IOQM MEGA TEST 2026 PAPER-2

ANSWERKEY

1.	(04)	**16.**	(18)
2.	(62)	**17.**	(27)
3.	(04)	**18.**	(96)
4.	(02)	**19.**	(02)
5.	(03)	**20.**	(43)
6.	(06)	**21.**	(02)
7.	(06)	**22.**	(02)
8.	(06)	**23.**	(20)
9.	(52)	**24.**	(51)
10.	(11)	**25.**	(64)
11.	(11)	**26.**	(04)
12.	(06)	**27.**	(26)
13.	(05)	**28.**	(09)
14.	(10)	**29.**	(02)
15.	(14)	**30.**	(84)

HINTS & SOLUTIONS

1. (04)

Since $2002 \equiv 4(\text{mod}9) \equiv 1(\text{mod}9)$ and $2002 = 667 \times 3 + 1$, it follows that $2002^{2002} \equiv 4^{667\times3+1} \equiv 4(\text{mod}9)$. Observe that for positive integers x, the possible residues modulo 9 for x^3 are $0\ \pm1$. Therefore, none of the following

$x_1^3\ x_1^3+x_2^3\ x_1^3+x_2^3+x_3^3$

can have a residue of 4 modulo 9. However, since $2002=10^3+10^3+1^3+1^3$, it follows that

$$2002^{2002}=2002\cdot\left(2002^{667}\right)^3$$

$$=\left(10^3+10^3+1^3+1^3\right)\left(2002^{667}\right)^3$$

$$=\left(10\cdot2002^{667}\right)^3+\left(10\cdot2002^{667}\right)^3+\left(2002^{667}\right)^3+\left(2002^{667}\right)^3$$

This shows that $x_1^3+x_2^3+x_3^3+x_4^4=2002^{2002}$ is indeed solvable. Hence the least integral value of n is 4.

2. (62)

Since $n^3-1=(n-1)\left(n^2+n+1\right)$, we know that $n^2 + n + 1$ divides $n^3 - 1$.

Also, since $n^{2013}-1=\left(n^3\right)^{671}-1$, we also know that $n^2 + n + 1$ divides $n^{2013}-1$.

As $n^{2013}+61=n^{2013}-1+62$

we must have that n^2+n+1 divides $n^{2013}+61$ if and only if n^2+n+1 divides 62 .

Case (i) : If $n^2+n+1=1$, then $n=0,-1$.

Case (ii) : If $n^2+n+1=2$, there is no integer solution for n.

Case (iii) : If $n^2+n+1=31$, then n = 6, –5.

Case (iv) : If $n^2+n+1=62$, there is no integer solution for n.

Thus, all the integer values of n are 0, –1, 6, –5.

Hence the sum of squares is 1 +36 + 25 = 62.

3. (04)

By setting p(0) = 1, we may write S = p(000) + p(001) + p(002) + … + p(999) – p(000)

Since we are now computing the product of non-zero digits only, we may change all the 0's to 1's, i.e. S = p(111) + p(111) + p(112) + … + p(999) – p(111).

Each term is the product of three numbers, and each multiplicand runs through 1, 1, 2, 3, 4, 5, 6, 7, 8, 9(note that 1 occurs twice as all 0's have been changed to 1's). Hence we see that

S = (1 + 1 + 2 + 3 + … + 9) × (1 + 1 + 2 + 3 + … + 9) × (1 + 1 + 2 + 3 + .. + 9) – 1

$= 46^3 - 1$

$= (46 - 1) \times (46^2 + 46 + 1)$

$= 3^2 \times 5 \times 2163$

$= 3^2 \times 5 \times 3 \times 7 \times 103$

It follows that the answer is 103.

4. (02)

The condition is equivalent to $2^{m-1}-1=\dfrac{n!}{2}$.

Since $\dfrac{n!}{2}$ is non-zero, it must be odd, thus n can only be 1, 2 or 3. Trying each case we find the only solutions are m = n = 2 or m = n = 3.

We have $2^m - 2 = n!$ or $2^m = 2 + n!$. If $n \geq 4$, then $2 + n! \geq 26$, thus m > 4. Hence, $2^m\equiv0(\text{mod}4)$ and $n!+2\equiv2(\text{mod}4)$, a contradiction.

Therefore, $n \leq 3$. By direct calculation we get (m, n) = (3, 3) or (m, n) = (2, 2).

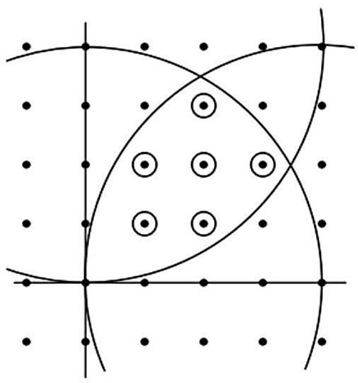
5. (03)

The first interesting numbers are 1005 and 1206. Since $1005 = 3\cdot5\cdot67$ and $1206 = 2\cdot3^2\cdot67$, their greatest common divisor is $3\cdot67 = 201$. Thus, d is a divisor of 201.

On the other hand, if x is an interesting number whose last two digits form the number k, then the first two digits of x form the number 2k, so x = 200k + k = 201k. Thus every interesting number is divisible by 201. Consequently, d = 201.

6. (06)

This is easiest to see by simply graphing the inequalities. They correspond to the (strict) interiors of circles of radius 4 and centers at (0, 0), (4, 0), (0, 4), respectively. So we can see that there are 6 lattice points in their intersection (circled in the figure).



7. (06)

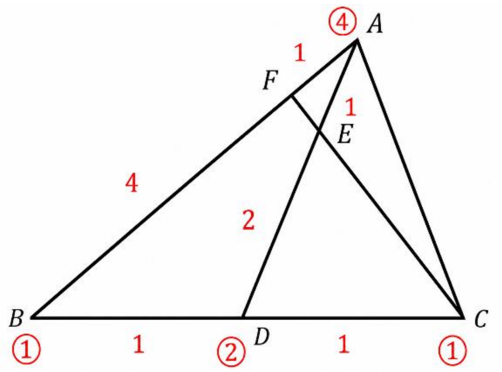
For positive integers a, b, we have

$a! \mid b! \Leftrightarrow a! \le b! \Leftrightarrow a \le b.$

Thus, $((n!)!) \mid (2004!)! \Leftrightarrow (n!)! \le 2004!$

$\Leftrightarrow n! \le 2004 \Leftrightarrow n \le 6.$

8. (06)



Using mass point Geometry

Let the mass at point B be 1.

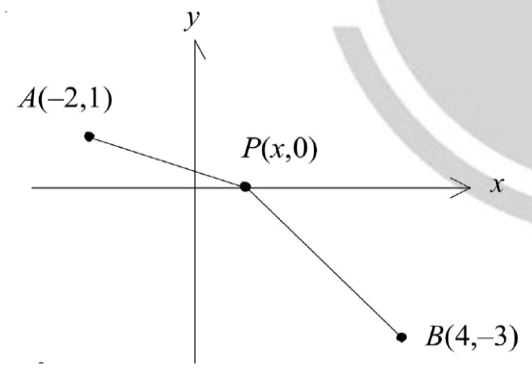
Because AD is the median, BD = CD. Hence, the mass at point C is 1 and the mass at point D is 2.

Because AE/AD = 1 : 3, AE/ED = 1 : 2. Thus the mass at point A is 4.

The ratio of the mass at A to the mass at B is 4 : 1, hence FB = 4AF.

Therefore, AB = AF + FB = 5AF = 6 inches

9. (52)



Rewrite $\sqrt{x^2+4x+5}+\sqrt{x^2-8x+25}$ into the form $\sqrt{(x+2)^2+(0-1)^2}+\sqrt{(x-4)^2+(0+3)^2}$.

If we consider the three points A(–2, 1), B(x, 0) and P(4, –3), then the first term is the distance between A and P while the second is the distance between P and B. The minimum of the sum this occurs when A, P, B are collinear, and the minimum sum is equal to the distance between A and B, which is

$\sqrt{(-2-4)^2+[1-(-3)]^2}=2\sqrt{13}$

$=N^2=(2\sqrt{13})^2$

$\Rightarrow N^2=52$

10. (11)

Let X denote the desired sum. Note that

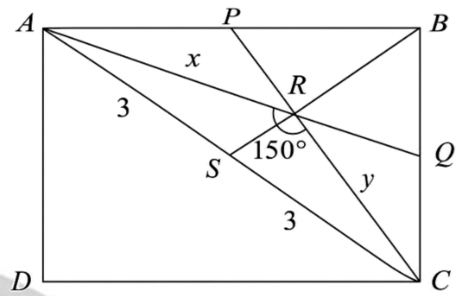
$$X=\frac{1}{4^2}+\frac{1}{4^3}+\frac{2}{4^4}+\frac{3}{4^5}+\frac{5}{4^6}+\ldots$$

$$4X=\frac{1}{4^1}+\frac{1}{4^2}+\frac{2}{4^3}+\frac{3}{4^4}+\frac{5}{4^5}+\frac{8}{4^6}+\ldots$$

$$16X=\frac{1}{4^0}+\frac{1}{4^1}+\frac{2}{4^2}+\frac{3}{4^3}+\frac{5}{4^4}+\frac{8}{4^5}+\frac{13}{4^6}+\ldots$$

so that X + 4X = 16X – 1, and X = 1/11.

11. (11)



Let S be the mid-point of AC. Then R lies on BS with BR : RS = 2 : 1 since R is the centroid of ΔABC. We have BS = 3 (as S is the centre and AB is a diameter of the circumcircle of ΔABC) and so RS = 1. Let AR = x and CR = y. Since the area of ΔARC is $\frac{1}{2}xy\sin 150^\circ=\frac{1}{4}xy$, the area of ΔABC is $\frac{3}{4}xy$ and hence the area of ABCD is $\frac{3}{2}xy$,

To find xy, we first apply the cosine law in ΔARC, ΔARS and ΔCRS to get

$36=x^2+y^2-2xy\cos 150^\circ$

$=\left(3^2+1^2-2\cdot3\cdot1\cdot\cos\angle RSA\right)+$

$\left(3^2+1^2-2\cdot3\cdot1\cdot\cos\angle RSC\right)+\sqrt{3}xy$

$=20+\sqrt{3}xy$

which gives (Note that $\cos\angle RSA=-\cos\angle RSC$)It follows that $xy=\frac{16}{\sqrt{3}}$ and hence the answer is

$\frac{3}{2}\left(\frac{16}{\sqrt{3}}\right)=8\sqrt{3}$.

12. (06)

Write $x:=x-\lfloor x\rfloor$. Then

$$\left\lfloor\frac{1}{2}+\left\{\frac{x}{2}\right\}\right\rfloor=\begin{cases}0, & \text{if}\ \frac{x}{2}<\frac{1}{2}\\ 1, & \text{otherwise}\end{cases}=\left\lfloor 2\left\{\frac{x}{2}\right\}\right\rfloor.$$

Thus, we have

$$\left\lfloor \frac{1}{2}+\frac{x}{2}\right\rfloor=\left\lfloor 2\left\{\frac{x}{2}\right\}\right\rfloor+\left\lfloor\frac{x}{2}\right\rfloor=\lfloor x\rfloor-\left\lfloor\frac{x}{2}\right\rfloor$$

Applying the above result for $x=\frac{n}{2^k}$,

$$\sum_{k=0}^{\infty}\left\lfloor\frac{n+2^k}{2^{k+1}}\right\rfloor=\sum_{k=0}^{\infty}\left(\left\lfloor\frac{n}{2^k}\right\rfloor-\left\lfloor\frac{n}{2^{k+1}}\right\rfloor\right)$$

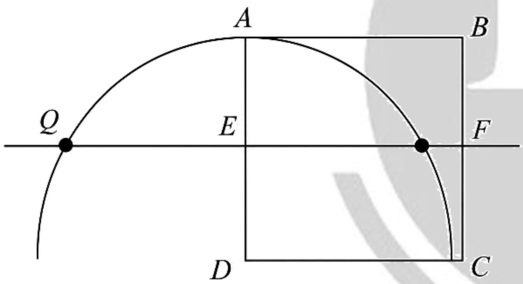
$$=\left\lfloor\frac{n}{2^0}\right\rfloor$$

$= n$

In particular, when n = 20121, the infinite series converges to 20121 .

13. **(05)**

As Q is the circumcentre of ΔBPC, it lies on the perpendicular bisector of BC, and we have PQ = CQ. As D is the circumcentre of ΔPQA, we have DQ = DA = 1 and hence Q lies on the circle with centre D and radius 1. These two loci intersect at two points, giving two possible positions of Q. We choose the one outside ΔABC in order to maximise PQ(= CQ) .



Let E and F be the midpoints of AD and BC respectively. Since ΔAPQ is equilateral with side length 1, we have $EQ=\frac{\sqrt{3}}{2}$.

Appling Pythagoras Theorem in ΔCFQ,we have Join QD toget EQ

$$\rightarrow PQ^2=CQ^2=CF^2+FQ^2$$

$$=\left(\frac{1}{2}\right)^2+\left(1+\frac{\sqrt{3}}{2}\right)^2=2+\sqrt{3}$$

14. **(10)**

Introduce letters for the digits of n and m so the problem has the form

$$\begin{array}{r} abcd \\ \underline{-abc} \\ 2010 \end{array}$$

Since abc is at most 999, the digit a must be 2 or 3. However, if a = 3, then n – m > 3000 – 399 = 2601 > 2010, so a = 2. Now the problem has the form

$$\begin{array}{r} 2bcd \\ \underline{-2bc} \\ 2010 \end{array}$$

Therefore, b must be 2 or 3. If b = 3, then n – m > 2300 – 239 = 2061 > 2010, so b = 2, and the problem has the form

$$\begin{array}{r} 22cd \\ \underline{-22c} \\ 2010 \end{array}$$

Again, c must be 3 or 4, but if c = 4, then n – m > 2240 – 224 = 2016 > 2010, so c = 3, and the problem has the form

$$\begin{array}{r} 223d \\ \underline{-223} \\ 2010 \end{array}$$

Now d must be 3. Thus n = 2233.

Sum of the digits of n = 10

15. **(14)**

The given condition is equivalent to

$$100P+10E+R+100P+10R+U+100P+10U+E+2009$$
$$=1000P+100E+10R+U$$

$$700P+89E=2009+\overline{UR}$$

Since $2009+\overline{UR}\le 2009+98$ and $89E\ge 89$, we deduce that $P\le 2$.

If P = 1, then we have $89E=1309+\overline{UR}$. We do not obtain a solution because $89EC\le 89\cdot 9=801$

If P = 2, then we obtain $89E=609+\overline{UR}$.

Since $609+\overline{UR}\ge 609+12=621$ and $609+\overline{UR}\le 609+98=707$,

we deduce that $\frac{621}{89}\le \overline{E}\le\frac{707}{89}$, hence E = 7.

It results that $\overline{PERU}=2741$

and P + E + R + U = 14.

16. **(18)**

Since D < A, when A is subtracted from D in the ones' column, there will be a borrow from C in the tens' column. Thus, D + 10 – A = C. Next, consider the subtraction in the tens' column, (C – 1) – B. Since C < B, there will be a borrow from the hundreds' column, so (C – 1 + 10) – B = A. In the hundreds' column, B – 1 ≥ C, so we do not need to borrow from A in the thousands' column. Thus, (B – 1) – C = D and A – D = B. We thus have a system of four equations in four variables A, B, C, D, and solving by standard methods (e.g. substitution) produces (A, B, C, D) = (7, 6, 4, 1).

17. (27)

We claim that for any odd n, $a_n = n$. The proof is by induction. To get the base cases n = 1, 3, we compute $a_1 = 1, a_2 = \lfloor 2^3/1 \rfloor = 8, a_3 = \lfloor 3^3/8 \rfloor = 3$.

And if the claim holds for odd $n \ge 3$,

then $a_{n+1} = \lfloor (n+1)^3/n \rfloor = n^2 + 3n + 3$, so

$$a_{n+2} = \left\lfloor (n+2)^3 / \left(n^2+3n+3\right) \right\rfloor$$

$$= \left\lfloor \left(n^3+6n^2+12n+8\right) / \left(n^2+3n+2\right) \right\rfloor$$

$$= \left\lfloor n+2+\frac{n^2+3n+2}{n^2+3n+3} \right\rfloor = n+2.$$

So the claim holds, and in particular, $a_{999} = 999$.

18. (96)

Let AB = c cm, BC = a cm and CA = b cm. By the cosine rule,

$$a^2 = b^2 + c^2 - 2bc\cos 60^\circ = b^2 + c^2 - bc = (b-c)^2 + bc$$

$$= 4^2 + bc = 16 + bc$$

Let O be the centre of the circle. Then $\angle BOC = 2\angle A = 120^\circ$.

Moreover, OB = OC = 4 cm. Therefore, by the cosine rule,

$$a^2 = 4^2 + 4^2 - 2\times 4\times 4\cos 120^\circ = 48 \Rightarrow a = 4\sqrt{3}.$$

Then $bc = a^2 - 4 = 48 - 16 = 32$. Thus,

$$\text{area}(\Delta ABC) = \frac{1}{2}bc\sin 60^\circ = \frac{1}{2}\times 32\times\frac{\sqrt{3}}{2} = 8\sqrt{3}.$$

Therefore, $x = 8\sqrt{3}$ and $x^2 = (8\sqrt{3})^2 = 192.$

$$\Rightarrow \frac{x^2}{2} = 96$$

19. (02)

For reference the given equation is

$n! + 8 = 2^k.$

Case-1: $n \ge 6$

Observe that n! is a multiple of $6! = 2^4 \times 3^2 \times 5$. Hence n! = 16x for some positive integer x. Therefore

$n! + 8 = 8(2x + 1)$

But the RHS of the above equation cannot be a power of 2 because 2x + 1 is an odd integer that is greater than 1. Hence there are no solutions in this case.

Case-2: $n \le 5$

We simply tabulate the values of n! + 8 and check which ones are powers of 2.

n	n! + 8	power of 2?
1	9	no
2	10	no
3	14	no
4	32	2^5
5	128	2^7

They there are only 2 points.

20. (43)

Let M = a + b + c + d and N = $a^2 + b^2 + c^2 + d^2$; we have $MN^2 = 2023$. Factoring $2023 = 7\cdot 17^2$, we have $(M, N) \in \{(2023,1),(7,17)\}$. The former is clearly absurd, so M = 7 and N = 17.

We now look for ways to write 17 as a sum of four squares. WLOG $a \le b \le c \le d$. Clearly $3 \le d \le 4$.

If d = 4, we must have $1 = a^2 + b^2 + c^2$ which only admits (a, b, c) = (0, 0, 1), but $0+0+1+4 \ne 7$, so this case fails.

If d = 3, we must have $8 = a^2 + b^2 + c^2$. It's easy to see we must have (a, b, c) = (0, 2, 2), and indeed 0 + 2 + 2 + 3 = 7.

Thus (a, b, c, d) = (0, 2, 2, 3), and thus $a^3 + b^3 + c^3 + d^3 = 43$.

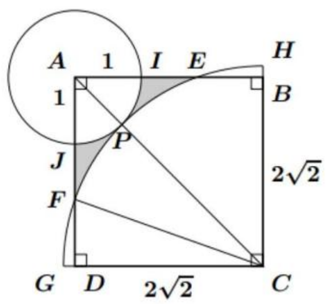
Remark: Rearranging this gives the rather pretty $(2+0+2+3)\left(2^2+0^2+2^2+3^2\right) = 2023.$

21. (02)

From the Pythagorean Theorem we get $AC = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{8+8} = \sqrt{16} = 4$.

Since CPP is a radius of the larger circle, and AP = 1 we get CP = CA – PA = 4 – 1 = 3, therefore the radius of the large circle is 3.

Using [A] to represent the area of figure A we get



$[GDF] = [FCG] - [FCD]$

$= \frac{\alpha}{2\pi}\cdot\pi(3)^2 - \frac{\sqrt{3^2-(2\sqrt{2})^2}\cdot 2\sqrt{2}}{2}$

$= \frac{9}{2}\alpha - \sqrt{2}$

where $\alpha = \angle FCG = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$. Next, we have

$[PCDF] = [PCG] - [GDF] = \frac{9\pi}{8} - \frac{9}{2}\alpha + \sqrt{2}$.

Thus $[JPF] = [ACD] - [APJ] - [PCDF]$

$= 4 - \frac{\pi}{8} - \left(\frac{9\pi}{8} - \frac{9}{2}\alpha + \sqrt{2}\right) = 4 - \sqrt{2} - \frac{5\pi}{4} + \frac{9}{2}\alpha$.

By symmetry [IPE] = [JPE] thus the area of the regions inside the square but not inside a circle is 2[JPE], or

$8 - 2\sqrt{2} - \frac{5\pi}{2} + 9\alpha$

$= 8 - 2\sqrt{2} - \frac{5\pi}{2} + 9\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

22. (02)

Let p, q be prime numbers. If we require that $p + q > 2$ has to be prime, either $p = 2$ or $q = 2$ to ensure that $p + q$ is odd. Let us consider two cases separately:

Case 1: $q = 2$

In this case, p must be an odd prime number $p \geq 3$. Clearly, $p = 3$ is a solution since $p + q = 5$, $p + q^2 = 7$, $p + q^3 = 11$ and $p + q^4 = 19$ are prime numbers. For $p > 3$ we will study the values of the expressions modulo 3. Let $p \equiv t \pmod{3}$ where $t = 1$ or $t = 2$ since $p > 3$ is a prime. Then

$p + q \equiv t + 2 \pmod{3}$

$p + q^2 \equiv t + 4 \equiv t + 1 \pmod{3}$

and therefore, either $p + q$ or $p + q^2$ is divisible by three, thus they both cannot be prime.

Case 2: $p = 2$

In this case, q must be an odd prime number $q \geq 3$. Clearly, $q = 3$ is a solution since $p + q = 5$, $p + q^2 = 11$, $p + q^3 = 29$ and $p + q^4 = 83$ are prime numbers. For $q > 3$ we will study the values of the expressions modulo 3. Let $q \equiv t \pmod{3}$ where $t = 1$ or $t = 2$ since $q > 3$ is a prime, then

$p + q \equiv 2 + t \pmod{3}$

$p + q^2 \equiv 2 + t^2 \pmod{3}$

If $t = 1$, $p + q$ is divisible by three and if $t = 2$, $p + q^2$ is divisible by three, therefore, either $p + q$ or $p + q^2$ is divisible by three, thus they both cannot be prime.

Summarizing, the only pairs (p, q) of prime numbers which satisfy the condition of the problem are (2, 3) and (3, 2).

23. (20)

The solutions are 1, 3, 4, and 12.

Note that if m is a positive integer and $m = ab$ where $1 \leq a \leq b \leq m$, then $a \leq \sqrt{m}$. Thus, for each $d = 1, 2, \ldots, \lfloor\sqrt{m}\rfloor$, there is at most one positive integer d' such that dd' = m. Hence, we have

$\tau(m) \leq 2\lfloor\sqrt{m}\rfloor \leq 2\sqrt{m}$ …(1)

Furthermore, if d is a positive divisor of m with $d < m$, then clearly $d \leq \frac{m}{2}$, so $d \leq \left\lfloor\frac{m}{2}\right\rfloor$.

Hence,

$$\sigma(m) \leq m + \sum_{k=1}^{\left\lfloor\frac{m}{2}\right\rfloor} k \leq m + \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{m}{2}+1\right) = \frac{1}{8}\left(m^2 + 10m\right)$$ …(2)

Now, if n is a solution to the given problem, then by (1), (2) and the fact that $\frac{1}{8}\left(m^2+10m\right)$ is an increasing function, we have

$n = \sigma(\tau(n)) \leq \max_k\{\sigma(k) | 1 \leq k \leq 2\sqrt{n}\}$

$\leq \frac{1}{8}\left((2\sqrt{n})^2 + 10(2\sqrt{2})\right) = \frac{1}{2}\left(n + 5\sqrt{n}\right)$

from which we easily deduce that $n \leq 25$.

Direct checking for n = 1, 2, ….,25 then reveal that only 1, 3, 4 and 12 satisfy the given condition and our proof is complete.

24. (51)

Each term in the sum has the form

$k!\left(k^2+k+1\right) = k!\left(\left(k^2+2k+1\right)-k\right)$

$= k!\left((k+1)^2 - k\right)$

$= k!(k+1)^2 - k!k$

$= (k+1)!(k+1) - k!k$

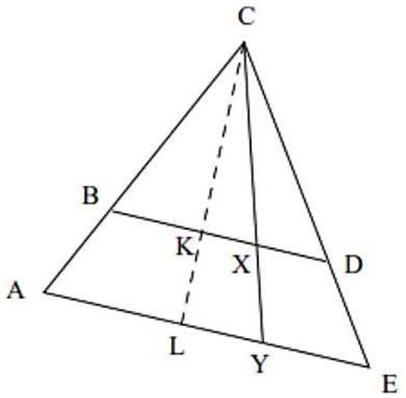
Therefore the sum is telescoping:

$S = (2!\cdot 2 - 1!\cdot 1) + (3!\cdot 3 - 2!\cdot 2)$

$$+\cdots+(100!\cdot 100-99!\cdot 99)+(101!\cdot 101-100!\cdot 100)$$
$$=101!\cdot 101-1!\cdot 1=101!\cdot 101-1$$

so $S + 1 = 101!\cdot 101$, thus $\dfrac{S+1}{101!}=101$.

25. (64)



Draw the altitude from C to AE, intersecting line BD at K and line AE at L. Then CK is the altitude of triangle BCD, so triangles CKX and CLY are similar. Since CY/CX = 8/5, CL/CK = 8/5. Also triangles CKB and CLA are similar, so that CA/CB = 8/5, and triangles BCD and ACE are similar, so that AE/BD = 8/5. The area of ACE is (1/2)(AE)(CL), and the area of BCD is (1/2)(BD)(CK), so the ratio of the area of ACE to the area of BCD is 64/25. Therefore, the ratio of the area of ABDE to the area of BCD is 39/25.

26. (04)

The number of zeroes in base 7 is the total number of factors of 7 in $1\cdot 2\cdots n$, which is

$$\lfloor n/7\rfloor+\lfloor n/7^2\rfloor+\lfloor n/7^3\rfloor+\cdots$$

The number of zeroes in base 8 is $\lfloor a\rfloor$, where

$$a=\left(\lfloor n/2\rfloor+\lfloor n/2^2\rfloor+\lfloor n/2^3\rfloor+\cdots\right)/3$$

is one-third the number of factors of 2 in the product n!. Now $\lfloor n/2^k\rfloor/3\ge\lfloor n/7^k\rfloor$ for all k, since $\left(n/2^k\right)/3\ge n/7^k$. But n can only be picante if the two sums differ by at most 2/3, so in particular this requires

$$\left(\lfloor n/2^2\rfloor\right)/3\le\lfloor n/7^2\rfloor+2/3$$

$\Leftrightarrow\lfloor n/4\rfloor\le 3\lfloor n/49\rfloor+2$. This cannot happen for $n\ge 12$; checking the remaining few cases by hand, we find n = 1, 2, 3, 7 are picante, for a total of 4 values.

27. (26)

Cross-multiplying gives (x + 7)P(2x) = 8xP(x + 1). If P has degree n and leading coefficient c, then the leading coefficients of the two sides are $2^n c$ and 8c, so n = 3. Now x = 0 is a root of the right-hand side, so it's a root of the left-hand side, so that P(x) = xQ(x) for some polynomial Q $\Rightarrow$ 2x(x + 7)Q(2x) = 8x(x + 1) Q(x + 1) or (x + 7)Q(2x) = 4(x + 1)Q(x + 1). Similarly, we see that x = –1 is a root of the left-hand side, giving Q(x) = (x + 2)R(x) for some polynomial R $\Rightarrow$ 2(x + 1)(x + 7)R(2x) = 4(x + 1)(x + 3)R(x + 1), or (x + 7)R(2x) = 2(x + 3) R(x + 1). Now x = –3 is a root of the left-hand side, so R(x) = (x + 6)S(x) for some polynomial S.

At this point, P(x) = x(x + 2)(x + 6)S(x), but P has degree 3, so S must be a constant. Since P(1) = 1, we get S = 1/21, and then P(–1) = (–1)(1)(5)/21 = –5/21.

28. (09)

Method 1

If $n^2 + 2016n = m^2$, where n and m are positive integers, then m = n + k for some positive integer k. Then $n^2 + 2016n = (n + k)^2$. So 2016 n = 2nk + k^2, or n = $k^2/(2016 - 2k)$. Since both n and k^2 are positive, we must have 2016 – 2k > 0, or 2k < 2016. Thus $1\le k\le 1007$.

As k increases from 1 to 1007, k^2 increases and 2016 – 2k decreases, so n increases. Conversely, as k decreases from 1007 to 1, k^2 decreases and 2016 – 2k increases, so n decreases. If we take k = 1007, then n = $1007^2/2$, which is not an integer. If we take k = 1006, then n = $1006^2/4 = 503^2$. So $n \le 503^2$.

If k = 1006 and n = 503^2, then $(n + k)^2 = (503^2 + 1006)^2 = (503^2 + 2 \times 503)^2$
$= 503^2(503 + 2)^2$
$= 503^2(503^2 + 4 \times 503 + 4)$
$= 503^2(503^2 + 2016) = n^2 + 2016n$.

So $n^2 + 2016n$ is indeed a perfect square. Thus 503^2 is the largest value of n such that $n^2 + 2016n$ is a perfect square.

Since $503^2 = (500 + 3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2$ = 250000 + 3000 + 9 = 253009, the remainder when n is divided by 1000 is 9.

Method 2

If $n^2 + 2016n = m^2$, where n and m are positive integers, then $m^2 = (n + 1008)^2 - 1008^2$.

So $1008^2 = (n + 1008 + m)(n + 1008 - m)$ and both factors are even and positive.

Hence $n + 1008 + m = 1008^2/(n + 1008 - m) \leq 1008^2/2$.

Since m increases with n, maximum n occurs when $n + 1008 + m$ is maximum.

If $n + 1008 + m = 1008^2/2$, then $n + 1008 - m = 2$. Adding these two equations and dividing by 2 gives $n + 1008 = 504^2 + 1$ and $n = 504^2 - 1008 + 1 = (504 - 1)^2 = 503^2$.

If $n = 503^2$, then $n^2 + 2016n = 503^2(503^2 + 2016)$. Now $503^2 + 2016 = (504 - 1)^2 = 2016 = 504^2 + 1008 + 1 = (504 + 1)^2 = 505^2$.

So $n^2 + 2016n$ is indeed a perfect square. Thus 503^2 is the largest value of n such that $n^2 + 2016n$ is a perfect square.

Since $503^2 = (500 + 3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009$, the remainder when n is divided by 1000 is 9.

29. (02)

Let p_n be the number of n-digit 'good' positive integers ending with 1 and q_n be the number of n-digit 'good' positive integers ending with 2. Then $a_n = p_n + q_n$. Furthermore a 'good' positive integer must end with 21, 211, 2111, 12 or 122. This leads to the recurrence relations $p_n = q_{n-1} + q_{n-2} + q_{n-3}$ as well as $q_n = p_{n-1} + p_{n-2}$. It follows that

$$p_n = (q_{n-2} + q_{n-3}) + (q_{n-3} + q_{n-4}) + (q_{n-4} + q_{n-5})$$
$$= q_{n-2} + 2q_{n-3} + 2q_{n-4} + q_{n-5}$$
$$q_n = (p_{n-2} + p_{n-3} + p_{n-4}) + (p_{n-3} + p_{n-4} + p_{n-5})$$
$$= p_{n-2} + 2p_{n-3} + 2p_{n-4} + p_{n-5}$$

Adding gives $a_n = a_{n-2} + 2a_{n-3} + 2a_{n-4} + a_{n-5}$.

In particular $a_{10} = a_8 + a_5 + 2(a_7 + a_6)$ and so

$$\frac{a_{10} - a_8 - a_5}{a_7 + a_6} = 2$$

30. (84)

Suppose Ann has the last turn. Let n be the number of turns that Bob has. Then the number of sweets that he takes is $2 + 4 + 6 + \ldots + 2n = 2(1 + 2 + \ldots + n) = n(n + 1)$. So the total number of sweets is $2n(n + 1)$.

Suppose Bob has the last turn. Let n be the number of turns that Ann has. Then the number of sweets that she takes is $1 + 3 + 5 \ldots + (2n - 1) = n^2$. So the total number of sweets is $2n^2$.

So half the number of sweets is $n(n + 1)$ or n^2. Applying the same sharing procedure to half the sweets gives, for some integer m, one of the following four cases:

1. $n(n + 1) = 2m(m + 1)$
2. $n(n + 1) = 2m^2$
3. $n^2 = 2m(m + 1)$
4. $n^2 = 2m^2$.

In the first two cases we want n such that $n(n + 1) < 500$. So $n \leq 21$.

In the first case, since 2 divides m or m + 1, we also want 4 to divide $n(n + 1)$. So $n \leq 20$.

Since $20 \times 21 = 420 = 2 \times 14 \times 15$, the total number of sweets could be $2 \times 420 = 840$.

In the second case $\frac{1}{2}n(n+1)$ is a perfect square.

So $n < 20$. In the last two cases we look for n so that $n^2 > 840/2 = 420$.

We also want n even and $n^2 < 500$. So $n = 22$.

In the third case, $m(m+1) = \frac{1}{2} \times 22^2 = 242$ but $15 \times 16 = 240$ while $16 \times 17 = 272$.

In the fourth case, $m^2 = 242$ but 242 is not a perfect square.

So the maximum total number of sweets is 840.