KCET PYQ K (2022)

Kannada

MATHEMATICS

Part-A

Answer any ten questions: 10×1 = 10

1. Show that the relation R in the set {1, 2, 3} given by R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} is not transitive.

2. Let * be the binary operation on N given by $a*b$=L.C.M. of a and b. Find 5*7.

3. Write the principal value branch of $y = \cot^{-1} x$.

4. Find the value of $\cos(\sec^{-1} x+\operatorname{cosec}^{-1}x)$, $|x|\geq 1$.

5. Define a diagonal matrix.

6. Find the value of x, if $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$.

7. If $y = \sin(ax+b)$, find $\frac{dy}{dx}$.

8. Differentiate $y = e^{x^3}$ with respect to x.

9. Find $\int \sec x(\sec x + \tan x)dx$.

10. Evaluate $\int_2^3 x^2 dx$.

11. Find the unit vector in the direction of the vector $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$.

12. Define Collinear Vectors.

13. Write the direction cosines of y-axis.

14. Define feasible region in a linear programming problem.

15. If $P(A) = 0.6$, $P(B) = 0.3$ and $P(A\cap B) = 0.2$, find $P(A|B)$.

Part-B

Answer **any ten** questions: **10 × 2 = 20**

16. Verify whether the operation * defined on the set of rational Q by $a * b = \frac{ab}{2}$ is associative or not.

17. Show that

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$$

18. Find the value of $\tan^{-1}\left(\sqrt{3}\right) - \cot^{-1}\left(-\sqrt{3}\right)$.

19. If $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, find X and Y.

20. Find the area of the triangle whose vertices are (2, 7), (1, 1) and (10, 8) using determinants.

21. If $2x + 3y = \sin x$, find $\frac{dy}{dx}$.

22. Differentiate $x^{\sin x}$, $x > 0$ with respect to x.

23. Find $\frac{dy}{dx}$, if $y = \log_7(\log x)$.

24. Find the approximate value of $\sqrt{25.3}$.

25. Evaluate $\int x^2 \log x dx$

26. Find $\int \frac{\sin^2 x}{1+\cos x} dx$.

27. Evaluate $\int_0^{\frac{\pi}{4}} \sin 2x dx$.

28. Find the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$.

29. Find the position vector of a point R which divides the line joining two points P and Q whose position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio 2:1 internally.

30. Find the area of the parallelogram whose adjacent sides are given by $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

31. Find the distance of a point (3, -2, 1) from the plane 2x - y + 2z + 3 = 0.

32. Find the angle between the pair of lines given by $\vec{r} = \left(3\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$ and $\vec{r} = \left(5\hat{i} - 2\hat{j}\right) + \mu\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)$.

33. The random variable x has a probability distribution $P(x)$ of the following form, where k is some number: $P(x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$ Determine the value of k.

Part-C

Answer **any ten** questions **10 × 3 = 30**

34. Show that the relation R in the set Z of integers given by R = {(a,b): 2 divides(a – b)} is an equivalence relation.

35. Solve $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

36. Express the matrix $A = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

37. Without expanding and using the property of determinants, prove that

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0.$$

38. Find $\frac{dy}{dx}$, if $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $(0 < x < 1)$.

39. If $x = a(\theta - \sin\theta)$ and $y = a(1+\cos\theta)$, find $\frac{dy}{dx}$

40. Verify Mean value theorem for the function $f(x) = x^2$ in the interval [2, 4].

41. Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is
(a) increasing
(b) decreasing

42. Find $\int \frac{x}{(x+1)(x+2)} dx$.

43. Evaluate $\int \frac{\tan^{-1} x}{1+x^2} dx$.

44. Find the integral of $\frac{(x-3)}{(x-1)^3} e^x$ with respect to x.

45. Find the area of the region bounded by the curve $y^2 = x$ and the lines $x = 1$, $x = 4$ and the x-axis in the first quadrant.

46. Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis

47. Find the equation of the curve passing through the point (1, 1) whose differential equation is $x\,dy = (2x^2 + 1)dx$, $(x \neq 0)$.

48. Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=1, |\vec{b}|=4$ and $|\vec{c}|=2$

49. Prove that $\left[\vec{a},\vec{b},\vec{c}+\vec{d}\right]=(\left[\vec{a},\vec{b},\vec{c}\right]+\left[\vec{a},\vec{b},\vec{d}\right]$.

50. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point (2, 2, 1).

51. A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.

Part-D

Answer any six questions: **6 × 5 = 30**

52. Verify whether the function $f:R\to R$ defined by $f(x) = 1+ x^2$ is one-one, onto or bijective. Justify your answer.

53. Consider $f: R\to R$ given by $f(x) = 4x +3$. Show that f is invertible. Find the inverse of f.

54. If $A=\begin{bmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{bmatrix}$, $B=\begin{bmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{bmatrix}$ and $C=\begin{bmatrix}4 & 1 & 2\\ 0 & 3 & 2\\ 1 & -2 & 3\end{bmatrix}$, then compute $(A+B)$ and $(B-C)$.

Also, verify that $A+(B-C) = (A+B)-C$.

55. Solve the following system of equations by matrix method: $3x-2y + 3z = 8$ $2x + y - z = 1$$4x - 3y + 2z = 4$.

56. If $y = \left(\tan^{-1} x\right)^2$, show that

$\left(x^2+1\right)^2 y_2 + 2x\left(x^2+1\right)y_1 = 2$.

57. The length x of a rectangle is decreasing at the rate of 3 cm/min and the width y is increasing at the rate of 2 cm/min. When $x = 10$ cm and $Y = 6$ cm, find the rates of change of

(a) the perimeter and

(b) the area of the rectangle

58. Find the integral $\frac{1}{x^2-a^2}$ with respect to x and hence evaluate $\int\frac{1}{x^2-16}dx$.

59. Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ using integration.

60. Find the general solution of the differential equation $\cos^2 x \frac{dy}{dx}+y=\tan x$, $(0\leq x<\frac{\pi}{2})$.

61. Derive the equation of a line in a space through a given point and parallel to a given vector $\vec{b}$ both in vector and Cartesian form.

62. A bag contains 4 red and 4 black balls; another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

63. If a fair coin is tossed 10 times, find the probability of a) exactly six heads b) at least six heads

Part-E

Answer any one question: **1 × 10 = 10**

64. **(**a) Solve the following linear programming problem graphically:

Maximize $Z = 4x + y$,

Subject to the constraints:

$x + y \leq 50,\ 3x + y \leq 90,\ x \geq 0,\ y \geq 0$

(b) If the matrix A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = 0$, where I is 2 x 2 identity matrix and 0 is 2 x 2 zero matrix. Using this equation, find A^{-1}.

65. **(**a) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ and hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$. (6)

(b) Find the value of k so that the function $f(x) = \begin{cases} kx+1, & \text{if} x \leq \pi \\ \cos x, & \text{if} x > \pi \end{cases}$ is continuous at $x = \pi$. (4)

66. **(**a) Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area. (6)

(b) Prove that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$ (4)

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Mathematics

Hint & Solution

1. For a relation to be transitive, if $(a, b) \in R$, then (a, c) must belong to R. From the given relation R:
• $(1, 2) \in R$
• $(2, 3) \in R$
However, $(1, 3) \notin R$,
Therefore, the relation R is not transitive.

2. Given operation: a * b = L.C.M. of a and b
For 5 * 7:
5 * 7 = L.C.M. of 5 and 7
Since 5 and 7 are co-prime numbers, their L.C.M. is their product:
$5 \times 7 = 35$

3. The principal value branch (range) of the inverse cotangent function is: $(0, \pi)$

4. Using the inverse trigonometric identity:
$$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$$
Substitute this value into the expression:
$$\cos\left(\sec^{-1} x + \csc^{-1} x\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

5. A square matrix is said to be a diagonal matrix if all its non-diagonal elements are zero.

6. Evaluate the determinants on both sides:
$$(2\times5)-(3\times4)=(x\times5)-(3\times2x)$$
$$10-12=5x-6x$$
$$-2=-x$$
$$x=2$$

7. Using the chain rule of differentiation:
$$\frac{dy}{dx}=\frac{d}{dx}[\sin ax+b]$$
$$\frac{dy}{dx}=\cos(ax+b).\frac{d}{dx}(ax+b)$$
$$\frac{dy}{dx}=\cos(ax+b).(a)$$
$$\frac{dy}{dx}=a\cos(ax+b)$$

8. Using the chain rule of differentiation:
$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{x^3}\right)$$
$$\frac{dy}{dx}=e^{x^3}.\frac{d}{dx}(x^3)$$
$$\frac{dy}{dx}=e^{x^3}.(3x^2)$$
$$\frac{dy}{dx}=3x^2e^{x^3}$$

9. Expand the terms inside the integral:
$$\int\left(\sec^2 x+\sec x\tan x\right)dx$$
Integrate term by term using standard identities:
$$\int\sec^2 x dx+\int\sec x\tan x\, dx$$
$$=\tan x+\sec x+C$$
(Where C is the constant of integration)

10. Integrate using the power rule:
$$\int_2^3 x^2dx=\left[\frac{x^3}{3}\right]_2^3$$
Substitute the upper and lower limits:
$$=\frac{1}{3}[27-8]=\frac{19}{3}$$

11. The unit vector $\vec{a}$ is given by $\frac{\vec{a}}{|\vec{a}|}$.

1. Find the magnitude $|\vec{a}|$:

$$|\vec{a}| = \sqrt{2^2+3^2+1^2} = \sqrt{4+9+1} = \sqrt{14}$$

2. Compute the unit vector:

$$\hat{a} = \frac{2\hat{i}+3\hat{j}+k}{\sqrt{14}} = \frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}$$

12. Two or more vectors are said to be collinear vectors if they are parallel to the same straight line, irrespective of their magnitudes and directions.

13. The *y*-axis makes an angle of 90° with the x-axis, and 90° with the *z*-axis.
Direction cosines (*l*, *m*, *n*) are:

$l = \cos 90° = 0$

$m = \cos 0° = 1$

$n = \cos 90° = 0$

(0, 1, 0)

14. The common region determined by all the constraints including non-negative constraints of a linear programming Problem is called the feasible region.

15. Using the formula for conditional probability:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Substitute the given values:

$$P(A/B) = \frac{0.2}{0.3} = \frac{2}{3}$$

16. For associativity, we must verify if $(a*b)*c = a*(b*c)$ for all *a*, *b*, *c* $\in Q$

1. Calculate Left-Hand Side (LHS) :

$$(a*b)*c = \left(\frac{ab}{2}\right)*c$$

$$= \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

2. Calculate Right-Hand Side (RHS)

$$(a*b)*c = a*\left(\frac{bc}{2}\right)$$

$$= \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

Since LHS = RHS, the binary operation * is associative.

17. Let $\sin^{-1} x = \theta \Rightarrow x = \sin\theta$

Substitute $x = \sin\theta$ into the Left-Hand Side (LHS):

LHS = $\sin^{-1}(2\sin\theta)\sqrt{1-\sin^2\theta})$

$= \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

$= \sin^{-1}(2\sin\theta\cos\theta)$

Using the double-angle identity
$2\sin\theta\cos\theta = \sin 2\theta$

$= \sin^{-1}(\sin 2\theta)$

$= 2\theta$

Substitute $\theta = \sin^{-1} x$ back

$= 2\sin^{-1} x = RHS$

18. 1. Find individual values using principal value branches:

1. $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

2. $\cot^{-1}\left(\sqrt{3}\right) = \pi - \cot^{-1}\left(\sqrt{3}\right) = \pi - \frac{\pi}{6} - \frac{5\pi}{6}$

2. Substitute into the expression:

$$\tan^{-1}\left(-\sqrt{3}\right) - \cot^{-1}\left(-\sqrt{3}\right) = \frac{\pi}{3} = \frac{5\pi}{6}$$

$$= \frac{2\pi - 5\pi}{6} = \frac{3\pi}{6} = -\frac{\pi}{2}$$

19. Let the equation be:

$$X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \quad \ldots..(1)$$

$$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \ldots\ldots(2)$$

1. Add Equation 1 and Equation 2 to find *X*:

$$2X = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

2. Subtract Equation2 from Equation 1 to find Y:

$$2Y = \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 2 & 2 \end{bmatrix}$$

$$Y = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

20. The area of a triangle formula with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is:

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Substitute the points $(2,7), (1,1), (10,8)$:

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

Expand along the first row (R_1):

$$= \frac{1}{2}[2(1-8) - 7(1-10) + 1(8-10)]$$

$$= \frac{1}{2}[2(-7) - 7(-9) + 1(-2)]$$

$$= \frac{1}{2}[-14 + 63 - 2]$$

$$= \frac{1}{2}[47] = 23.5$$

21. Differentiate both side with respect to x:

$$\frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(\sin x)$$

$$2 + 3\frac{dy}{dx} = \cos x$$

$$3\frac{dy}{dx} = \cos x - 2$$

$$\frac{dy}{dx} = \frac{\cos x - 2}{3}$$

22. Let $y = x^{\sin x}$

Take natural logarithm (log) on both sides:

$$\log y = \log(x \sin x)$$

$$\log y = \sin x \cdot \log x$$

Differentiate both sides with respect to x using the product rule on the RHS:

$$\frac{1}{y}\frac{dy}{dx} = \sin x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x\left(\frac{1}{x}\right) + \log x(\cos x)$$

$$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right]$$

Substitute $y = x^{\sin x}$ back

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

23. Using base change formula, rewrite y:

$$y = \frac{\log(\log x)}{\log 7}$$

Differentiate with respect to x using chain rule:

$$\frac{dy}{dx} = \frac{1}{\log 7} \cdot \frac{d}{dx}[\log(\log x)]$$

$$\frac{dy}{dx} = \frac{1}{\log 7} \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \frac{1}{\log 7} \cdot \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x \log x \log 7}$$

24. Let $f(x) = \sqrt{x}$.. Take $x = 25$ and $\Delta x = 0.3$

1. Find $f(x)$:

$$f(25) = \sqrt{25} = 5$$

2. Find the derivative $f'(x)$:

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10} = 0.1$$

3. Compute approximate formula:

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

$\sqrt{25.3} \approx 5 + (0.1)(0.3)$

$\sqrt{25.3} \approx 5 + 0.03 = 5.03$

25. Using integration by Parts formula:

$\int uv\,dx = u\int v dx - \int (u' \int v dx) dx.$

Following ILATE rule, choose

$u = \log x, \qquad dv = x^2 dx.$

Then,

$\frac{du}{dx} = \frac{1}{x},$

and

$v = \int x^2 dx = \frac{x^3}{3}.$

Applying the formula

$$\int x^2 \log x dx = (\log x)\left(\frac{x^3}{3}\right) - \int \left(\frac{1}{x} \cdot \frac{x^3}{3}\right) dx$$

$$= \frac{x^3 \log x}{3} - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3 \log x}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + C$$

$$= \frac{x^3 \log x}{3} - \frac{x^3}{9} + C$$

26. Using the identity $\sin^2 x = 1 - \cos^2 x,$

$$\int \frac{1-\cos^2 x}{1+\cos x} dx.$$

Factorise the numerator using

$a^2 - b^2 = (a-b)(a+b):$

$$= \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x} dx.$$

$$= \int (1-\cos x) dx.$$

$$= x - \sin x + C.$$

27. Integrate the function:

$$\int_0^{\pi/4} \sin 2x dx = \left[-\frac{\cos 2x}{2}\right]_0^{\pi/4}$$

Substitute the upper and lower limits:

$$= -\frac{1}{2}\left[\cos\left(2 \cdot \frac{\pi}{4}\right) - \cos(0)\right]$$

$$= -\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right]$$

$$= -\frac{1}{2}[0-1] = \frac{1}{2}$$

28. 1. Order: The highest derivative present is $\frac{d^2 y}{dx}$.

Thus, Order = 2

2. Degree: The differential equation is not a polynomial in its derivatives due to the $\sin\left(\frac{dy}{dx}\right)$ term. Thus, Degree is not defined.

29. Let $\vec{a} = \hat{i} + 2\hat{j} - \hat{k},\ \vec{b} = -\hat{i} + \hat{j} + \hat{k},\ m = 2,$ and $n = 1.$

The section formula for internal division is:

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m+n}.$$

Substitute the vectors and values:

$$\vec{r} = \frac{2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})}{2+1}$$

$$= \frac{(-2\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})}{3}$$

$$= \frac{-\hat{i} + 4\hat{j} + \hat{k}}{3} \qquad \vec{r} = -\frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

30. The area of a parallelogram is given by $|\bar{a} \times \bar{b}|$.

1. Find cross product $\bar{a} \times \bar{b}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}\big(1(-4)\big) - \hat{j}\big(3-4\big) + \hat{k}\big(-3-1\big)$$

$$= 5\hat{i} + \hat{j} - 4\hat{k}.$$

2. Calculate the magnitude:

$$|\vec{a} \times \vec{b}| = \sqrt{5^2 + 1^2 + (-4)^2}$$

$$= \sqrt{25+1+16} = \sqrt{42}.$$

31. The perpendicular distance d from a point (x_1, y_1, z_1) to a plane $Ax + B_y + Cz + D = 0$ is

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Substitute the values (3, –2, 1) and plane parameters $A = 2$, $B = -1$, $C = 2$, $D = 3$:

$$d = \frac{|2(3) - (-2) + 2(1) + 3|}{\sqrt{2^2 + (-1)^2 + 2^2}}.$$

$$d = \frac{|6 + 2 + 2 + 3|}{\sqrt{4 + 1 + 4}} = \frac{13}{\sqrt{9}} = \frac{13}{3}.$$

32. The directional vectors of the lines are

$\vec{b}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 2\hat{j} + 6\hat{k}$.

The angle θ between them satisfies:

$$\cos\theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}.$$

1. Calculate dot product:

$$\vec{b}_1 \cdot \vec{b}_2 = (1)(3) + (2)(2) + (2)(6)$$
$$= 3 + 4 + 12 = 19$$

2. Calculate magnitudes:

$$|\vec{b}_1| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3.$$

$$|\vec{b}_2| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7.$$

3. Compute angle:

$$\cos\theta = \frac{19}{3 \times 7} = \frac{19}{21} \Rightarrow \theta = \cos^{-1}\left(\frac{19}{21}\right).$$

33. The sum of all probabilities in a probability distribution must equal 1:

$$\sum P(x) = 1.$$

$$P(0) + P(1) + P(2) = 1.$$

$$k + 2k + 3k = 1.$$

$$6k = 1 \Rightarrow k = \frac{1}{6}.$$

34. To prove R is an equivalence relation, we must verify that it is Reflexive, Symmetric, and Transitive.

1. Reflexive:

For any $a \in Z$. we have $a - a = 0$

Since 0 is divisible by 2, we can write 2 divides $(a, -a)$

$$\Rightarrow (a, a) \in R, \forall a \in Z$$

Therefore, R is reflexive.

2. Symmetric:

Let $(a, b) \in R$

$\Rightarrow$ 2 divides $(a - b)$, which means $a - b = 2k$ for some integer k.

Rearranging gives $b - a = -(a - b) = -2k$ $= 2(-k)$.

Since $-k$ is also an integer, 2 divides $(b - a)$

$$\Rightarrow (b, a) \in R$$

Therefore, R is symmetric.

3. Transitive:

Let $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow$ 2 divides $(a - b)$ and 2 divides $(b - c)$.

Thus, $a - b = 2\text{m}$ and $b - c = 2n$ for some integers m, n.

Adding both equations:

$$(a - b) + (b - c) = 2m + 2n$$

$$a - c = 2(m + n)$$

Since $(m + n)$ is an integer, 2 divides $(a - c)$.

$$\Rightarrow (a, c) \in R$$

Therefore, R is transitive.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

35. Using the identity

$$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A + B}{1 - AB}\right):$$

$$\tan^{-1}\left(\frac{2x + 3x}{1 - (2x)(3x)}\right) = \frac{\pi}{4}.$$

$$\frac{5x}{1 - 6x^2} = \tan\left(\frac{\pi}{4}\right).$$

Since, $\tan\left(\frac{\pi}{4}\right) = 1,$

$$\frac{5x}{1 - 6x^2} = 1$$

$$5x = 1 - 6x^2$$

$$6x^2+5x-1=0$$
$$6x^2+6x-x-1=0.$$
$$6x(x+1)-1(x+1)=0.$$
$$(x+1)(6x-1)=0.$$
$$x=-1 \quad \text{or} \quad x=\frac{1}{6}.$$

If $x=-1$, then $\tan^{-1}(-2)\tan^{-1}(-3)$ becomes a negative value, which cannot equal $\frac{\pi}{4}$. Thus $x=-1$ is discarded.

36. Any square matrix A can be expressed as $A = P + Q$, where:

$P=\frac{1}{2}(A+A')$ is a symmetric matrix.

$Q=\frac{1}{2}(A+A')$ is a skew-symmetric matrix.

First, compute the transpose of matrix A:

$$A'=\begin{bmatrix}1 & -1\\ 5 & 2\end{bmatrix}$$

1. Compute P.

$$A+A'=\begin{bmatrix}1+1 & 5+(-1)\\ -1+5 & 2+2\end{bmatrix}=\begin{bmatrix}2 & 4\\ 4 & 4\end{bmatrix}$$

$$P=\frac{1}{2}\begin{bmatrix}2 & 4\\ 4 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$$

2. Compute Q:

$$A-A'=\begin{bmatrix}1-1 & 5-(-1)\\ -1-5 & 2-2\end{bmatrix}=\begin{bmatrix}0 & 6\\ -6 & 0\end{bmatrix}$$

$$Q=\frac{1}{2}\begin{bmatrix}0 & 6\\ -6 & 0\end{bmatrix}=\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}$$

Conclusion:

$$A=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}+\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}$$

37. Let $\Delta=\begin{vmatrix}2 & 7 & 65\\ 3 & 8 & 75\\ 5 & 9 & 86\end{vmatrix}$

Apply the column operation $C_3 \to C_3-9C_2$.

For row 1: $65-9(7)=65-63=2$

For row2: $75-9(8)=75-72=3$

For row 3: $86-9(9)=86-81=5$

Substituting these values into column 3:

$$\Delta=\begin{vmatrix}2 & 7 & 2\\ 3 & 8 & 3\\ 5 & 9 & 5\end{vmatrix}.$$

Since column 1(C_1) and column 3(C_3) are identical, the value of the determinant is automatically zero by properties of determinants.

$\Delta=0$

38. Substitute $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$.

$$y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right).$$

Using the double-angle identity

$$\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta,$$

$$y=\cos^{-1}(\cos 2\theta)=2\theta.$$

Substituting $\theta=\tan^{-1}x$ back into the expression,

$$y=2\tan^{-1}x.$$

Differentiating with respect to x.

$$\frac{dy}{dx}=2\cdot\frac{1}{1+x^2}=\frac{2}{1+x^2}.$$

39. This is a parametric differentiation problem.

Differentiate both variables with respect to θ.

1. Differentiate x:

$$\frac{dx}{d\theta}=\frac{d}{d\theta}\left[a(\theta-\sin\theta)\right]=a(1-\cos\theta).$$

2. Differentiate y:

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left[a(1+\cos\theta)\right]$$
$$=a(0-\sin\theta)=-a\sin\theta.$$

3. Compute $\frac{dy}{dx}$:

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-\sin\theta}{1-\cos\theta}.$$

Using trigonometric half-angle identities

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \qquad 1-\cos\theta = 2\sin^2\frac{\theta}{2},$$

$$\frac{dy}{dx} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\left(\frac{\theta}{2}\right).$$

40. 1. Verify Conditions:

1. $f(x) = x^2$ is a polynomial function, so it is continuous on [2, 4]
2. Its derivative $f'(x) = 2x$ exists for all values in (2, 4) making it differentiable on (2, 4).

2. Apply the theorem Formula:
By mean value theorem, there must exist at least one value of $c \in (2, 4)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

3. Compute boundary constraints (a = 2, b = 4):

1. $f(2) = 2^2 = 4.$
2. $f(4) = 4^2 = 16.$

4. Solve for c:

$$2c = \frac{16-4}{4-2}$$

$$2c = \frac{12}{2} = 6$$

$$c = 3.$$

Since $c = 3 \in (2, 4)$, the Mean value Theorem is verified.

41. First, find the first derivative $f'(x)$:

$f'(x) = 2x - 4$

Set $f'(x) = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2.$. This splits the domain into two intervals: $(-\infty, 2)$ and $(2, \infty)$

(a) For increasing function:

We need $f'(x) > 0.$

$2x - 4 > 0 \Rightarrow 2x > 4 \Rightarrow x > 2.$

Therefore, the function is strictly increasing in the interval $(2, \infty)$.

(b) For Decreasing Function:

We need $f'(x) < 0.$

Therefore, the function is strictly decreasing in the interval $(-\infty, 2)$.

42. Resolve the expression using Partial Fractions:

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

$x = A(x+2) + B(x+1).$

Set $x = -1 \Rightarrow -1 = A(-1+2) \Rightarrow A = -1$

Set $x = -2 \Rightarrow -2 = B(-2+1) \Rightarrow B = 2.$

Substitute the coefficients back into the integral:

$$= \int\left(-\frac{1}{x+1} + \frac{2}{x+2}\right)dx.$$

$$= -\log|x+1| + 2\log|x+2| + C.$$

Using log properties $(2\log|x+2| = \log\left((x+2)^2\right),$

$$= \log\left|\frac{(x+2)^2}{x+1}\right| + C.$$

43. Use integration by substitution. Let

$$t = \tan^{-1}x \Rightarrow dt = \frac{1}{1+x^2}dx.$$

Change limits:

When $x = 0$, $t = \tan^{-1}(0) = 0$

When $x = 1$, $t = \tan^{-1}(1) = \frac{x}{4}$

Substitute t and dt into the definite integral:

$$\int_0^{\pi/4} t\,dt = \left[\frac{t^2}{2}\right]_0^{\pi/4}.$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - 0\right] = \frac{1}{2}\cdot\frac{\pi^2}{16} = \frac{\pi^2}{32}.$$

44. Rearrange the algebraic terms to match the structure of the denominator:

$$\int e^x\left[\frac{x-1-2}{(x-1)^3}\right]dx = \int e^x\left[\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3}\right]dx$$

$$= \int e^x \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right] dx.$$

This matches the standard identity form

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C.$$

Let $f(x) = \frac{1}{(x-1)^2}, \quad f'(x) = -\frac{2}{(x-1)^3}$

$$= \frac{e^x}{(x-1)^2} + C$$

45. The equation (taking positive root for the first quadrant evaluation).

The boundaries range along the horizontal direction from $x = 1$ to $x = 4$.

The area A is computed as:

$$A = \int_1^4 y\,dx = \int_1^4 x^{1/2}\,dx.$$

$$A = \left[\frac{x^{3/2}}{3/2} \right]_1^4 = \frac{2}{3} \left[x^{3/2} \right]_1^4.$$

$$A = \frac{2}{3} \left(4^{3/2} - 1^{3/2} \right).$$

Note that $4^{3/2} = \left(2^2 \right)^{3/2} = 2^3 = 8.$

$$A = \frac{2}{3}(8-1) = \frac{2}{3}(7) = \frac{14}{3}.$$

$Answer: \frac{14}{3}$ sq. units.

46. The general equation representing a family of parabolas centered at the origin with its axis aligned symmetrically along the positive x-axis is:

$y^2 = 4a$ ………..(1)

(where a is the arbitrary constant)

Differentiating both sides with respect to x:

Differentiating both sides with respect to x:

$2y \frac{dy}{dx} = 4a$ ……(2)

Isolate 4a from equation 1 : $4a = \frac{y^2}{x}$. Substitute this expression into Equation (2):

$$2y \frac{dy}{dx} = \frac{y^2}{x}$$

Divide both sides by y (where $y \neq 0$):

$$2 \frac{dy}{dx} = \frac{y}{x} \Rightarrow 2x \frac{dy}{dx} - y = 0$$

47. Isolate variables to separate formats on both sides:

$$dy = \frac{2x^2 + 1}{x} dx \Rightarrow dy = \left(2x + \frac{1}{x} \right) dx$$

Integrate both side of the equation:

$$\int dy = \int \left(2x + \frac{1}{x} \right) dx$$

$$y = x^2 + \log|x| + C$$

Since the curve passes through the specific coordinate point (1, 1), substitute $x = 1$ and $y = 1$ to compute the constant C:

$$1 = (1)^2 + \log|1| + C$$

Since log 1 = 0

$$1 = 1 + 0 + C \Rightarrow C = 0$$

Substitute $C = 0$ back into the general expression:

$$y = x^2 + \log|x|$$

48. Given structural vector statement:

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

Take the dot product of this vector expression with itself.

$$(\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = \vec{0} \cdot \vec{0}$$

$$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

Substitute the given scalar magnitudes

$$(1)^2 + (4)^2 + (2)^2 + 2\mu = 0$$

$$1 + 16 + 4 + 2\mu = 0 \Rightarrow 21 + 2\mu = 0$$

$$\Rightarrow \mu = -10.5 \text{ (or } -\frac{21}{2})$$

49. By definition, the scalar triple product $\left| \vec{x}, \vec{y}, \vec{z} \right|$ is expressed as $\vec{x}.\left(\vec{y} \times \vec{z} \right)$.

Evaluate the Left Hand Side (LHS) :

$$\text{LHS} = \vec{a} \cdot [\vec{b} \times (\vec{c} + \vec{d})]$$

Using the distributive property of vector cross products over addition:

$$= \vec{a} \cdot [(\vec{b} \times \vec{c}) + (\vec{b} \times \vec{d})]$$

Using the distributive property of dot products over vector addition:

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{d})$

Convert back into the standard bracket scalar triple notation format:

$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{d}] =$ RHS (Hence Proved)

50. The equation of any plane passing through the intersecting boundaries of two planes is represented as:

$$(3x - y + 2z - 4) + \lambda(x + y + z - 2) = 0$$

Since this custom plane passes directly through the point context (2, 2, 1), substitute these values to isolate parameters:

$$[3(2) - 2 + 2(1) - 4] + \lambda(2 + 2 + 1 - 2) = 0$$

$$[6 - 2 + 2 - 4] + \lambda(3) = 0$$

$$2 + 3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}$$

Substitute $\lambda = -\frac{2}{3}$ back into the primary intersection profile equation:

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

Multiply the entire equation parameters by 3 to simplify the fractions out:

$$3(3x - y + 2z - 4) - 2(x + y + z - 2) = 0$$

$$9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$$

$$7x - 5y + 4z - 8 = 0$$

51. The total sample space S for tossing a coin and rolling a die has $2 \times 6 = 12$ total outcomes

$S = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

1. Event A (Head appears on the coin):

   $A = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\}$

   $\Rightarrow P(A) = \frac{6}{12} = \frac{1}{2}$

2. Event B (3 on the die):

   $B = \{(H,3),(T,3)\} \Rightarrow P(B) = \frac{2}{12} = \frac{1}{6}$

3. Intersection Event $A \cap B$ (Head on coin and 3 on die):

   $A \cap B = \{(H,3)\} \Rightarrow P(A \cap B) = \frac{1}{12}$

4. Test Independence Condition ($P(A \cap B) = P(A) \cdot P(B)$):

   $P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

Since $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{12}$, the events A and B are independent events.

52.

1. One-one (Injectivity): Let $x_1 = 1$ and $x_2 = -1$. Both belong to domain R.

   $f(1) = 1 + (1)^2 = 2$ and $f(-1) = 1 + (-1)^2 = 2$

   Since $f(1) = f(-1) = 2$ but $1 \neq -1$, the function is not one-one.

2. Onto (Surjectivity): Let $y = -3 \in R$ (co-domain). Set $y = f(x)$:

   $-3 = 1 + x^2 \Rightarrow x^2 = -4 \Rightarrow x = \pm 2i \notin R$.

   Negative real numbers have no pre-images in the domain. Thus, it is not onto.

Conclusion: Since it is neither one-one nor onto, it is not bijective.

53.

1. One-one: Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

   $4x_1 + 3 = 4x_2 + 3 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$

   Thus, it is one-one.

2. Onto: Let $y \in R$ (co-domain). Set $y = f(x)$: $y = 4x + 3 \Rightarrow 4x = y - 3 \Rightarrow x = \frac{y-3}{4}$.

   For any $y \in R, x = \frac{y-3}{4} \in R$ (domain) exists such that $f(x) = y$. Thus, it is onto.

3. Inverse: Since f is one-one and onto, it is invertible. Its inverse is: $f^{-1}(x) = \frac{x-3}{4}$.

54. 1. Compute $(A + B)$:

$$A+B=\begin{bmatrix}1+3 & 2-1 & -3+2\\5+4 & 0+2 & 2+5\\1+2 & -1+0 & 1+3\end{bmatrix}=\begin{bmatrix}4 & 1 & -1\\9 & 2 & 7\\3 & -1 & 4\end{bmatrix}$$

2. Compute $(B - C)$:

$$B-C=\begin{bmatrix}3-4 & -1-1 & 2-2\\4-0 & 2-3 & 5-2\\2-1 & 0+2 & 3-3\end{bmatrix}=\begin{bmatrix}-1 & -2 & 0\\4 & -1 & 3\\1 & 2 & 0\end{bmatrix}$$

3. Verify LHS = $A + (B - C)$:

$$LHS=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}+\begin{bmatrix}-1 & -2 & 0\\4 & -1 & 3\\1 & 2 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & -3\\9 & -1 & 5\\2 & 1 & 1\end{bmatrix}$$

4. Verify RHS = $(A + B) - C$:

$$RHS=\begin{bmatrix}4 & 1 & -1\\9 & 2 & 7\\3 & -1 & 4\end{bmatrix}-\begin{bmatrix}4 & 1 & 2\\0 & 3 & 2\\1 & -2 & 3\end{bmatrix}=\begin{bmatrix}0 & 0 & -3\\9 & -1 & 5\\2 & 1 & 1\end{bmatrix}$$

Since LHS = RHS, the property is verified.

55. Write in matrix form $AX = B: A$

$$A=\begin{bmatrix}3 & -2 & 3\\2 & 1 & -1\\4 & -3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}8\\1\\4\end{bmatrix}$$

1. Determinant $|A|$:

$$|A|=3(2-3)-(-2)(4+4)+3(-6-4)$$

$$=-3+16-30=-17\neq 0$$

(Unique solution exists).

2. Cofactors of A:

$$A_{11}=-1, A_{12}=-8, A_{13}=-10; A_{21}=-5, A_{22}=-6, A_{23}=1; A_{31}$$

$$=-1, A_{32}=9, A_{33}=7$$

3. Adjoint matrix adj(A):

$$adj(A)=\begin{bmatrix}-1 & -5 & -1\\-8 & -6 & 9\\-10 & 1 & 7\end{bmatrix}$$

4. Compute $X=\dfrac{1}{|A|}adj(A)B$:

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{-17}\begin{bmatrix}-1 & -5 & -1\\-8 & -6 & 9\\-10 & 1 & 7\end{bmatrix}\begin{bmatrix}8\\1\\4\end{bmatrix}=\frac{1}{-17}\begin{bmatrix}-8-5-4\\-64-6+36\\-80+1+28\end{bmatrix}$$

$$=\frac{1}{-17}\begin{bmatrix}-17\\-34\\-51\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Answer: $x = 1$, $y = 2$, $z = 3$

56. Given: $y=(\tan^{-1}x)^2$

Differentiating w.r.t x using chain rule:

$$y_1=2(\tan^{-1}x)\cdot\frac{1}{x^2+1}\Rightarrow(x^2+1)y_1=2\tan^{-1}x$$

Differentiating both sides w.r.t x again using the product rule:

$$(x^2+1)y_2+y_1(2x)=2\cdot\frac{1}{x^2+1}$$

Multiplying the entire equation by (x^2+1) to eliminate the denominator:

$$(x^2+1)^2y_2+2x(x^2+1)y_1=2$$

(Hence Proved)

57. Given data:

- Rate of change of length: $\dfrac{dx}{dt}=-3\,\text{cm/min}$

(negative sign indicates a decrease)

- Rate of change of width: $\dfrac{dy}{dt}=2$ cm/min
- Current dimensions: $x = 10$ cm and $y = 6$ cm

Rate of change of perimeter (P):

The perimeter of a rectangle is given by:

$P = 2(x + y)$

Differentiating both sides with respect to time t:

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

Substitute the given values:

$$\frac{dP}{dt} = 2(-3+2) = 2(-1) = -2 \text{ cm/min}$$

The perimeter is decreasing at a rate of 2 cm/min.

(c) Rate of change of area (A):

The area of a rectangle is given by:

$$A = x \cdot y$$

Differentiating both sides with respect to time t using the product rule:

$$\frac{dA}{dt} = x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}$$

Substitute the given values:

$$\frac{dA}{dt} = (10)(2) + (6)(-3) = 20 - 18 = 2\ cm^2 / min$$

The area is increasing at a rate of 2 cm^2 / min.

58. Part 1: Derivation of the standard formula

Let $I = \int \frac{1}{x^2 - a^2} dx$.

Resolve the integrand using partial fractions:

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right]$$

$$= \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

Now, integrate both terms:

$$I = \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

$$I = \frac{1}{2a}[\log|x-a| - \log|x+a|] + C$$

Using the logarithmic property

$$\log M - \log N = \log\left(\frac{M}{N}\right):$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

Part 2: Evaluation of the given integral

Given: $\int \frac{1}{x^2 - 16} dx$

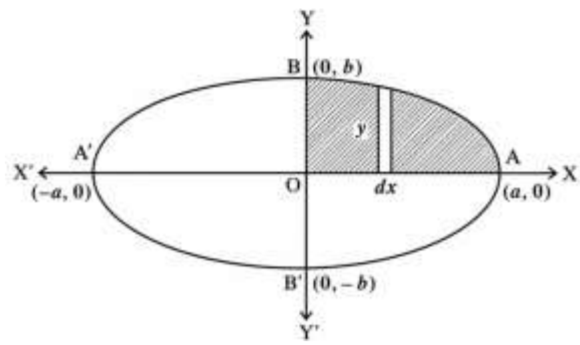
Rewrite the denominator as a difference of squares:

$$\int \frac{1}{x^2 - 4^2} dx$$

Apply the derived standard formula with $a = 4$:

$$= \frac{1}{2(4)}\log\left|\frac{x-4}{x+4}\right| + C = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + C$$

59. An ellipse is symmetrical about both the x-axis and y-axis. Therefore, the total area A is equal to 4 times the area of the region lying in the first quadrant (from $x = 0$ to $x = a$).



1. Express y in terms of x:

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$$

2. Set up the definite integral for the total area:

$$A = 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

3. Use the standard integration formula

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right):$$

$$A = \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

4. Substitute the limits:
   1. Upper limit ($x = a$):

   $$\frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1)\right] = \frac{4b}{a}\left[\frac{a^2}{2} \cdot \frac{\pi}{2}\right]$$

   $$= \frac{4b}{a}\left[\frac{\pi a^2}{4}\right] = \pi ab$$

   2. Lower limit ($x = 0$): Results in 0.

The total area enclosed is πab sq. units.

60. Divide by $\cos^2 x \Rightarrow \frac{dy}{dx}+(\sec^2 x)y=\sec^2 x\tan x$

(Standard form $\frac{dy}{dx}+Py=Q$)

Integrating Factor (I.F.) = $e^{\int Pdx}=e^{\int \sec^2 xdx}=e^{\tan x}$

General Solution: $y\cdot(I.F.)=\int Q\cdot(I.F.)dx+C$

$y\cdot e^{\tan x}=\int(\sec^2 x\tan x)e^{\tan x}dx+C$

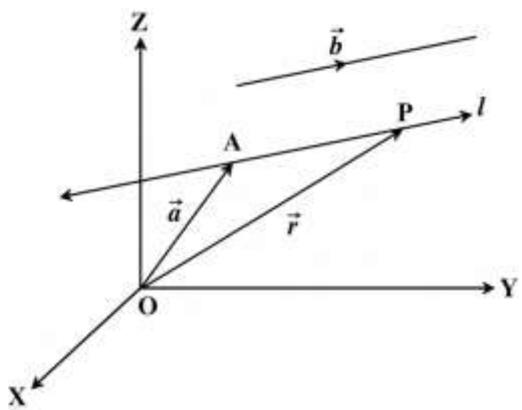
Substitute

$t=\tan x\Rightarrow dt=\sec^2 xdx\Rightarrow\int te^t dt=te^t-e^t$

$=e^{\tan x}(\tan x-1)$

$\Rightarrow y=\tan x-1+Ce^{-\tan x}$

61. Let line pass through point $A(\vec{a})$ and be parallel to vector $\vec{b}$. Let $P(\vec{r})$ be any point on the line.



Vector Form: Vector $\overrightarrow{AP}$ is parallel to $\vec{b}\Rightarrow\overrightarrow{AP}=\lambda\vec{b}$

$\vec{r}-\vec{a}=\lambda\vec{b}\Rightarrow\vec{r}=\vec{a}+\lambda\vec{b}$

Cartesian Form:

Let $A(x_1,y_1,z_1),\vec{b}=a\hat{i}+b\hat{j}+c\hat{k}$, and $P(x,y,z)$.

$(x\hat{i}+y\hat{j}+z\hat{k})=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$

Equating components:

$x=x_1+\lambda a, y=y_1+\lambda b, z=z_1+\lambda c$

Eliminating $\lambda\Rightarrow\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

62. Let E_1,E_2 be choices of Bag 1 and Bag 2 ($P(E_1)=P(E_2)=\frac{1}{2}$). Let A be drawing a red ball.

$P(A|E_1)=\frac{4}{8}=\frac{1}{2}$ and $P(A|E_2)=\frac{2}{8}=\frac{1}{4}$

By Bayes' Theorem:

$$P(E_1|A)=\frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)}$$

$$P(E_1|A)=\frac{\frac{1}{2}\times\frac{1}{2}}{(\frac{1}{2}\times\frac{1}{2})+(\frac{1}{2}\times\frac{1}{4})}=\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{8}}=\frac{\frac{1}{4}}{\frac{3}{8}}=\frac{2}{3}$$

63. Binomial distribution parameters:

$n=10, p=\frac{1}{2},q=\frac{1}{2}.P(X=x)=\binom{10}{x}(\frac{1}{2})^{10}$

(a) Exactly 6 heads (X = 6):

$$P(X=6)=\binom{10}{6}(\frac{1}{2})^{10}=\frac{210}{1024}=\frac{105}{512}$$

(b) At least 6 heads ($X\ge 6$):

$P(X\ge 6)=P(6)+P(7)+P(8)+P(9)+P(10)$

$=[\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10}]\frac{1}{1024}$

$=[210+120+45+10+1]\frac{1}{1024}=\frac{386}{1024}=\frac{193}{512}$

64. (a) Solve the following linear programming problem graphically: Maximize $Z = 4x + y$ subject to constraints:

$x + y \le 50, 3x + y \le 90, x \ge 0, y \ge 0$

1. Find boundary line intercepts:
    1. Line 1: x + y = 50 passes through (50, 0) and (0, 50).
    2. Line 2: 3x + y = 90 passes through (30, 0) and (0, 90).
2. Find the intersection point of the lines:

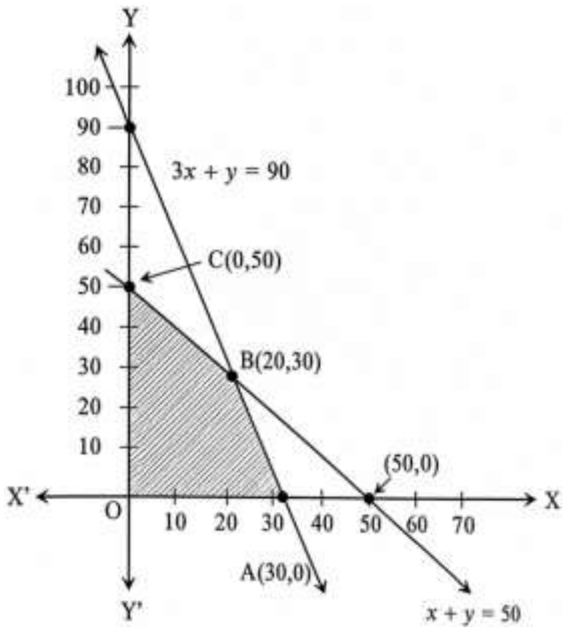
    Subtracting the two equations:

    $(3x+y)-(x+y)=90-50\Rightarrow 2x=40$

    $\Rightarrow x=20$.

Substitute $x = 20$ into $x + y = 50 \Rightarrow y = 30$.

The lines intersect at (20, 30).



3. Identify the corner points of the feasible region:
   Since constraints are "$\le$" and non-negative, the bounded area in the first quadrant has corner points: O(0, 0), A(30, 0), B(20, 30), and C(0, 50).
4. Evaluate objective function $Z = 4x + y$:
   1. At $O(0, 0)$: $Z = 4(0) + 0 = 0$
   2. At $A(30, 0)$: $Z = 4(30) + 0 = 120$
   3. At $B(20, 30)$: $Z = 4(20) + 30 = 110$
   4. At $C(0, 50)$: $Z = 4(0) + 50 = 50$

The maximum value of Z is 120, occurring at point (30, 0).

(b) If the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = 0$, find A^{-1} using this equation

Given matrix equation: $A^2 - 4A + I = 0$

Rearranging to isolate the identity matrix I:

$$I = 4A - A^2$$

Multiply both sides by A^{-1}:

$$A^{-1} \cdot I = A^{-1}(4A - A^2)$$

$$A^{-1} = 4(A^{-1}A) - A^{-1}A \cdot A$$

$$A^{-1} = 4I - A$$

Substitute matrices I and A:

$$A^{-1} = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

65. (a) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ and hence evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ (p. 18).

Proof:

Let $I = \int_0^a f(x)dx$. Put $t = a - x \Rightarrow dx = -dt$.

- When $x = 0 \Rightarrow t = a$
- When $x = a \Rightarrow t = 0$

Substitute into the integral:

$$I = \int_a^0 f(a-t)(-dt) = -\int_a^0 f(a-t)dt$$

$$= \int_0^a f(a-t)dt$$

Changing the dummy variable t to x:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

(Hence Proved)

Evaluation:

Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ --- (Equation 1)

Apply the proven property ($x \to \frac{\pi}{2} - x$):

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \text{--- (Equation 2)}$$

Adding Equation 1 and Equation 2:

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$= \int_0^{\pi/2} 1dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

(b) For continuity at $x = \pi$: LHL = RHL = $f(\pi)$

- LHL = $\lim_{x \to \pi^-} (kx + 1) = k\pi + 1$
- RHL = $\lim_{x \to \pi^+} (\cos x) = \cos \pi = -1$

Equating LHL and RHL:

$$k\pi + 1 = -1 \Rightarrow k\pi = -2 \Rightarrow k = -\frac{2}{\pi}$$

66. **(a)** Let a rectangle with length x and width y be inscribed inside a circle of fixed radius R. By Pythagoras' theorem, the diagonal of the rectangle matches the diameter

$$x^2 + y^2 = (2R)^2 = 4R^2$$

$$\Rightarrow y = \sqrt{4R^2 - x^2}$$

The area of the rectangle is $A = x \cdot y = x\sqrt{4R^2 - x^2}$. Squaring both sides to simplify:

$$S = A^2 = x^2(4R^2 - x^2) = 4R^2x^2 - x^4$$

Differentiate S with respect to x:

$$\frac{dS}{dx} = 8R^2x - 4x^3$$

For maxima/minima, set

$$\frac{dS}{dx} = 0 \Rightarrow 4x(2R^2 - x^2) = 0.$$

$$\Rightarrow x^2 = 2R^2 \Rightarrow x = \sqrt{2}R$$

Find the second derivative:

$$\frac{d^2S}{dx^2} = 8R^2 - 12x^2$$

Substitute

$$x^2 = 2R^2 \Rightarrow \frac{d^2S}{dx^2} = 8R^2 - 12(2R^2) = -16R^2 < 0$$

(Area is maximized).

Find y:

$$y = \sqrt{4R^2 - (\sqrt{2}R)^2} = \sqrt{4R^2 - 2R^2} = \sqrt{2}R.$$

Since length x equals width y, the rectangle is a square.

(b) Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Apply row operations $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$:

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$$

Take out common factors $(b-a)$ from R_2 and $(c-a)$ from R_3:

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

Expanding along the first column (C_1):

$$\Delta = (b-a)(c-a) \cdot 1 \cdot [(c+a) - (b+a)]$$

$$= (b-a)(c-a)(c-b)$$

Rearranging negative signs to match the required form:

$$\Delta = [-(a-b)][-(b-c)](c-a)$$

$$= (a-b)(b-c)(c-a)$$

(Hence Proved)