Hyperbola

01 Definition of Hyperbola

A hyperbola is the particular case of the conic

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

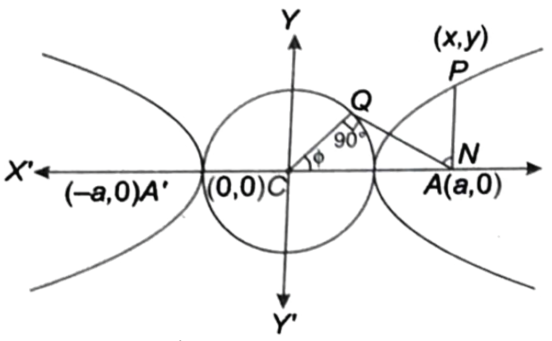
When, $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

i.e., $\Delta \neq 0$ and $h^2 > ab$

03 Auxiliary Circle of Hyperbola

Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola with centre C and transverse axis $A'A$. Therefore, circle drawn with centre C and segment $A'A$ as a diameter is called auxiliary circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Equation of the auxiliary circle is $x^2 + y^2 = a^2$.



Let $\angle QCN = \phi$

Here, P and Q are the corresponding points on the hyperbola and the auxiliary circle $(0 \le \phi < 2\pi)$.

Note : Here ϕ is called eccentric angle of point P .

04 Parametric equations of hyperbola

The equations $x = a\ sec\ \phi$ and $y = b\ tan\ \phi$ are known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

This $(a\ sec\ \phi,\ b\ tan\ \phi)$ lies on the hyperbola for all values of ϕ.

Position of points Q on auxiliary circle and the corresponding point P which describes the hyperbola and $0 \le \phi < 2\pi$

ϕ varies from	Q(a co ϕ, b sin ϕ)	P($a\ sec\phi$, $b\ tan\phi$)
$0\ to\ \frac{\pi}{2}$	I	I
$\frac{\pi}{2}\ to\ \pi$	II	III
$\pi\ to\ \frac{3\pi}{2}$	III	II
$\frac{3\pi}{2}\ to\ 2\pi$	IV	IV

02 Hyperbola

Hyperbola Fundamentals	Hyperbola	CONJUGATE HYPERBOLA
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ or$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Centre	$(0,0)$	$(0,0)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Eccentricity	$e = \sqrt{\left(\frac{a^2 + b^2}{a^2}\right)}$	$e = \sqrt{\left(\frac{a^2 + b^2}{b^2}\right)}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parametric co-ordinates	$(a\ sec\ \phi,\ b\tan\ \phi)$, $0 \le \phi < 2\pi$	$(b\ sec\ \phi,\ a\tan\ \phi)$, $0 \le \phi < 2\pi$
Focal radii	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii $(S'P - SP)$	$2a$	$2b$
Tangents at the vertices	$x = -a, x = a$	$y = -b,\ y = b$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

- If e and e' are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.
- The foci of a hyperbola and its conjugate are concyclic.

05 Position of a Point with Respect to a Hyperbola

Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then , $P(x_1, y_1)$ will lie inside, on or outside the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive, zero or negative.

06 Equation of the Chord joining Two Points on the Hyperbola

The equation of the chord joining the points $P(a\sec\phi_1, b\tan\phi_1)$ and $(a\sec\phi_2, b\tan\phi_2)$

$$y - b\tan\phi_1 = \frac{b\tan\phi_2 - b\tan\phi_1}{a\sec\phi_2 - a\sec\phi_1}(x - a\sec\phi_1) \quad \frac{x}{a}\cos\left(\frac{\phi_1-\phi_2}{2}\right) - \frac{y}{b}\sin\left(\frac{\phi_1+\phi_2}{2}\right) = \cos\left(\frac{\phi_1+\phi_2}{2}\right)$$

Note

- If the chord joining two points $(a\sec\phi_1, b\tan\phi_1)$ and $(a\sec\phi_2, b\tan\phi_2)$ passes through the focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = \frac{1-e}{1+e}$ or $\frac{1+e}{1-e}$.

07 Intersection of a Line and a Hyperbola

The straight line $y = mx + c$ will cut the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in two points may be real, coincident or imaginary according as $c^2 >, =, < a^2m^2 - b^2$.

08 Condition for tangency

If straight line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c^2 = a^2m^2 - b^2$.

9 Equations of Tangent in Different Forms

01 **Point form :** The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

02 **Parametric form :** The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec\phi_1, b\tan\phi_1)$ is $\frac{x}{a}\sec\phi - \frac{y}{b}\tan\phi = 1$

Note

Point of intersection of tangents drawn at point on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\left(\frac{a\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}, \frac{b\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}\right).$$

03 **Slope form :** The equations of tangents of slope m to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y = mx \pm \sqrt{a^2m^2-b^2}$ and the co-ordinates of points of contacts are $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$ where $c^2 = a^2m^2 - b^2$.

Clearly for the existence of tangent with slope m to the hyperbola $|m| > \frac{b}{a}(\text{where } a, b > 0)$.

Note

- If the straight line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $a^2l^2 - b^2m^2 = n^2$.
- If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.
- Two tangents can be drawn from an outside point to a hyperbola.

10 Equation of Pair of Tangents

If $P(x_1, y_1)$ be any point outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then a pair of tangents PQ, PR can be drawn to it from P

The equation of pair of tangents PQ and PR is $SS_1 = T^2$, where,

$$S = \frac{x^2}{a^2}-\frac{y^2}{b^2} - 1,\ S_1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1,\ T = \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1.$$

Director circle : The director circle is the locus of points from which perpendicular tangents are drawn to the given hyperbola. The equation of the directo r circle of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2-b^2$.

11 Equations of Normal in Different Forms

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Point form : The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2}{y_1}=a^2+b^2.$$

02

Parametric form : The equation of normal at $(a\ sec\theta, b\ tan\theta)$, to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$ax\ cos\theta + by cot\theta = a^2+b^2.$$

03

Slope form : The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of the slope m of

$$y = mx \mp \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

04

Condition of normality : If $y = mx + c$ is the normal of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $c = \mp\frac{m(a^2+b^2)}{\sqrt{a^2-m^2b^2}}$ or $c^2=\frac{m^2(a^2+b^2)^2}{(a^2-m^2b^2)}$, which is condition of normality.

12 Important Tips

01

In general, four normals can be drawn to a hyperbola from any point and if $\alpha,\beta,\gamma,\delta$ be the eccentric angles of these four co-normal points, then $\alpha+\beta+\gamma+\delta$ is an odd multiple of π.

02

if α,β,γ are the eccentric angles of three points on the hyperbola. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, at which the normals are concurrent, then

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$$

03

If the normal at P meets the transverse axis in G, then $SG = e.SP$.

Also the tangent and normal bisect the angle between the focal distances of P

04

If the normal at P meets the transverse axis in G and conjugate axis at g, then

$$PG : Pg = b^2 : a^2.$$

13 Equation of Chord of Contact of Tangents drawn from a Point to a Hyperbola

Let PQ and PR be tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ drawn from any external point $P(x_1, y_1)$.

Then equation of chord of contact QR is or $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

or $T=0\ (At\ x_1, y_1)$

14 Equation of the Chord of the Hyperbola whose Mid-point (x_1, y_1) is given

Equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, bisected at the given point (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1 = \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1, \quad \text{i.e., } T=S_1.$$

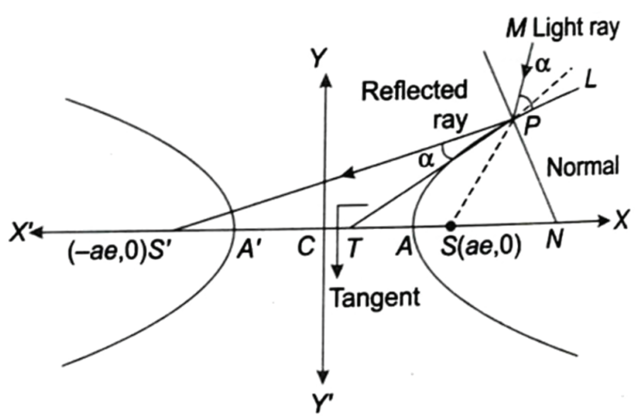
Note

The length of chord cut off by hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ from the line

$$y = mx + c \text{ is } \frac{2ab\sqrt{\left[c^2-(a^2m^2-b^2)\right](1+m)^2}}{(b^2-a^2m^2)}.$$

15 Reflection Property of the Hyperbola

If an incoming light ray passing through one focus (S) strike convex side of the hyperbola, then it will get reflected towards other focus (S'). $\angle TPS' = \angle LPM = \alpha$



Note

Hyperbola and ellipse are called orthogonal curves to each other iff they are confocal (i.e., they have the same foci).

16 Asymptotes of a Hyperbola

The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$$y=\pm\frac{b}{a}x \text{ or } \frac{x}{a}\pm\frac{y}{b}=0.$$

Important Tips

The product of length of perpendiculars drawn from any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to the asymptotes is $\frac{a^2b^2}{a^2+b^2}$.

- The tangent at any point P on hyperbola if meet its asymptotes at Q and R, then :

(i) the midpoint of QR is always P, (ii)area of triangle QCR is always *"ab"* where C is the of hyperbola and 2a= lengthof transverse axis, 2b= length of conjugate axis of hyperbola.

17 Rectangular or Equilateral Hyperbola

(i) Definition : A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola. The eccentricity of rectangular hyperbola is always $\sqrt{2}$. The general equation of second degree represents a rectangular hyperbola if $\Delta \neq 0, h^2 > ab$ and coefficient of x^2+ coefficient of $y^2 = 0$.

18 Parametric coordinates of a point ton he hyperbola $xy = c^2$

If t is non-zero variable, the coordinates of any point on the rectangular hyperbola $xy = c^2$ can be written as $\left(ct, \frac{c}{t}\right)$. The point $\left(ct, \frac{c}{t}\right)$ on the hyperbola $xy = c^2$ is generally referred as the point 't'.

For rectangular hyperbola the coordinates of foci are $(\pm a\sqrt{2}, 0)$ and directrices are $x = \pm a\sqrt{2}$.

For rectangular hyperbola $xy = c^2$, the coordinates of foci are $(\pm c\sqrt{2}, \pm c\sqrt{2})$ and directrices are $x + y = \pm c\sqrt{2}$.

19 Equation of the chord joining points t_1 and t_2

The equation of the chord joining two points $\left(ct_1, \frac{c}{t_1}\right)$ and $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy = c^2$ is $y - \frac{c}{t_1} = \dfrac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}(x - ct_1)$

$$\Rightarrow x + yt_1t_2 = c(t_1 + t_2).$$

20 Equation of tangent in different forms

(i) Point form : The equation of tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is $xy_1 + yx_1 = 2c^2$ or $\frac{x}{x_1} + \frac{y}{y_1} = 2$.

(ii) Parametric form : The equation of the tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $\frac{x}{t} + yt = 2c$ On replacing x_1 by ct and y_1 by $\frac{c}{t}$ on the equation of the tangent at (x_1, y_1), $i.e., xy_1 + yx_1 = 2c^2$, we get $\frac{x}{t} + yt = 2c$.

Note

Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

21 Equation of the chord joining points t_1 and t_2

(i) Point form : The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is $xx_1 - yy_1 = x_1^2 - y_1^2$.

(ii) Parametric form : The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is $xt^3 - yt - ct^4 + c = 0$.

Note

- The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree in t. So, in general,
- If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again in 't' then $t' = -\frac{1}{t^3}$.
- Point of intersection of normals at 't_1' and 't_2' is $\left(\frac{c\{t_1t_2(t_1^2 + t_1t_2 + t_2^2) - 1\}}{t_1t_2(t_1 + t_2)}, \frac{c\{t_1^3t_2^3(t_1^2 + t_1t_2 + t_2^2)\}}{t_1t_2(t_1 + t_2)}\right)$.
- A triangle has its vertices on a rectangular hyperbola; then the orthocentre of the triangle also lies on the same hyperbola
- All conics passing through the intersection of two rectangular hyperbolas are themselves rectangular hyperbolas.

22 Intersection of a Circle and a Rectangular Hyperbola

If a circle $x^2 + y^2 + 2gx + 2fy + k = 0$ cuts a rectangular hyperbola $xy = c^2$ in A, B, C and D and the parameters of these four points t_1, t_2, t_3 and t_4 respectively, then :

01

(i) $\sum t_1 = -\frac{2g}{c}$

(ii) $\sum t_1t_2 = \frac{k}{c^2}$

(iii) $\sum t_1t_2t_3 = \frac{-2f}{c}$

(iv) $t_1t_2t_3t_4 = 1$

(v) $\sum \frac{1}{t_1} = -\frac{2f}{c}$