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ADVANCE 2026

Mathematics | Lecture - 02

Integration, D.E. & Area

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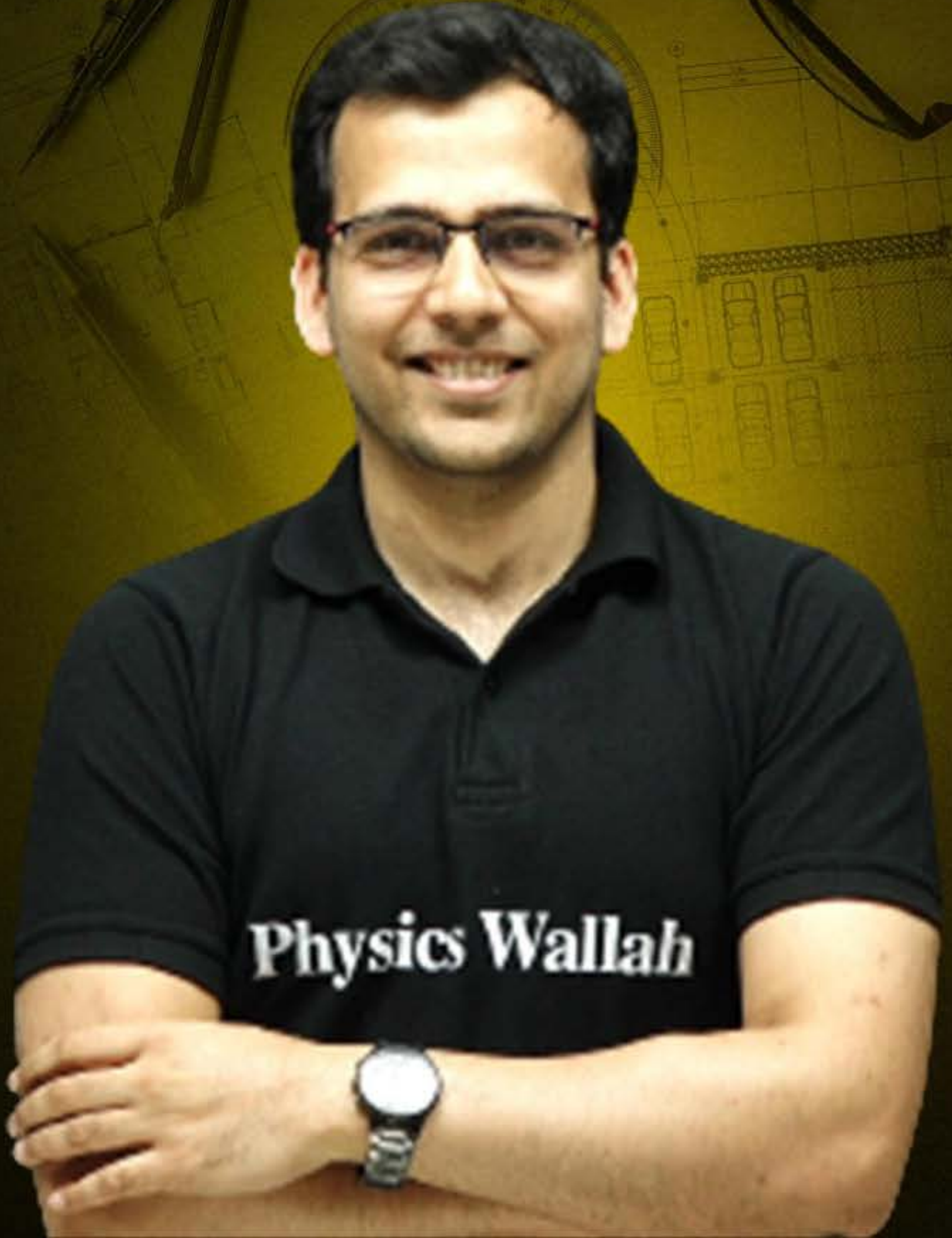


Topics *To be covered*

1. Definite Integration
2. Important Brain Teasers

Definite Integration

Brain Teaser-14

[Ans. A]

The value of $\int_0^\infty \frac{(x^2+4)\ln x\,dx}{(x^4+16)}$ is equal to:

$\boxed{x = 2t}$

(A) $\frac{\pi}{2\sqrt{2}}\ln 2$ ✓

(B) $\frac{\pi}{2}\ln 2$

(C) $\pi \ln 2$

(D) $\frac{\pi}{\sqrt{2}}\ln 2$

$$\int_0^\infty \frac{(4t^2+4)}{(16t^4+16)}\ln(2t)\cdot 2\,dt$$

$$\frac{8}{16}\int_0^\infty \frac{(t^2+1)}{(t^4+1)}(\ln 2+\ln t)\,dt$$

$$\frac{1}{2}\underbrace{\int_0^\infty \frac{(t^2+1)\ln 2}{t^4+1}}_{I_1} + \frac{1}{2}\underbrace{\int_0^\infty \frac{(t^2+1)\ln t}{(t^4+1)}\,dt}_{I_2}$$

$$I_2 = \int_0^\infty \frac{(t^2+1)\ln t}{t^4+1}\,dt$$

$t = 1/z$

$$\int_\infty^0 \frac{\left(\frac{1}{z^2}+1\right)\ln\frac{1}{z}}{\left(\frac{1}{z^4}+1\right)}\left(-\frac{1}{z^2}\right)dz$$

$$I_2 = -\int_0^\infty \frac{(1+z^2)\ln z}{(z^4+1)}\,dz$$

$I_2 = -I_2$

$\Rightarrow 2I_2 = 0$

$I_2 = 0$

$$\text{Ans} = \frac{\ln 2}{2}\int_0^{\infty} \frac{t^2+1}{t^4+1}\,dt$$

$$\frac{\ln 2}{2}\int_0^{\infty} \frac{1+1/t^2}{(t-1/t)^2+2}\,dt$$

$$t - 1/t = \alpha$$

$$\frac{\ln 2}{2}\int_{-\infty}^{\infty} \frac{d\alpha}{\alpha^2+2}$$

$$\frac{\ln 2}{2}\,\frac{1}{\sqrt{2}}\tan^{-1}\frac{\alpha}{\sqrt{2}}\Big|_{-\infty}^{\infty}$$

$$\frac{\ln 2}{2\sqrt{2}}\left[\pi/2 - (-\pi/2)\right]$$

$$\frac{\pi}{2\sqrt{2}}\ln 2 \quad \text{Ans}$$

Theory

Let $I = \int_0^\infty \frac{\ln x}{(1+x)^2}\,dx$.

$x = 1/t$

$dx = -\frac{1}{t^2}\,dt$

$I = -\int_\infty^0 \frac{\ln 1/t}{\left(1+\frac{1}{t}\right)^2} \cdot \frac{1}{t^2}\,dt$

$I = \int_\infty^0 \frac{\ln t}{(t+1)^2}\,dt$

$I = -\int_0^\infty \frac{\ln x}{(x+1)^2}\,dx$

$I = -I \Rightarrow 2I = 0$

$I = 0$

$I = \int_0^\infty \frac{\ln x\,dx}{ax^2+bx+a} = 0$

$x \to 1/t$

$I = -I$

$2I = 0$

$I = 0$

Theory

$$\int_0^\infty \frac{\ln x}{(x^2+8x+16)}\,dx$$

$$\int_0^\infty \frac{\ln x \, dx}{(x+4)^2}$$

$x = 4t$

$$\int \frac{\ln(4t)\; 4\,dt}{(4t+4)^2}$$

$$\frac{4}{16}\int \frac{\ln(4t)}{(t+1)^2}\,dt$$

$$\frac{1}{4}\int \frac{\ln 4 + \ln t}{(t+1)^2}$$

$$\frac{\ln 4}{4}\int_0^\infty \frac{dt}{(t+1)^2} + \frac{1}{4}\int_0^\infty \frac{\ln t\, dt}{(t+1)^2} \rightarrow 0$$

$$\frac{\ln 4}{4}\left(\frac{1}{t+1}\right)\Big|_\infty^0$$

$$\frac{\ln 4}{4}\left[\frac{1}{1} - \frac{1}{\infty}\right]$$

$$\frac{\ln 4}{4} \quad \text{Ans}$$

Question [JEE Adv. 2025]

[Ans. 21]

If $\alpha = \int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{2x^2-3x+2} dx$, then the value of $\sqrt{7}\tan\left(\frac{2\alpha\sqrt{7}}{\pi}\right)$ is _____.

(Here, the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

$x = 1/t$ ✓

Brain Teaser-16

Let $f(\alpha)=\int_{\alpha^{-1}}^{\alpha}\frac{1}{x}\cot^{-1}\left(\frac{x^2-x+1}{2x-3x^2}+\frac{x^2-x+1}{3-2x}\right)dx$

and $g(\alpha)=\int_{\ln\frac{1}{\alpha}}^{\ln\alpha}\left(\frac{|x^2-3x+2|-|(x+1)(x+2)|+|x+1|+|x-1|}{|x+1|+|x-1|}\right)dx$

where $\alpha\in(0,\infty)-\left\{\frac{2}{3},\frac{3}{2}\right\}$. If $f(200)-\frac{\pi}{2}g(50)=\frac{a\pi}{3}\ln b$, then the value of $a+b$ is

[Ans.

$\cot^{-1}(-x)=\pi-\cot^{-1}x$

$x=1/t$

$$f(\alpha)=\int_{\alpha}^{1/\alpha}t\cdot\cot^{-1}\left[\frac{\frac{1}{t^2}-\frac{1}{t}+1}{2/t-3/t^2}+\frac{\frac{1}{t^2}-\frac{1}{t}+1}{3-2/t}\right]\left(-\frac{1}{t^2}\right)dt$$

$$=\int_{1/\alpha}^{\alpha}\frac{1}{t}\cot^{-1}\left[\frac{t^2-t+1}{2t-3}+\frac{t^2-t+1}{(3t-2)t}\right]dt$$

$$\int_{1/\alpha}^{\alpha}\frac{1}{t}\cot^{-1}-\left\{\frac{t^2-t+1}{3-2t}+\frac{t^2-t+1}{2t-3t^2}\right\}dt$$

$$f(\alpha) = \int_{1/\alpha}^{\alpha} \frac{1}{x}\left(\pi - \cot^{-1}\left(\frac{x^2-x+1}{3-2x} + \frac{x^2-x+1}{2x-3x^2}\right)\right) dx$$

$$f(\alpha) = \int_{1/\alpha}^{\alpha} \pi/x \, dx - f(\alpha)$$

$$2f(\alpha) = \pi \ln x \Big|_{1/\alpha}^{\alpha}$$

$$2f(\alpha) = \pi\left(\ln\alpha - \ln\tfrac{1}{\alpha}\right)$$

$$\boxed{f(\alpha) = \pi \ln\alpha}$$

$$f(200) = \pi \ln 200$$

Brain Teaser-16

[Ans. 7] or 8

$\cot^{-1}(-x) = \pi - \cot^{-1}x$

Let $f(\alpha) = \int_{\alpha^{-1}}^{\alpha} \frac{1}{x}\cot^{-1}\left(\frac{x^2-x+1}{2x-3x^2} + \frac{x^2-x+1}{3-2x}\right)dx$

and $g(\alpha) = \int_{\ln\frac{1}{\alpha}}^{\ln\alpha}\left(\frac{|x^2-3x+2| - |(x+1)(x+2)| + |x+1| + |x-1|}{|x+1|+|x-1|}\right)dx$

where $\alpha \in (0,\infty) - \left\{\frac{2}{3}, \frac{3}{2}\right\}$. If $f(200) - \frac{\pi}{2}g(50) = \frac{a\pi}{3}\ln b$ then the value of $a+b$ is

$$g(\alpha) = \int_{-\ln\alpha}^{\ln\alpha}\left(\frac{|x^2-3x+2| - |x^2+3x+2| + |x+1| + |x-1|}{|x+1|+|x-1|}\right)dx$$

$x \to -x$

$$g(\alpha) = \int_{-\ln\alpha}^{\ln\alpha}\left(\frac{|x^2+3x+2| - |x^2-3x+2| + |x-1| + |x+1|}{|x+1|+|x-1|}\right)dx$$

$$\cancel{2}\, g(\alpha) = \int_{-\ln\alpha}^{\ln\alpha} \cancel{2}\, dx \Rightarrow g(\alpha) = 2\ln\alpha$$

$$g(\alpha) = 2\ln\alpha$$

$$g(50) = 2\ln 50$$

$$\pi/_2\, g(50) = \pi \ln 50$$

$$f(200) - \pi/_2\, g(50) = \pi\ln 200 - \pi\ln 50$$

$$\pi\left[\ln\frac{200}{50}\right]$$

$$\pi\ln(4)$$

$$3\pi/_3 \ln 4 \quad \text{or} \quad 6\pi/_3 \ln 2$$

$$a = 3,\ b = 4 \qquad a = 6,\ b = 2$$

Brain Teaser-17

[Ans. 2]

Let $J_n = \int_0^{\pi/2} (1-\sin x)^n \sin 2x \, dx$. Find $\sum_{n=0}^{\infty} J_n$.

$$= \int_0^{\pi/2} (1-\sin x)^n \cdot 2\sin x \cos x \, dx$$

$$\boxed{\sin x = t}$$

$$J_n = 2\int_0^1 (1-t)^n \, t \, dt$$

$$t \to 1-t$$

$$J_n = 2\int_0^1 t^n (1-t)\, dt$$

$$= 2\left[\int (t^n - t^{n+1})\, dt\right]$$

$$= 2\left[\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2}\right]$$

$$J_n = 2\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$J_0 = 2\left[\frac{1}{1} - \cancel{\frac{1}{2}}\right]$$

$$\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$

$$2\left[\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right]$$

$$\Sigma J_n = 2\left[1 - \cancel{\frac{1}{n+2}}\right]$$

$$S_\infty = 2$$

Brain Teaser-18

[Ans. 2025]

If $\int_0^{\pi} \frac{\sin x(\sin x+1)e^{\sin x+\cos x}}{e^{\cos x}+1}dx = a + b\int_0^{\pi} e^{\sin x}dx$ where a and b are positive rational numbers, then find the value of $1620(a^2+b^2)$.

$\longrightarrow (1)$

$x \to \pi - x$

$$I = \int_0^{\pi} \frac{\sin x(\sin x+1)e^{\sin x - \cos x}}{e^{-\cos x}+1}dx$$

$$= \int_0^{\pi} \frac{\sin x(\sin x+1)e^{\sin x}}{1+e^{\cos x}}dx \longrightarrow (2)$$

$(1)+(2)$

$$2I = \int_0^{\pi} \frac{\sin x(\sin x+1)}{e^{\cos x}+1}\left(e^{\sin x+\cos x}+e^{\sin x}\right)dx$$

$$2I = \int_0^{\pi} \sin x(\sin x+1)e^{\sin x}dx$$

(P-6)

$$I = \int_0^{\pi/2}(\sin^2 x+\sin x)e^{\sin x}dx$$

$$\int_0^{\pi/2}(1-\cos^2 x+\sin x)e^{\sin x}dx$$

$$\int_0^{\pi/2}e^{\sin x}dx - \int_0^{\pi/2}\cos^2 x\, e^{\sin x}dx + \int_0^{\pi/2}\sin x\, e^{\sin x}dx \to I_3$$

$$= \frac{1}{2}\int_0^{\pi}\sin x\, dx + 1$$

$$I_3 = \int_0^{\pi/2} \underbrace{\sin x}_{II}\, \underbrace{e^{\sin x}}_{I}\, dx$$

$$e^{\sin x}(-\cos x)\Big|_0^{\pi/2} + \int_0^{\pi/2} e^{\sin x}\cos x \cdot \cos x\, dx$$

$$e^{\sin x}(\cos x)\Big|_{\pi/2}^{0}$$

$$(1-0)$$

$$I_3 = 1 + \int_0^{\pi/2} e^{\sin x}\cos^2 x\, dx$$

$a = 1$

$b = 1/2$

$a^2 + b^2 = 5/4$

$1620(a^2+b^2) = 5/4 \times 1620$

$= 5 \times 405$

$\underline{2025}$

Brain Teaser-19

[Ans. 7]

The value of $\int_0^{\frac{\pi}{4}} e^{\sec x} \frac{\sin\left(x+\frac{\pi}{4}\right)}{(1-\sin x)\cos x} dx$ can be expressed as $\frac{(a+\sqrt{b})e^{\sqrt{c}}-e}{\sqrt{d}}$, then find the value of $(a+b+c+d)$.

$$\frac{1}{\sqrt{2}}\int_0^{\pi/4} e^{\sec x}\frac{(\sin x+\cos x)}{(1-\sin x)\cos x}\,dx$$

divide N^r & D^r with $\cos^2 x$

$$\frac{1}{\sqrt{2}}\int_0^{\pi/4} e^{\sec x}\left[\frac{\sec x\tan x+\sec x}{(\sec x-\tan x)}\right]dx$$

$$\frac{1}{\sqrt{2}}\left\{\underbrace{\int_0^{\pi/4} \frac{e^{\sec x}(\sec x\tan x)}{(\sec x-\tan x)}}_{I_1}+\int \frac{e^{\sec x}\sec x}{\sec x-\tan x}dx\right.$$

$$I_1 = \int_0^{\pi/4} \underbrace{\frac{1}{\sec x - \tan x}}_{\mathrm{I}} e^{\sec x} \underbrace{(\sec x \tan x)}_{\mathrm{II}} \, dx$$

$$= \left. \frac{1}{\sec x - \tan x} e^{\sec x} \right|_0^{\pi/4} + \int_0^{\pi/4} \frac{1}{(\sec x - \tan x)^2} (\sec x \tan x - \sec^2 x) \, e^{\sec x} \, dx$$

$$I_1 = \frac{e^{\sqrt{2}}}{\sqrt{2}-1} - \frac{e^1}{1-0} - \int_0^{\pi/4} \frac{\sec x (\sec x - \tan x)}{(\sec x - \tan x)^2} e^{\sec x} \, dx$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \left[\frac{e^{\sqrt{2}}}{\sqrt{2}-1} - e \right] = \left[\frac{e^{\sqrt{2}}(\sqrt{2}+1) - e}{\sqrt{2}} \right]$$

$a = 1$, $b = 2$, $c = 2$

$d = 2$

Brain Teaser-20

[Ans. 2027]

Let $I_1 = \int_1^3 (x^2 + x + 3) f(x^3 - 2x^2 - 5x + 2026)\,dx$ and

$I_2 = \int_1^3 (4x^2 - 3x - 2) f(x^3 - 2x^2 - 5x + 2026)\,dx$.

If the value of $\frac{2026 I_1}{I_2} = \frac{a}{b}$ where a and b are co-prime, then find the value of $(a + b)$.

$$I_2 - I_1 = \int_1^3 (3x^2 - 4x - 5) f(x^3 - 2x^2 - 5x + 2026)\,dx$$

$$\frac{d}{dx}(x^3 - 2x^2 - 5x + 2026)$$

$$(3x^2 - 4x - 5)$$

$$\boxed{x^3 - 2x^2 - 5x + 2026 = t}$$

$$(3x^2 - 4x - 5)\,dx = dt$$

$x = 1$

$1 - 2 - 5 + 2026$

$= 2020$

$x = 3$

$27 - 18 - 15 + 2026$

$- 6 + 2026$

$= 2020$

$$I_2 - I_1 = \int_{2020}^{2020} f(t)\,dt$$

$$\Rightarrow I_2 - I_1 = 0$$

$$\Rightarrow \boxed{I_2 = I_1}$$

$$\frac{2026}{1} = a/b$$

Brain Teaser-21

[Ans. 2]

Let $I(n)=\int_0^{\pi}\ln(1-2n\cos x+n^2)dx$. Find the value of $\frac{I(2025)}{I(45)}$.

$x \to \pi - x$

$I(n) = \int_0^{\pi} \ln(1+2n\cos x+n^2)\,dx$

$2I(n) = \int_0^{\pi} \ln\left((1+n^2)^2-(2n\cos x)^2\right)dx$

$\int_0^{\pi} \ln(n^4+1+2n^2-4n^2\cos^2 x)\,dx$

$2I(n) = \int_0^{\pi} \ln\left(n^4+1-2n^2(\cos 2x)\right)dx$

Let $2x=\theta$

$2I(n) = \int_0^{2\pi} \ln(n^4+1-2n^2\cos\theta)\frac{d\theta}{2}$

$2I(n) = \cancel{2}\int_0^{\pi} \ln(n^4+1-2n^2\cos\theta)\frac{d\theta}{\cancel{2}}$

$2I(n) = \int_0^{\pi} \ln(n^4+1-2n^2\cos\theta)\,d\theta$

$I(n^2)$

$2I(n) = I(n^2)$

$n=45$

$2I(45) = I(2025)$

$2 = \frac{I(2025)}{I(45)}$

Brain Teaser-22

[Ans. 6]

If $\int_0^{1/2} \tan^{-1}\left(\frac{1}{2}\left(\sqrt{\frac{1+x}{x}} - \sqrt{\frac{x}{1+x}}\right)\right) dx = p\ln(2+\sqrt{3}) - \frac{\pi}{q}$, then pq is equal to ______.

$$\int_0^{1/2} \tan^{-1}\left[\frac{1}{2}\,\frac{(1+\cancel{x}) - \cancel{x}}{\sqrt{x}\sqrt{1+x}}\right] dx$$

$$\int_0^{1/2} \tan^{-1}\left(\frac{1}{2\sqrt{x(1+x)}}\right) dx$$

$$\int_0^{1/2} \tan^{-1}\left(\frac{1}{\sqrt{4x^2+4x}}\right) dx$$

$$\int \tan^{-1}\left(\frac{1}{\sqrt{(2x+1)^2-1}}\right) dx$$

$$\int_0^{1/2} \sin^{-1}\left(\frac{1}{2x+1}\right) dx$$

$$\sin^{-1}\left(\frac{1}{2x+1}\right) = \theta$$

$$\sin\theta = \frac{1}{2x+1}$$

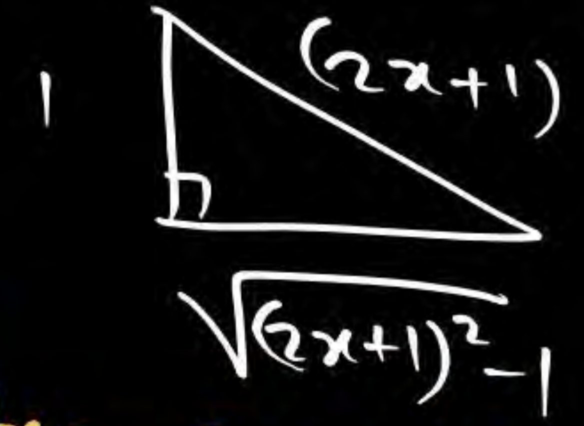
$$\boxed{2x+1 = \operatorname{cosec}\theta}$$

$$\frac{1}{2}\int_{\pi/2}^{\pi/6} \underbrace{\theta}_{\text{I}}\,\underbrace{(-\operatorname{cosec}\theta\cot\theta)}_{\text{II}}\, d\theta$$

$$\frac{1}{2}\left\{\theta\cdot\operatorname{cosec}\theta\Big|_{\pi/2}^{\pi/6} - \int_{\pi/2}^{\pi/6} 1\cdot\operatorname{cosec}\theta\, d\theta\right\}$$

$$\tan^{-1}\frac{1}{\sqrt{(2x+1)^2-1}} = \theta$$

$$\tan\theta = \frac{1}{\sqrt{(2x+1)^2-1}}$$



$$2\,dx = -\operatorname{cosec}\theta\cot\theta\, d\theta$$

$$\frac{1}{2}\left(\frac{\pi}{6}\operatorname{cosec}\frac{\pi}{6} - \frac{\pi}{2}\operatorname{cosec}\frac{\pi}{2}\right) + \frac{1}{2}\int_{\pi/6}^{\pi/2}\operatorname{cosec}\theta\, d\theta$$

$$\frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{2}\right] + \frac{1}{2}\ln\left(\operatorname{cosec}\theta - \cot\theta\right)\Big|_{\pi/6}^{\pi/2}$$

$$-\frac{\pi}{12} + \frac{1}{2}\left[\cancel{\ln(1-0)} - \ln\left(2-\sqrt{3}\right)\right]$$

$$-\frac{\pi}{12} - \frac{1}{2}\ln\left(2-\sqrt{3}\right)$$

$$-\frac{\pi}{12} + \frac{1}{2}\ln\left(2+\sqrt{3}\right)$$

$$p = \frac{1}{2},\ q = 12$$

Brain Teaser-23

[Ans. 2028]

Let f be a differentiable function defined in $[0,1]$ such that $\boxed{f(f(x)) = x}$ and $\boxed{f(0) = 1.}$ If the value of $\int_0^1 (x - f(x))^{2026} dx = \frac{p}{q}$ where p and q are relatively prime, then find the value of $(p+q)$.

$$ff(0) = 0$$

$$\boxed{f(1) = 0}$$

$$I = \int_0^1 (x - f(x))^{2026} dx \longrightarrow (1)$$

$$\boxed{x = f(t)} \Rightarrow dx = f'(t)\,dt$$

$$\int_1^0 (f(t) - t)^{2026} f'(t)\,dt$$

$$-\int_0^1 (t - f(t))^{2026} f'(t)\,dt$$

$$I = -\int_0^1 (x - f(x))^{2026} f'(x)\,dx \longrightarrow (2)$$

$$2I = \int_0^1 (x - f(x))^{2026} (1 - f'(x))\,dx$$

Let $x - f(x) = z$

$$(1 - f'(x))\,dx = dz$$

$$2I = \int_{-1}^{1} z^{2026} dz$$

$$\cancel{2} I = \cancel{2}\int_0^1 z^{2026} dz = \frac{z^{2027}}{2027}\Big|_0^1$$

$$I = \frac{1}{2027}$$

Brain Teaser-24

If $I_r = \int_a^{\pi+a} \left|\frac{1}{r}\sin x + \frac{1}{r+1}\cos x\right| dx$ where $a \in R$ and $r \in N$, [Ans. 4]

then $\lim_{n\to\infty} \sum_{r=1}^{n} \left(I_r^2 - \frac{8}{(r+1)^2}\right)$ equals ________.

$$I_r = \int_a^{\pi+a} (\quad) dx$$

$$I_r = \int_0^{\pi} \left|\frac{1}{r}\sin x + \frac{1}{r+1}\cos x\right| dx$$

$x \to \pi - x$

$$I_r = \int_0^{\pi} \left|\frac{1}{r}\sin x - \frac{1}{r+1}\cos x\right| dx$$

$$I_r = \int_0^{\theta} \left(-\frac{1}{r}\sin x + \frac{1}{r+1}\cos x\right) + \int_{\theta}^{\pi} \left(\frac{1}{r}\sin x - \frac{1}{r+1}\cos x\right) dx$$

$f(x) = |a\sin x + b\cos x|$

Period π

$$\int_a^{a+\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx$$

If $\frac{1}{r}\sin x - \frac{1}{r+1}\cos x > 0$

$\frac{1}{r}\sin x > \frac{1}{r+1}\cos x$

$\tan x > \frac{r}{r+1}$

$\tan x = \frac{r}{r+1}$

$\theta = \tan^{-1} r/_{r+1}$

$$I_r = \frac{1}{r}\cos x + \frac{1}{r+1}\sin x \Big|_0^{\theta} - \left[\frac{1}{r}\cos x + \frac{1}{r+1}\sin x\right]\Big|_{\theta}^{\pi}$$

$$\left(\frac{1}{r}\cos\theta + \frac{1}{r+1}\sin\theta\right) + \frac{1}{r}\cos\theta + \frac{1}{r+1}\sin\theta - \cancel{\frac{1}{r}\cos\pi}$$

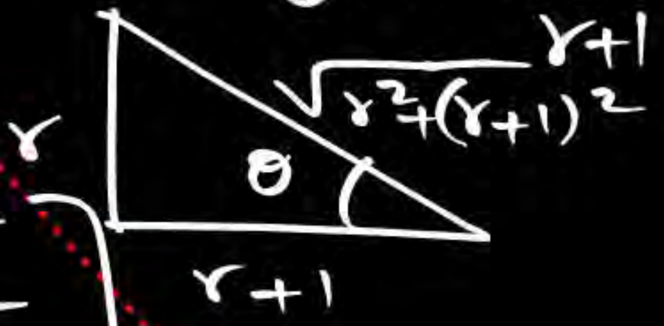
$$-\cancel{\frac{1}{r}}$$

$$\left(\frac{2}{r}\cos\theta + \frac{2}{r+1}\sin\theta\right)$$

$$2\cos\theta\left[\frac{1}{r} + \frac{\tan\theta}{r+1}\right]$$

$$I_r = \frac{2(r+1)}{\sqrt{r^2+(r+1)^2}}\left[\frac{1}{r} + \frac{r}{(r+1)^2}\right]$$

$$\tan\theta = \frac{r}{r+1}$$



$$I_r = \frac{2\cancel{(r+1)}}{\sqrt{r^2+(r+1)^2}}\left[\frac{(r+1)^2+r^2}{r\,(r+1)^2}\right]$$

$$I_r = \frac{2\sqrt{(r+1)^2+r^2}}{r(r+1)} = 2\sqrt{\frac{(r+1)^2+r^2}{r^2(r+1)^2}}$$

$$I_r = 2\sqrt{\frac{1}{r^2}+\frac{1}{(r+1)^2}}$$

$$T_r = (I_r)^2 - \frac{8}{(r+1)^2}$$

$$4\left(\frac{1}{r^2} + \frac{1}{(r+1)^2}\right) - \frac{8}{(r+1)^2}$$

$$\frac{4}{r^2} - \frac{4}{(r+1)^2} = 4\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right]$$

$$T_1 = 4\left[\frac{1}{1^2} - \frac{1}{2^2}\right]$$

$$T_2 = 4\left[\frac{1}{2^2} - \frac{1}{3^2}\right]$$

$$\vdots$$

$$T_n = 4\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]$$

$$S_\infty = 4[1-0]$$
$$= 4$$

Brain Teaser-25

[Ans. 6]

Let $S_n = \sum_{x=1}^{n} x!;\ n \geq 6,\ T = \arcsin\left(\sin\left(\boxed{S_n - 7\left[\frac{S_n}{7}\right]}\right)\right)$.

If $\int_0^1 \frac{T}{\sqrt{1-x^2}}\,dx = \frac{a\pi}{b} - \pi^c$ where $a, b, c \in W;\ b \neq 0$, then find $\left(\frac{b}{c} + a\right)$.

$$S_n - 7\left[\frac{S_n}{7}\right]$$

$$7\left(\frac{S_n}{7} - \left[\frac{S_n}{7}\right]\right)$$

$$7\left\{\frac{S_n}{7}\right\}$$

$$\frac{S_n}{7} = \frac{1! + 2! + 3! + 4! + 5! + 6!}{7} + \left(\frac{7! + \ldots n!}{7}\right)$$

$$\frac{S_n}{7} = \frac{1 + 2 + 6 + 24 + 120 + 720}{7} + I$$

$$\left\{\frac{S_n}{7}\right\} = \left\{\frac{873}{7}\right\} = \left\{\frac{7 \times 124 + 5}{7}\right\} = \left\{124 + 5/7\right\}$$

$$\Rightarrow 5/7$$

$$\begin{array}{r} 124 \\ 7\,\overline{)\,873} \\ \underline{7} \\ 17 \\ \underline{14} \\ 33 \\ \underline{28} \\ 5 \end{array}$$

$$T = \sin^{-1}\sin(7 \times 5/7)$$

$$T = \sin^{-1}\sin 5 = (5-2\pi)$$

$$\int_0^1 (5-2\pi)\frac{dx}{\sqrt{1-x^2}}$$

$$(5-2\pi)\ \sin^{-1}x\Big|_0^1$$

$$(5-2\pi)\,\pi/2$$

$$5\pi/2 - \pi^2$$

$a=5$

$b=2$

$c=2$

$b/c + a = 1+5 = 6$

Brain Teaser-26

[Ans. 6]

If $\int_{-n}^{n} \frac{3\{x\}+1}{\{3x\}+1} dx = 6\ln(4e^2)$, then find the value of n.

[**Note:** $\{k\}$ denotes the fractional part function of k.]

$\{x\} \to 1$
$\{3x\} \to 1/3$ $\Big\}$ LCM $= 1$

$\int_{mT}^{nT} f(t)dt = (n-m)\int_0^T f(t)dt$

$x \in (0,1)$

$3x \in (0,3)$

$$\int_{-n\times 1}^{n\times 1} \frac{3\{x\}+1}{\{3x\}+1} dx$$

$$2n\int_0^1 \frac{3\{x\}+1}{\{3x\}+1} dx$$

$$2n\int_0^1 \frac{3x+1}{3x-[3x]+1} dx$$

$$2n\left[\int_0^{1/3} \frac{3x+1}{3x+1} dx + \int_{1/3}^{2/3} \frac{3x+1}{3x-1+1} dx + \int_{2/3}^{1} \frac{3x+1}{3x-2+1} dx\right]$$

$$2n\left[\frac{1}{3}+\int_{1/3}^{2/3}\left(1+\frac{1}{3x}\right)dx \quad \int_{2/3}^{1}\frac{\cancel{3x-1}+2}{\cancel{3x-1}}\,dx\right]$$

$$2n\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\ln x\Big|_{1/3}^{2/3}+\frac{1}{3}+\frac{2\ln(3x-1)}{3}\Big|_{2/3}^{1}\right.$$

$$2n\left(1+\frac{1}{3}\ln 2+\frac{2}{3}\ln 2\right)$$

$$2n\left(1+\ln 2\right)$$

$$2n\left(\ln e+\ln 2\right)$$

$$2n\left(\ln 2e\right)$$

$$n\left(\ln 4e^2\right)$$

$$\Rightarrow n=6$$

Brain Teaser-27

[Ans. D]

If $\int_{-2}^{-1} \frac{\alpha^2 d\alpha}{\sqrt{\alpha^4 - \alpha^2 + 1}} = I$, then the value of $\int_{-2}^{-1} \frac{x\,dx}{\sqrt{(x - x^{-1})^2 + 1}} + \int_{-1}^{-1/2} \frac{dx}{x^3\sqrt{x^2 + x^{-2} - 1}}$ is equal to:

(A) 0

(B) I

(C) 2I

(D) −2I

$$I_1 = \int_{-2}^{-1} \frac{x\,dx}{\sqrt{\left(x - \frac{1}{x}\right)^2 + 1}}$$

$$I_1 = \int_{-2}^{-1} \frac{x\,dx}{\sqrt{x^2 + \frac{1}{x^2} - 1}}$$

$$I_1 = \int_{-2}^{-1} \frac{|x|\,x\,dx}{\sqrt{x^4 + 1 - x^2}}$$

$$I_1 = -\int_{-2}^{-1} \frac{x^2\,dx}{\sqrt{x^4 + 1 - x^2}}$$

$$\Rightarrow I_1 = -I$$

$$\int_{-1}^{-1/2} \frac{dx}{x^3\sqrt{x^2 + \frac{1}{x^2} - 1}}$$

$$\int_{-1}^{-1/2} \frac{-x\,dx}{x^3\sqrt{x^4 + 1 - x^2}}$$

$$-\int_{-1}^{-1/2} \frac{\frac{1}{x^2}\,dx}{\sqrt{x^4 + 1 - x^2}}$$

$$\boxed{\frac{1}{x} = t}$$

$$-\int_{-1}^{-2} \frac{t^2\left(-\frac{1}{t^2}\right)dt}{\sqrt{\frac{1}{t^4} + 1 - \frac{1}{t^2}}}$$

$$I_2 = \int_{-1}^{-2} \frac{\sqrt{t^4}\, dt}{\sqrt{1+t^4-t^2}}$$

$$\int_{-1}^{-2} \frac{|t^2|\, dt}{\sqrt{t^4-t^2+1}}$$

$$I_2 = -\int_{-2}^{-1} \frac{t^2\, dt}{\sqrt{t^4-t^2+1}}$$

$$I_2 = -I$$

Brain Teaser-28

[Ans. B, C]

Let $I(n)=\int_{2026}^{2026+\frac{1}{n}} x\cos^2(x-2026)\,dx$, then which of the following is/are correct?

(A) $I\left(\frac{1}{\pi}\right)=\frac{(\pi+2026)\pi}{2}$

(B) $I\left(\frac{1}{\pi}\right)=\frac{(\pi+4052)\pi}{4}$

(C) $\lim_{n\to\infty} n.I(n)=2026$

(D) $\lim_{n\to\infty} n.I(n)=2027$

Let $x-2026=\theta$

$\int_0^{1/n}(\theta+2026)\cos^2\theta\,d\theta$

$n=1/\pi$

$I\left(\frac{1}{\pi}\right)=\int_0^{\pi}(\theta+2026)\cos^2\theta\,d\theta \rightarrow (1)$

P-5

$I\left(\frac{1}{\pi}\right)=\int_0^{\pi}(\pi-\theta+2026)\cos^2\theta\,d\theta \rightarrow (2)$

$2I=\int_0^{\pi}(\pi+4052)\cos^2\theta\,d\theta$

$I=(\pi+4052)\int_0^{\pi/2}\cos^2\theta\,d\theta$

$I=(\pi+4052)\pi/4$

$$n\,I(n) = \lim_{n\to\infty} n\int_0^{1/n}(\theta+2026)\cos^2\theta\,d\theta$$

$$\lim_{n\to\infty}\frac{\int_0^{1/n}(\theta+2026)\cos^2\theta\,d\theta}{1/n}$$

$\boxed{\text{Let } 1/n = x}$

$$\lim_{x\to 0}\frac{\int_0^{x}(\theta+2026)\cos^2\theta\,d\theta}{x} = \frac{(x+2026)\cos^2 x\times 1 - 0}{1}$$

L'Hospital's Rule

$$= 2026 \checkmark$$

Brain Teaser-29

[Ans. 168]

If f and g are two functions such that $2f(1) = g(2) = 4$ and $2f(9) = g(10) = 20$ and $\int_0^2 \left(x^2 g\left(f(x^3+1)\right) f'(x^3+1) - 3x^2\right) dx = 0$, then find the value of $\int_4^{20} g^{-1}(x)\,dx$.

$$\int_0^2 \underline{x^2}\, g\left(f(x^3+1)\right) f'\underline{(x^3+1)}\, dx - \int_0^2 3x^2\, dx = 0$$

$$\boxed{f(x^3+1) = t}$$

$$f'(x^3+1)\cdot 3x^2\, dx = dt$$

$$\frac{1}{3}\int_2^{10} g(t)\, dt - x^3\Big|_0^2 = 0$$

$$\frac{1}{3}\int_2^{10} g(t)\, dt - 8 = 0$$

$$\int_2^{10} g(t)\, dt = 24$$

$$\int_2^{10} g(t)\, dt + \int_4^{20} g^{-1}(x)\, dx = 200 - 8$$
$$= 192$$

$$24 + \int_4^{20} g^{-1}(x)\, dx = 192$$

$$I = 192 - 24$$
$$= 180 - 12 = 168.$$

Note

If $g(x)$ is the inverse of a bijective function $f(x)$ in $[a, b]$ such that $f(a) = c$ and $f(b) = d$ then the value of $\boxed{\int_a^b f(x)dx + \int_c^d g(y)dy = (bd - ac)}$

Adv-2022

$$\int_1^2 \underbrace{\log_2(x^3+1)}dx + \int_1^{\log_2 9} (2^x-1)^{\frac{1}{3}}dx$$

$$\int f(x)\,dx + \int f^{-1}(x)\,dx$$

$$= 2 \times \log_2 9 - 1 \times 1$$

$$= 2\log_2 9 - 1$$

$$\boxed{\log_2(x^3+1) = y}$$

$$x^3 + 1 = 2^y$$

$$x = (2^y - 1)^{1/3}$$

$$f^{-1}(x) = (2^x - 1)^{1/3}$$

$$\int_e^{e^2} \ln x\,dx + \int_1^2 e^x\,dx$$

$$= 2e^2 - 1 \times e = \boxed{2e^2 - e}$$

✓

Proof:

$$\int_a^b f(x)\,dx + \int_c^d g(y)\,dy \qquad y = f(x)$$

$$\int_a^b f(x)\,dx + \int_a^b x f'(x)\,dx$$

$$\int_a^b \left(f(x) + x f'(x)\right) dx$$

$$x f(x)\Big|_a^b$$

$$b\underbrace{f(b)} - a f(a)$$

$$bd - ac$$

Brain Teaser-30

[Ans.

Consider the integral $I_1 = \int_1^e (1+x)(x+\ell n x)^{2026} dx$ and $I_2 = \int_{\sin^{-1}\left(\frac{1}{e}\right)}^{\frac{\pi}{2}} (1 + e\sin x + \ln \sin x)^{2027} \cos x dx$, then $I_1 + \frac{eI_2}{2027}$ is equal to:

(A) $\frac{(e+1)^{2027}}{2027}$

(B) $\frac{e(e+1)^{2027}-1}{2027}$ ✓

(C) $\frac{(e+1)^{2026}}{2026}$

(D) $\frac{e(e+1)^{2026}-1}{2026}$

$$I_2 = \int_{\sin^{-1} 1/e}^{\pi/2} (\ln e + e\sin x + \ln \sin x)^{2027} dx$$

$$\int_{\sin^{-1} 1/e}^{\frac{\pi}{2}} (\ln(e\sin x) + e\sin x)^{2027} \cos x dx$$

$$\boxed{e\sin x = t}$$

$$I_2 = \int_1^e \underbrace{1}_{II} \cdot \underbrace{(\ln t + t)^{2027}}_{I} dt/e$$

$$eI_2 = (\ln t + t)^{2027} \cdot t \Big|_1^e - \int 2027 (\ln t + t)^{2026}\left(\frac{1}{t}+1\right) t\, dt$$

$$eI_2 = (e+1)^{2027}e - 1 - 2027\,I_1$$

$$eI_2 + 2027\,I_1 = (e+1)^{2027}e - 1$$

$$\frac{eI_2}{2027} + I_1 = \frac{(e+1)^{2027}e - 1}{2027}$$

(B) ✓

Brain Teaser-31

[Ans. (4)]

If $\int_0^{4\pi} \ln \left|13 \sin y + 3\sqrt{3} \cos y\right| dy = \alpha\pi \ln 7$, then the value of α is ______.

$$\int_0^{\pi/2} \ln(\sin x)\,dx = -\frac{\pi}{2}\ln 2$$

Homework

Redo

& L-3 Attempt

THANK YOU