dx = \begin{cases} 2\int_0^a f(x)dx, & f(-x) = f(x) \\ 0, & f(-x) = -f(x) \end{cases}$
5. $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$
6. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
7. $\int_0^{2a} f(x)dx = \int_0^a (f(x) + f(2a-x))dx$

$$= \begin{cases} 2\int_0^a f(x)dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

8. If $f(x)$ is a periodic function with period T, then

$$\int_0^{nT} f(x)dx = n\int_0^T f(x)dx, n \in Z,$$

$$\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx, n \in Z, a \in R$$

$$\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx, m, n \in Z,$$

$$\int_{nT}^{a+nT} f(x)dx = \int_0^a f(x)dx, n \in Z, a \in R$$

$$\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx, n \in Z, a, b \in R$$

9. If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$, then

$$\int_a^b \Psi(x)dx \le \int_a^b f(x)dx \le \int_a^b \phi(x)dx$$

Leibnitz Theorem:

$$If\ F(x) = \int_{g(x)}^{h(x)} f(t)dt$$

then $\dfrac{dF(x)}{dx} = h'(x)f(h(x)) - g'(x)f(g(x))$

Walli's Formula:

1. $\displaystyle\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{(n-1)(n-3)\ldots.(1 or 2)}{n(n-2)\ldots..(1 or 2)}K$

where $K = \begin{cases} \pi/2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

2. $\displaystyle\int_0^{\pi/2} \sin^n x.\cos^m x\, dx$

$$= \frac{[(n-1)(n-3)(n-5)\ldots.1 or 2][(m-1)(m-3)\ldots.1 or 2]}{(m+n)(m+n-2)(m+n-4)\ldots.1 or 2}K$$

where $K = \begin{cases} \dfrac{\pi}{2} & \text{if both } m \text{ and } n \text{ are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$

Definite Integral as Limit of a Sum:

$$\int_a^b f(x)dx = \lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+\overline{n-1}h)]$$

$$\Rightarrow \lim_{h\to\infty} h\sum_{n-1}^{r=0} f(a+rh) = \int_0^1 f(x)dx \text{ where } b - a = nh$$

If $a = 0$ and $b = 1$ then, $\displaystyle\lim_{n\to\infty} h\sum_{n-1}^{r=0} f(rh) = \int_0^1 f(x)dx;$ where $nh = 1$

OR $\displaystyle\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{n-1}^{r=1} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx.$

Estimation of Definite Integral:

1. If $f(x)$ is continuous in $[a, b]$ and it's range in this interval is $[m, M]$, them $m\,(b-a) \le \int_a^b f(x)dx \le M(b-a)$

2. If $f(x) \le \phi(x)$ for $a \le x \le b$ then $\int_a^b f(x)dx \le \int_a^b \phi(x)dx$

3. $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx.$

4. If $f(x) \ge 0$ on the interval $[a,b]$, then $\int_a^b f(x)dx \ge 0.$

5. $f(x)$ and $g(x)$ are two continuous function on $[a, b]$ then
$$\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\int_a^b f^2(x)dx \int_a^b g^2(x)dx}$$

Some Standard Results:

1. $\int_0^{\pi/2} \log \sin x\,dx = -\frac{\pi}{2}\log 2 = \int_0^{\pi/2} \log \cos x\,dx$

2. $\int_a^b \frac{|x|}{x}dx = |b| - |a|.$